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Commutative Semigroups with Few Fully Invariant Congruences I.

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Simple objects in the class of semimodules over a semigroup are studied.

Simple objects in the classes of chains, semilattices and, more generally, commutative semigroups with a given automorphism group were studied in [1] – [7]. The aim of the present paper is to study commutative semigroups that are congruence-simple over an endomorphism semigroup.

1. Semigroups - preliminaries

Let S be a semigroup. We denote by $(\mathscr{I}_l(S), \mathscr{I}_r(S))\mathscr{I}(S)$ the set of (left, right) ideals of S and we put $(\mathscr{I}_l^o(S) = \mathscr{I}_l(S) \cup \{\emptyset\}, \mathscr{I}_r^o(S) = \mathscr{I}_r(S) \cup \{\emptyset\})\mathscr{I}^o(S) = \mathscr{I}(S) \cup \{\emptyset\}$.

A semigroup S will be called

- ideal-free if $I = S$ for every $I \in \mathscr{I}(S)$;
- ideal-simple if $I = S$ for every $I \in \mathscr{I}(S)$ such that $|I| \geq 2$;
- left (right) uniform if $Sa \cap Sb \neq \emptyset$ $(aS \cap bS \neq \emptyset)$ for all $a, b \in S$;
- uniform if S is both left and right uniform;
- hereditarily left (right) uniform (or hl(hr)-uniform for short) if every subsemigroup of S is left (right) uniform;
- hereditarily uniform (h-uniform) if S is both hl- and hr-uniform.

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The following observations and examples are easy to check:

Lemma 1.1. *S is hl-uniform if and only if $A \cap B \neq \emptyset$ whenever A, B are subsemigroups od S such that $AB \subseteq B$ and $BA \subseteq A$.*

Lemma 1.2. *Suppose that S is right cancellative. Then:*
(i) *S is hl-uniform if and only if no subsemigroup of S is a free semigroup of rank (at least) 2.*
(ii) *S is hl-uniform, provided that S contains no infinite subset P such that $a^n \neq b^m$ for all $a, b \in P$, $a \neq b$, and $m, n \geq 1$.*

Corollary 1.3. *Suppose that S is cancellative. Then the following conditions are equivalent:*
(i) *S is hl-uniform.*
(ii) *S is hr-uniform.*
(iii) *S is h-uniform.*
(iv) *No subsemigroup of S is a free semigroup of rank 2.*
(v) *No subsemigroup of S is a free semigroup of rank at least 2.*
(vi) *No subsemigroup of S is a free semigroup of rank $\aleph_0$.*

Example 1.4. (i) All commutative semigroups are h-uniform.
(ii) All periodic groups are h-uniform.
(iii) All locally nilpotent groups (and their subsemigroups) are h-uniform.
(iv) There exist metabelian groups which are not h-uniform.

A semigroup S will be called
- left (right) subcommutative if $aS \subseteq Sa$ $(Sa \subseteq aS)$ for every $a \in S$;
- subcommutative if $Sa = aS$ for every $a \in S$.

Lemma 1.5. (i) *Every left (right) subcommutative semigroup is left (right) uniform.*
(ii) *Every subcommutative semigroup is uniform.*

Proof. (i) We have $ab \in aS \cap Sb \subseteq Sa \cap Sb$.
(ii) The assertion follows immediately from (i). □

Lemma 1.6. *If S is a left subcommutative, then $\mathscr{I}_l(S) = \mathscr{I}(S)$.*

Proof. Obvious. □

Corollary 1.7. *If S is subcommutative, then $\mathscr{I}_l(S) = \mathscr{I}(S) = \mathscr{I}_r(S)$.*

Let R be a subsemigroup of a semigroup S. Put $\alpha_S(R) = \{a \in S \mid R \cap Ra \neq \emptyset\}$ and $\beta_S(R) = \{a \in S \mid R \cap aR \neq \emptyset\}$. We say that R is a left (right) dense in S if $\alpha_S(R) = S\,(\beta_S(R) = S)$. The following two assertions are clear:

Lemma 1.8. *If S is left uniform, then $\alpha_S(R)$ is a subsemigroup of S and $\alpha_S(\alpha_S(R)) = \alpha_S(R)$.*

Lemma 1.9. *If S is cancellative and R is uniform, then R is left dense in S if and only if R is right dense is S.*

Now, denote by $\mathscr{I}_l(R, S)(\mathscr{I}_r(R, S))$ the set of non-empty subsets A of S such that $RA \subseteq A(AR \subseteq A)$ and put $\mathscr{I}_l^o(R, S) = \mathscr{I}_l(R, S) \cup \{\emptyset\}$ $(\mathscr{I}_r^o(R, S) = \mathscr{I}_r(R, S) \cup \{\emptyset\})$.

Lemma 1.10. (i) $\{R, S\} \subseteq \mathscr{I}_l(R, S) \cap \mathscr{I}_r(R, S)$.

(ii) *The sets* $\mathscr{I}_l^o(R, S)$ *and* $\mathscr{I}_r^o(R, S)$ *are closed under arbitrary intersections and unions.*

(iii) *If* $A \in \mathscr{I}_l^o(R, S)$ *and* Z *is any subset of* S, *then* $AZ \in \mathscr{I}_l^o(R, S)$.

Proof. Obvious. □

Put $\mathscr{I}_l(R, S) = \{A \in \mathscr{I}_l(R, S) \mid Aa \subseteq R$ for at least one $a \in S\}$, $\mathscr{I}_l^o(R, S) = \mathscr{I}_l(R, S) \cup \{\emptyset\}$, $\mathscr{I}_r(R, S) = \{A \in \mathscr{I}_r(R, S) \mid aA \subseteq R$ for at least one $a \in S\}$ and $\mathscr{I}_r^o(R, S) = \mathscr{I}_r(R, S) \cup \{\emptyset\}$.

Lemma 1.11. (i) $\mathscr{I}_l^o(R, S) \subseteq \mathscr{I}(R, S)$.

(ii) $(A : a)_r = \{b \in S \mid ba \in A\} \in \mathscr{I}_l^o(R, S)$ *for all* $a \in S$ *and* $A \in \mathscr{I}_l^o(R, S)$.

(iii) $\mathscr{I}_l^o(R, S)$ *is closed under arbitrary intersections.*

Proof. Easy. □

Put $A_l(R, S) = \bigcup \mathscr{I}_l(R, S)$ and $A_r(R, S) = \bigcup \mathscr{I}_r(R, S)$.

Lemma 1.12. (i) $A_l(R, S) = \{a \in S \mid R \cap aS \neq \emptyset\} = \{a \in S \mid Ra \cup \{a\} \in \mathscr{I}_l(R, S)\}$.

(ii) *If* $1_s \in R$, *then* $A_l(R, S) = \{a \in S \mid Ra \in \mathscr{I}_l(R, S)\}$

(iii) $R \subseteq A_l(R, S)$ *and* $A_l(R, S) \in \mathscr{I}_l(R, S)$.

(iv) *If* $S \neq A_l(R, S)$, *then* $S \setminus A_l(R, S)$ *is a right ideal of* S.

(v) $S \setminus A_l(R, S) \in \mathscr{I}_r^o(R, S)$.

Proof. Easy. □

2. Semimodules – introduction

Let S be a semigroup. By a (left) S-semimodule M we mean a commutative semigroup $M(+)$ equipped with a scalar multiplication $S \times M \to M$ such that $a(x + y) = ax + ay$ and $a(bx) = (ab)x$ for all $a, b \in S$ and $x, y \in M$. If $1_S \in S$ and $1_S x = x$ for every $x \in M$, then the semimodule M is said to be unitary.

A semimodule M is called

- an ip-semimodule (or idempotent) if $x + x = x$ for every $x \in M$;
- a up-semimodule (or unipotent) if $x + x = y + y$ for all $x, y \in M$;
- a zp-semimodule (or zeropotent) if $x + x = x + x + y$ for all $x, y \in M$;
- a zs-semimodule if M is zeropotent and $M + M = M$;
- a za-semimodule if $x + y = x + z$ for $x, y, z \in M$;
- a qza-semimodule if $x + y = x + z$ for all $x, y, z \in M$, $y \neq x \neq z$;

- a cn-semimodule (or cancellative) if $x + y \neq x + z$ for all $x, y, z \in M$, $y \neq z$;
- a module if $M(+)$ is an (abelian) group;
- faithful if for all $a, b \in S$, $a \neq b$, there exists $x \in M$ with $ax \neq bx$.

An element w of a semimodule M is said to be neutral (absorbing, resp.) if $w + x = x$ $(w + x = w)$ for every $x \in M$. If such an such an element exists in M, it will be denoted by 0 (o, rep.)

For a semimodule M, $Ann(M) = \{a \in S \mid |aM| = 1\}$.

Lemma 2.1. *If $Ann(M) \neq \emptyset$, ten it is an ideal of the semigroup S. That is, $Ann(M) \in \mathscr{I}^o(S)$.*

Proof. Easy. □

Proposition 2.2. *Suppose that S is a non-trivial ideal-simple semigroup. Let M be a semimodule and $A = Ann(M)$. Then just one of the following three cases takes place:*

1. $A = \emptyset$;
2. $A = \{q\}$, *where q is an absorbing erlement of S;*
3. $A = S$.

Proof. Use 2.1. □

Lemma 2.3. *Suppose that S is right subcommutative and let M be a semimodule with $A = Ann(M) \neq \emptyset$. Then there exists an element $w \in M$ such that $w = w + w$ and $AM = \{w\} = Sw$ (in particular, $\{w\}$ is a subsemimodule of M).*

Proof. Easy. □

Lemma 2.4. *Let N be a semimodule.*

(i) *If M is a up-semimodule and $w = 2x$, $x \in M$, then $Sw = \{w\}$ and $\{w\}$ is a subsemimodule of M.*

(ii) *If M is a za-semimodule, then $o = x + y$, $x, y \in M$, $S \cdot o = \{o\}$ and $\{o\}$ is a subsemimodule of M.*

(iii) *If M is a module, then $S \cdot 0 = \{0\}$ and $\{0\}$ is a submodule of M.*

Proof. Easy. □

Lemma 2.5. *Let M be a qza-semimodule. Then just one of the following two cases takes place:*

1. $M(+)$ is a two element group;

2. $o \in M$ and $x + y = o$ for all $x, y \in M$, $x \neq y$.

Proof. Easy. □

Lemma 2.6. *Let M be a zs-semimodule. Then $o \in M$ and $So = \{o\}$. If M is non-trivial, then M is infinite.*

Proof. Easy. □

Lemma 2.7. *Let M be a semimodule. Define a relation ϱ_M on S by $(a, b) \in \varrho_M$ is and only if $ax = bx$ for every $x \in M$. Then ϱ_M is a congruence of S and M becomes a faithful S/ϱ_M-semimodule.*

Proof. Easy. □

3. Two-element semimodules

3.1. Denote by $\mathscr{T}_1$ the set of (left S –) semimodules whose (underlying) additive semigroup is the following two-element za-semigroup T_1:

T_1	o	1
o	o	o
1	o	o

If $M \in \mathscr{T}_1$, then $I_M = \{a \in S \mid a1 = o\} \in \mathscr{I}^o(S)$. Conversely, if $I \in \mathscr{I}^o(S)$, then $M_I \in \mathscr{T}_1$, where a scalar multiplication is defined on T_1 by $ao = o = b1$ and $c1 = 1$, $a \in S$, $b \in I$, $c \in S \setminus I$.

The semimodules from $\mathscr{T}_1$ are pair-wise non-isomorphic and there is a biunique correspondence between the sets $\mathscr{T}_1$ and $\mathscr{I}^o(S)$ given by $M \to I_M$ and $I \to M_I$. Notice that $|\mathscr{T}_1| \geq 2$ and $|\mathscr{T}_1| = 2$ if and only if S is ideal-free. If $1_S \in S$, then M_I is unitary if and only if $I \neq S$.

3.2. Denote by $\mathscr{T}_2$ the set of semimodules whose additive semigroup is the following two-element semilattice T_2:

T_2	o	0
o	o	o
0	o	0

Let $\mathscr{A}(S)$ be te set of ordered triples (A, B, C), where A, B, C are pair-wise disjoint subsets of S such that $A \cup B \cup C = S$, $A \in \mathscr{I}_r^o(S)$, $B \in \mathscr{I}_r^o(S)$, $CA \subseteq A$, $CB \subseteq B$ and either $C = \emptyset$ or C is subsemigroup of S.

If $M \in \mathscr{T}_2$, then $(A_M, B_M, C_M) \in \mathscr{A}(S)$, where $A_M = \{a \in S \mid aM = o\}$, $B_M = \{b \in S \mid bM = 0\}$ and $C_M = \{c \in S \mid co = o,\ c0 = 0\}$. Conversely, if $(A, B, C) \in \mathscr{A}(S)$, then $M_{(A,B,C)} \in \mathscr{T}_2$, where $aM = o$, $co = o$, $bM = 0$, $c0 = 0$, $a \in A$, $b \in B$, $c \in C$.

The semimodules from $\mathcal{T}_2$ are pair-wise non-isomorphic and there is a biunique correspondence between the sets $\mathcal{T}_2$ and $\mathcal{A}(S)$ given by $M \to (A_M, B_M, C_M)$ and $(A, B, C) \to M_{(A,B,C)}$. Notice that $|\mathcal{T}_2| \geq 3$ and, if $1_S \in S$, then $M_{(A,B,C)}$ is unitary if ad only if $C \neq \emptyset$ (equivalently, $1_S \in C$).

3.3. Denote by $\mathcal{T}_3$ the set of (semi)modules whose additive (semi)group is the following two-element group T_3:

T_3	o	1
0	0	1
1	1	0

If $M \in \mathcal{T}_3$, then $I_{(M)} = \{a \in S \mid aM = \{0\}\} \in \mathcal{I}^o(S)$. Conversely, if $I \in \mathcal{I}^o(S)$, then $M_{(I)} \in \mathcal{T}_3$, where $ax = 0$ and $bx = x$, $a \in I$, $b \in S \setminus I$, $x \in T_3$. The modules from $\mathcal{T}_3$ are pair-wise non-isomorphic and there is a biunique correspondence between the sets $\mathcal{T}_3$ and $\mathcal{I}^o(S)$ given by $M \to I_{(M)}$ and $I \to M_{(I)}$. Notice that $|\mathcal{T}_3| \geq 2$ and $|\mathcal{T}_3| = 2$ if and only if S is ideal-free. If $1_S \in S$, then $M_{(I)}$ is unitary if and only if $I \neq S$.

Remark 3.4. T_1, T_2 and T_3 are (up to isomorphism) the only commutative two-elements semigroups.

Proposition 3.5. *The pair-wise non-isomorphic two-element semimodules M_I, $M_{(I)}$, $I \in \mathcal{I}^o(S)$, $M_{(A,B,C)}$, $(A, B, C) \in \mathcal{A}(S)$, are up to isomorphism the only two-element semimodules.*

Proof. Combine 3.1, 3.2, 3.3, and 3.4. □

Corollary 3.6. *There exist at least seven non-isomorphic two-element semimodules. If $1_S \in S$, then four of them are not unitary.*

4. Ideal-simple semimodules

A subset V of a semimodule M is said to be an ideal of M if V is a subsemimodule such that $V + M \subseteq V$ (i.e., V is both a subsemimodule of M and an ideal of $M(+)$).

A semimodule M is called ideal-free (ideal-simple) if M is non-trivial and $V = M$ whenever V is an ideal of M (with $|V| \geq 2$).

Proposition 4.1. *Let M be a non-trivial semimodule (with or without absorbing element). Then M is ideal-simple if and only if at least one (and then just one) of the following conditions takes place:*

1. $Sx + M = M$ *for every* $x \in M$;
2. $o \in M$, $SM = o$ *and* $M \setminus \{o\}$ *is a subgroup of* $M(+)$;
3. $o \in M$, $So = o$ *and* $Sx + M = M$ *for every* $x \in M$, $x \neq o$;
4. $o \in M$, $SM = o = M + M$ *and* $|M| = 2$;
5. $o \in M$, $So = o = M + M$ *and* $M \setminus \{o\} = Sx$ *for every* $x \in M$, $x \neq o$.

Proof. For every $x \in M$, the set $V_x = Sx + M$ is an ideal of M. The rest of the proof is divided into three parts.

(i) Assume that M is ideal-simple. Then, for every $x \in M$, either $|V_x| = 1$ or $V_x = M$. If M has no one-element ideal, then (1) is true. On the other hand, if $V_w = \{v\}$ is a one-element set for some $w \in M$, then $v = o$ is an absorbing elemen of $M(+)$ and $So = o$. In such a case, put $W = \{x \in M \mid V_x = o\}$. Clearly, $o \in W$ and W is an ideal of M. Thus either $W = o$ or $W = M$.

Assume, firstly, that $W = o$. Then $V_y = Sy + M = M$ for every $y \in M$, $y \neq 0$, and (3) takes place.

Next, assume that $W = M$, i.e., $Sx + M = o$ for every $x \in M$, $SM + M = o$. Put $Z = \{x \in M \mid Sx = o\}$. Then $o \in Z$ and Z is an ideal of M.

If $Z = M$, then $SM = o$ and M is ideal-simple if and only if the additive semigroup $M(+)$ is so. Thus if and only if (2) or (4) is true.

If $Z = o$, then $Sx \neq o$ for every $x \in M$, $x \neq o$. But $Sx \cup \{o\}$ is an ideal of M and it follows that $Sx \cup \{o\} = M$. That is, (5) is true.

(ii) Assume that at least one of the conditions (1) – (5) is true. Let U be an ideal of M with $|U| \geq 2$. Take $w \in U$, $w \neq o$. Then $V_w \subseteq U$, and so $U = M$, provided that (1) is satisfied. If (2) is true, then $w + M = M$ and, again, $U = M$. Similarly, if (3) is true. If (4) is satisfied, then M is ideal-simple, since it contains only 2 elements. Finally, if (5) is satisfied, then $M \subseteq Sw \cup \{o\} \subseteq U$.

(iii) The fact that any of the conditions (1), …, (5) excludes the remaining ones is easily seen. □

Proposition 4.2. *Suppose that S is right subcommutative. If M is an ideal-simple semimodule with* $A = Ann(M) \neq \emptyset$, *then at least one of the following two cases takes place:*

1. $0 \in M$ *and* $AM = 0 = S \cdot 0$;
2. $o \in M$ *and* $AM = o = S \cdot o$.

Proof. By 2.3, there is $w \in M$ such that $AM = w = Sw$. Now, the set $w + M$ is an ideal of M, and hence either $|w + M| = 1$ or $w + M = M$. In the first case, $w + M = w$ (2.3), and $w = o$. Then (2) is true. In the latter case, since $\{w\}$ is a subsemimodule, we have $w = 0$ and (1) is true. □

Lemma 4.3. *Suppose that S is left subcommutative. If M is ann ideal-simple semimodule with* $o \in M$ $So = o$ *and if* $a \in S$ *and* $x \in N$ *are such that* $ax = o \neq x$, *then* $a \in Ann(M)$ *and* $aM = o$.

Proof. The set $V = \{y \in M \mid ay = o\}$ is an ideal of M and $o,\ x \in V$. Thus $V = M$. □

Remark 4.4. Every two-element semimodule is ideal-simple.

5. Congruence-simple semimodules – introduction

A semimodule possessing just two congruence relations is called (congruence-) simple.

Theorem 5.1. *Let M be a simple semimodule. Then just one of the following four cases takes place:*
1. M is a za-semimodule;
2. M is a zs-semimodule;
3. M is an ip-semimodule;
4. M is a cn-semimodule.

Proof. It is essentially the same as that of [1, 2.1]. Whatwever, for benefit of a reader, an outline is given here.

Firstly, if M is neither unipotent nor idempotent, then $x \to 2x$ is an injective endomorphism of M and $r = M \times M$, where r is defined on M by $(x, y) \in r$ iff $2^i x = y + u$ and $2^i y = x + v$ for some $i \geq 0$ and $u, v \in M \cup \{0\}$.Now, it is easy to check that M is cancellative.

Similarly, if M is unipotent but not zeropotent, then $x \to 3x$ in injective and M is cancellative, too.

Finally, if M is zeropotent and $N = M + M \subsetneq M$, then N is a proper ideal of M, $(N \times N) \cup \mathrm{id}_M$ is a congruence of M, $N = \{o\}$and M is a za-semimodule. □

Proposition 5.2. *(i) Every two-element semimodule (see 3.5) is simple.*
(ii) Every simple semimodule is ideal-simple.

Proof. Easy. □

Proposition 5.3. *Assume that $1_S \in S$, Then every simple non-unitary semimodule containing at least three elements is a (finite) p-element module, where $SM = 0$ and p is a prime number, $p \geq 3$.*

Proof. Let M be a non-unitary simple semimodule with $|M| \geq 3$. Define a relation r on M by $(u, v) \in r$ iff $au = av$ for every $a \in S$. Then r is a congruence of M and we have $(x, 1_S x) \in r$ for every $x \in M$. Since M is not unitary, $r \neq \mathrm{id}_M$ and consequently $r = M \times M$. Now, every congruence of $M(+)$ is a congruence of M and it follows that $M(+)$ is congruence-simple. Since $|M| \geq 3$, $M(+)$ is a p-elment group for a prime $p \geq 3$. Thus M is a module and, of course, $S \cdot 0 = 0$. Since $r = M \times M$ we conclude $SM = 0$. □

Proposition 5.4. *Let M be a simple semimodule with $0 \in M$. Then just one of the following two cases takes place:*

1. M is a module;

2. M is an ip-semimodule.

Moreover, if S is left subcommutative and (2) is true, then $|M| = 2$ (see 3.2).

Proof. (i) According to 5.1, M is either idempotent or cancellative. Assume the latter to be true. If $a \in S$, then $0 + a0 = a(0 + 0) = a0 + a0$, and so $a0 = 0$; thus $S \cdot 0 = 0$. Further, $N = \{x \mid 0 \in M + x\}$ is a submodule of M and r is a congruence of M, where $(u, v) \in r$ iff $u + N = v + N$. Of course, if $r = M \times M$, then $N = M$ and M is a module. On the other hand, if $r = \mathrm{id}_M$, then $N = 0$ (since N is a submodule) and s is a congruence of M, where $(x, y) \in s$ iff $\{a \in S \mid ax = 0\} = \{a \in S \mid ay = 0\}$. Moreover, $(x, 2x) \in s$ for every $x \in M$. Consequently, $s \neq \mathrm{id}_M$, $s = M \times M$, $\{a \in S \mid ax = 0\} = \{a \in S \mid a0 = 0\} = 0$ and $SM = 0$. Now, it is clear that M is a p-element module, $p \geq 2$ being a prime number.

(ii) Assume that S is left subcommutative and M idempotent. Let $a \in S$ and $x \in M$ be such that $ax = 0 \neq x$. Then $0 = ax = a(x + 0) = ax + a0 = a0$ and $(x, 0) \in t$, where t is the congruence of M defined by $(u, v) \in t$ iff $au = av$ (use the left subcommutativity of S). Consequently, $t = M \times M$ and $aM = 0$. Using this observation, we conclude that $(P \times P) \cup \mathrm{id}_M$ is a congruence of M, where $P = M \setminus \{0\}$ and, since M is simple, we get $|M| = 2$ as desired. □

Lemma 5.5. *Let M be a simple semimodule such that $o \in M$ ($0 \in M$, resp.) and $S \cdot o \neq o$ ($S \cdot 0 \neq 0$). Then M is idempotent.*

Proof. Combine 5.1 and 5.4. □

Lemma 5.6. *Suppose that A is left subcommutative. If M is a simple semimodule and $a \in S \setminus Ann(M)$, then the mapping $x \to ax$, $x \in M$, is injective.*

Proof. The relation r defined by $(x, y) \in r$ iff $ax = ay$ is a congruence of M. □

Proposition 5.7. *Let M be a simple semimodule such that $A = Ann(M) \neq \emptyset$.*

(i) If $A = S$, then either $|M| = 2$ or M is a (finite) p-element module with $SM = 0$, $p \geq 2$ being a prime number.

(ii) If S is left subcommutative and $A \neq S$, then $R = S \setminus A$ is a subsemigroup of S and M is simple as an R-semimodule. Moreover, $Ann_R(M) = \emptyset$ and the mapping $x \to ax$, $x \in M$, is an injective endomorphism of $M(+)$ for every $a \in R$.

(iii) If S is subcommutative and $|M| \geq 3$, then either M is a module and $AM = 0 = S \cdot 0$ or $o \in M$ and $Am = o = S \cdot o$.

Proof. (i) The transformations $x \to ax$, $x \in M$, are constant, and hence $M(+)$ is congruence-simple.

(ii) Use 5.6.

(iii) Use 4.2 and 5.4. □

Lemma 5.8. *Let M be a simple semimodule such that $|M| \geq 3$ and M is not a p-element module with $SM = 0$ for any prime $p \geq 3$. Then, for all $u, v \in M$, $u \neq v$, there is $a \in S$ with $au \neq av$.*

Proof. Define a relation r on M by $(x, y) \in r$ iff $ax = ay$ for every $a \in S$. Then r is a congruence of M and the rest is clear. □

Lemma 5.9. *Let M be a simple semimodule such that M is not idempotent. Then the semigroup $M(+)$ is archimedean (i.e., for all $x, y \in M$ there are positive integers m, n such that $my \in M + x$ and $nx \in M + y$).*

Proof. Define a relation r on M by $(x, y) \in r$ iff $my \in M + x$ and $nx \in M + y$ for some positive integers m, n. Then r is a congruence of M and $(x, 2x) \in r$ for every $x \in M$. Since M is not idempotent, $r = M \times M$. □

Remark 5.10. Put $S_1 = S \cup \{e\}$, where S is a subsemigroup of S_1 and $w = 1_{S_1}$. If M is an S-semimodule, then M becomes a unitary S_1-semimodule. Clearly, ${}_sM$ is simple if and only if ${}_{S_1}M$ is simple.

Simple semimodules with absorbing element – introduction

Let M be a semimodule with $o \in M$. Define a relation $\sigma_1 (= \sigma_{M,1})$ on M by $(x, y) \in \sigma_1$ iff $\{(a, u\} \in S \times M \mid ax + u = 0\} = \{(a, u) \in S \times M \mid ay + u = o\}$. Further, define $\sigma_2 (\sigma_{M,2})$ by $(x, y) \in \sigma_2$ iff $\{a \in S \mid ax = o\} = \{a \in S \mid ay = 0\}$ and $\sigma_3 (\sigma_{M,3})$ by $(x, y) \in \sigma_3$ iff $\{u \in M \mid x + u = o\} = \{u \in M \mid y + u = o\}$.

Proposition 6.1. *The relations σ_1, $\sigma_1 \cap \sigma_2$, $\sigma_1 \cap \sigma_3$ and $\sigma_1 \cap \sigma_2 \cap \sigma_3$ are congruences of M.*

Proof. Easy to check. □

Proposition 6.2. *Assume that M is ideal-simple, $S \cdot o = o$ and $M + M \neq o \neq$ $\neq SM)$ (see 4.1). Then M/σ_1 is a simple semimodule, $\sigma_1 \subseteq \sigma_2 \cap \sigma_3$ and $\{x \in M \mid (x, o) \in \sigma_1\} = \{o\}$.*

Proof. In view of 4.1, M is of the type 4.1(3), and hence $(x, o) \notin \sigma_1$ for every $x \in M$, $x \neq o$. Consequently, $N = M/\sigma_1$ is a non-trivial semimodule, and so it is ideal-simple, too.

Let r be a congruence of M such that $\sigma_1 \subseteq r$ and $\sigma_1 \neq r$. Then there are $x, y, u \in M$ and $a \in S$ such that $(x, y) \in r$ and $o = ax + u \neq ay + u = z$. Clearly, $(z, o) \in r$ and we have $|V| \geq 2$, $V = \{v \mid (v, o) \in r\}$. Now, V, is an ideal of M, $V = M$ and $r = M \times M$. We have thus proved that σ_1 is a maximal congruence of M, i.e., N is a simple semimodule.

Finally, if $(x, y) \in \sigma_1$ and $ax = o$, then $(o, ay) \in \sigma_1$ and $ay = o$ (see the first part of the proof). Similarly, if $x + u = o$, then $(o, y + u) \in \sigma_1$ and $y + u = o$. Thus $\sigma_1 \subseteq \sigma_2 \cap \sigma_3$. □

Proposition 6.3. *Suppose that $|M| \geq 3$ and $S \cdot o = o \neq M + M$. Ten M is simple if and only if the following two conditions are satisfied:*

(a) For all $x, y \in M$, $x \neq o \neq y$, there exist $a \in S$ and $z \in M$ such that $ax + z = y$;

(b) For all $x, y \in M$, $o \neq x \neq y \neq o$, there exist $a \in S$ and $z \in M$ such that $ax + z \neq ay + z$ and either $ax + z = o$ or $ay + z = o$.

Proof. If M is simple, then M is ideal-simple and (a) follows from 4.1. Further, $\sigma_1 = \mathrm{id}_M$ and (b) is clear.

Conversely, if both (a) and (b) are true and r is a non-identical congruence of M, then $V = \{x \mid (x, o) \in r\}$ contains at least two elements by (b). Now, V is an ideal of M and $V = M$ by (a). Thus $r = M \times M$ and M is simple. □

Lemma 6.4. *Suppose that M is simple and $|M| \geq 3$. Then for every $x \in M$, $x \neq o$, there is $a \in S$ with $o \neq ax \neq ao$.*

Proof. By 5.8, $ax \neq ao$ for some $a \in S$. If $ax = o$, then $ao = a(x + o) = ax + ao = o + ao = ax$, a contradiction. Thus $ax \neq 0$. □

7. Simple za-semimodules

Proposition 7.1. *If M is a za-semimodule, then $o \in M$ and $S \cdot o = o = M + M$.*

Proof. Easy. □

Proposition 7.2. *Let M be a za-semimodule such that $|M| \geq 3$. Then M is simple if and only if the following two conditions are satisfied:*

(a) *For all $x, y \in M$, $x \neq o \neq y$, there is $a \in S$ with $ax = y$;*

(b) *For all $x, y \in M$, $o \neq x \neq y \neq o$, there is $a \in S$ with $ax \neq ay$ and $o \in \{ax, ay\}$.*

Proof. Similar to that of 6.3. □

Lemma 7.3. *Let M be a simple za-semimodule. Then either $|M| = 2$ or $Sx = M$ for every $x \in M$, $x \neq o$.*

Proof. Assume that $|M| \geq 3$. Now, with regard to 7.2(a), it remains to show that $o \in Sx$, $x \in M$, $x \neq o$. Let, on the contrary, $|V| \geq 2$, where $V = \{x \mid o \notin Sx\} \cup \{o\}$. Clearly, V is an ideal of M, and hence $V = M$. It follows that $SN \subseteq N$ and $r = (N \times N) \cup \mathrm{id}_M$ is a congruence of M, where $N = M \setminus \{0\}$. Then $r = \mathrm{id}_M$ and $|M| = 2$, a contradiction. □

Corollary 7.4. *If M is a simple za-semimodule, then* $|M| \leq max(2, |S|)$.

Proposition 7.5. *If S is left subcommutative, then* $|M| = 2$ *for eery simple za-semimodule M.*

Proof. Let $x, y \in M$ and $a \in S$ be such that $x \neq o \neq y$ and $ax = o \neq ay$ (7.2(b)). By 7.2(a), $y = bx$, $b \in S$, and we have $o \neq ay = abx = cax = co = o$, a contradiction. □

Example 7.6. Let $M(+)$ be a non-trivial za-semigroup (i.e., $M + M = o$). If $S = End(M(+))$, then M becomes a simple S-za-semimodule. Notice that if $|M| = n \geq 2$ is finite, then $|S| = n^{n-1}$.

8. Simple zs-semimodules

Proposition 8.1. *Let M be a non-trivial zs-semimodule. Then M is simple if and only if the following two conditions are satisfied:*

(a) *If $x, y \in M$, $x \neq o \neq y$, then $ax + z = y$ for some $a \in S$ and $z \in M$;*

(b) *If $x, y \in M$, $o \neq x \neq y \neq o$, then $ax + z \neq ay + z$ and $o \in \{ax + z, ay + z\}$ for some $a \in S$ and $z \in M$.*

Proof. Combine 2.6 and 6.3. □

Theorem 8.2. *There exist no simple zs-semimodules in each of the following two cases:*

1. *The semigroup S is hr-uniform;*
2. *S is finite.*

Proof. Let M be a simple zs-semimodule and let $x, y, z \in M$ be such that $x = y + z \neq o$. Put $A = \{a \in S \mid y \in M + ax\}$ and $B = \{b \in S \mid z \in M + bx\}$. By 8.1(a), we have $A \neq \emptyset \neq B$ and it is easy to check that $AA \cup AB \subseteq A$ and $BB \cup BA \subseteq B$. Now, by the dual of 1.1, we have $A \cap B \neq \emptyset$. If $c \in A \cap B$, then $y = cx + u$ and $z = cx + v$, $u, v \in M$, and we get $o \neq x = y + z = cx + cx + u + v = o + u + v = o$, a contradiction. Thus $A \cap B = \emptyset$ and S is not hr-uniform.

Further, take $w \in M$, $w \neq o$, and define a relation q on the set Sw by $(aw, bw) \in q$ iff either $aw = bw$ or $aw \in M + bw$. Clearly, q is both reflexive and transitive and if $aw = bw + x$ and $bw = aw + y$, then $aw = aw + x + y = aw + x + y + x + y = o$, and similarly, $bw = o$. It follows that q is an order on Sw. Now, by 8.1(a), $x = bw + u$, $y = cw + v$ and $(aw, bw) \in q$, $(aw, cw) \in q$. If $aw = bw$ and $aw = cw$, then $aw = aw + u + aw + v = o$, a contradiction. Thus either $aw \neq bw$ or $aw \neq cw$ and it follows that aw is not maximal in (Sw, q). We have shown that the ordered set Sw has no maximal elements. In particular, Sw is not finite and S is not finite either. □

Example 8.3. Let R be a subsemigroup of a left cancellative semigroup S such that $aS \cap bR$ is nonempty for all $a \in S$ and $b \in R$ (e.g., S a group), Define an addition on $\mathscr{I} = \mathscr{I}_r(R, S)$ by $A + B = A \cup B$ if $A \cap B = \emptyset$ and $A + B = S$ if $A \cap B \neq \emptyset$. Then $\mathscr{I}(+)$ is a commutatve zp-semigroup, where $o = S$, and ϱ is a congruence of $\mathscr{I}(+)$, where $(A, B) \in \varrho$ iff $\{C \in \mathscr{I} \mid A \cap C = \emptyset\} = \{C \in \mathscr{I} \mid B \cap C = \emptyset\}$. Now, we denote by $\mathscr{Z}(+)$ the factorsemigroup $\mathscr{I}(+)/\varrho$ and by π the natural projection of $\mathscr{I}$ onto $\mathscr{Z}$.

Lemma 8.3.1. *(i) $(aS, S) \in \varrho$ for every $a \in S$.*
(ii) If $(A, B) \in \varrho$, then $(aA, aB) \in \varrho$ for every $a \in S$.
(iii) If $A, B \in \mathscr{I}$ and $a \in S$, then $(a(A + B), aA + aB) \in \varrho$.

Proof. (i) We have $aS \cap bR \neq \emptyset$ for every $b \in R$.
(ii) If $C \in \mathscr{I}$ in such that $aA \cap C \neq \emptyset$, then $A \cap D \neq \emptyset$, $D = \{d \in S \mid ad \in C\} \in \mathscr{I}$, and so $B \cap D \neq \emptyset$ and $aB \cap C \neq \emptyset$.
(iii) Use (i). □

Now, due to the preceding lemma, we can define a scalar multiplication on $\mathscr{Z}$ by $a\pi(A) = \pi(aA)$ for all $a \in S$ and $A \in \mathscr{I}$. In this way, $\mathscr{Z}$ becomes an S-zp-semimodule.

Lemma 8.3.2. *Let η be a congruence of the semimodule Z such that $(\pi(R), \pi(S)) \in \eta$. Then $\eta = \mathscr{Z} \times \mathscr{Z}$.*

Proof. Put $\sigma = \pi^{-1}(\eta)$. Then σ is a congruence of $\mathscr{I}(+)$ and, since $(R, S) \in \sigma$, we have $(aR, S) \in \sigma$ for every $a \in S$. Consequently, if $a \in A \in \mathscr{I}$, $(aR, A) \in \sigma$, then $(A, S) \in \sigma$. On the other hand, if $(aR, A) \notin \sigma$, $B \in \mathscr{I}$ is maximal with respect to $B \subseteq A$ and $B \cap aR = \emptyset$, then $(A, B \cup aR) \in \sigma$, $(B \cup aR, S) = (B + aR, B + S) \in \sigma$ and, finally, $(A, S) \in \sigma$. □

Lemma 8.3.3. *If $(R, S) \in \varrho$, then $|\mathscr{Z}| = 1$, R is right uniform and R is right dense is S.*

Proof. Easy. □

In the remaining part of this example, assume that R is not right uniform. Then $\pi(R) \neq \pi(S)$ and there exists a congruence τ of $\mathscr{Z}$ maximal with respect to $(\pi(R), \pi(S)) \notin \tau$. Put $\mathscr{W} = \mathscr{Z}/\tau$.

Proposition 8.3.4. *$\mathscr{W}$ is a simple zs-semimodule.*

Proof. By 8.3.2 and the maximality of τ, $\mathscr{W}$ is a simple semimodule. By 5.1, $\mathscr{W}$ is either a za-semimodule or a zs-semimodule. Further, since R is not right uniform, there are right ideals A and B of R such that B is maximal with respect to $A \cap B = \emptyset$. Then $A + B = A \cup B$, $(A \cap B, R) \in \varrho$, $\pi(A) + \pi(B) = \pi(R)$, and so $(\pi(A) + \pi(B), \pi(S)) \notin \tau$. Thus $\mathscr{W}$ is not a za-semimodule and $\mathscr{W}$ is a simple zs-semimodule. □

Remark 8.4. Combining 1.3, 8.2 and 8.3, we et an equivalence of the following three conditions for a group S:

(i) No subsemigroup of S is free of rank (at least) 2;

(ii) S is h-uniform;

(iii) There exist no simple S-zs-semimodules.

9. Simple qza-semimodules

Proposition 9.1. *Let M be a simple qza-semimodule. Then just one of the following three cases takes place:*

1. M is a za-semimodule;

2. M is an ip-semimodule;

3. M is a two-element module.

Proof. Combine 5.1 and 2.5. □

An idempotent qza-semimodule will be called a qzaa-semimodule if $S \cdot o = o$. In the remaining part of this section, let M be an idempotent qza-semimodule with $|M| \geq 3$. Put $A = \{a \in S \mid ao = o\}$, $A_1 = \{a \in S \mid aM = o\} \subseteq A_1$ and $B = S \setminus A$.

Lemma 9.2. *(i) Either $A = \emptyset$ or A is a subsemigroup of S.*

(ii) Either $A_1 = \emptyset$ or A_1 is a right ideal of S.

(iii) Either $B = \emptyset$ or B is a right ideal of S.

(iv) $A_1 \cap B = \emptyset$ and $A_1 \cup B = Ann(M)$.

(v) $AA_1 \subseteq A$ and $BA_1 \subseteq B$.

Proof. Easy. □

Corollary 9.3. *Assume that S is right uniform. Then either M is a qzaa-semimodule or $Ann(M) = B \neq \emptyset$.*

Proposition 9.4. *Assume that S is right subcommutative (then it is right uniform). If M is ideal-simple, then M is a qzaa-semimodule (i.e., $A = S$).*

Proof. Assume, on the contrary, that $B \neq \emptyset$. By 9.3, $B = Ann(M)$ and it follows from 4.2 that $0 \in M$.Then $x = x + 0 = o$ for every $x \in M$, $x \neq 0$, and $|M| = 2$, a contradiction. □

Proposition 9.5. *The following conditions are equivalennt:*

(i) *M is a simple semimodule;*

(ii) *$A \neq \emptyset$ and M is a simple A–qzaa-semimodule.*

Proof. (i) implies (ii). If $A = \emptyset$, then $|M| = 2$, a contradiction. Thus $A \neq \emptyset$ and the rest is clear, since the map $x \to ax$, $x \in M$, is constant for every $a \in B$. □

Proposition 9.6. *M is simple if and only if $M\setminus\{o\} \subseteq Ax$ for every $x \in M$, $x \neq o$.*

Proof. Im view of 9.5, we can assume that $A = S$.

Firstly, let M be simple and $N = \{x \in M \mid Sx = o\}$. If $N \neq \emptyset$, then N is an ideal of M and we have $N = \{o\}$. Thus $N \subseteq \{o\}$ anyway and, if $x \in M$, $x \neq o$, the set $V = Sx \cup \{o\}$ is again an ideal of M, $|V| \geq 2$ and $V = M$.

Conversely, let $r \neq \mathrm{id}_M$ be a congruence of M and $U = \{x \in M \mid (x, o) \in M\}$. Then U is an ideal and, if $(u, v) \in r$, $u \neq v \neq o$, ten $(u, o) = (u + u, u + v) \in r$. Then $|U| \geq 2$ and $U = M$, M being ideal-simple. □

Corollary 9.7. *If M is simple, then $|M| \leq |S| + 1$.*

Remark 9.8. Suppose that M is a simple semimodule. Using 9.6, one can show that at least one of the following two conditions is true:

(a) $Sx = M$ for every $x \in M$, $x \neq o$;

(b) $o \neq Sx = M \setminus \{o\}$ for every $x \in M$, $x \neq o$.

Lemma 9.9. *Assume that S is right subcommutative. If $B \neq \emptyset$, then $A_1 = \emptyset$, $B = Ann(M)$ is an ideal of S and there is $w \in M$ such that $w \neq o$ and $BM = w = Sw$.*

Proof. Easy. □

Proposition 9.10. *Suppose that S is right subcommutative and M is simple. Then:*

(i) *M is qzaa-semimodule $(A = S)$.*

(ii) *$S \setminus A_1 = C \neq \emptyset$ and C is a subsemigroup of S.*

(iii) *$aM = M$ for every $a \in C$.*

(iv) *C operates transitively on $M \setminus \{0\}$.*

Proof. Firstly, $B = \emptyset$ by 9.8(a) and 9.9. Further, $C \neq \emptyset$, since $|M| \geq 3$. If $a \in C$, then aM is an ideal of M, $|aM| \geq 2$ and $aM = M$. □

Proposition 9.11. *Suppose that S is subcommutative and M is simple. Then the mapping $x \to ax$ is a permutation of M for every $a \in C$.*

Proof. See 9.10 and 5.6. □

Remark 9.12. Suppose that S is right subcommutative (see 9.10). Then M is a simple S-semimodule if and only if M is a simple C-semimodule.

Now, let M be simple and define a relation μ on S by $(a, b) \in \mu$ iff $ax = bx$ for every $x \in M$. Then μ is a congruence of S and the subset A_1 is contained in a block of μ. Moreover, if S is subcommtative, then C/μ is isomorphic to a subsemigroup of the automorphism group of $M(+)$. Finally, if S is commutative, then C/μ is an abelian group.

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