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A Note on Finitely Generated Commutative Rings

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If $R \subseteq S$ is an integral extension of commutative rings, where R is finitely generated, and if M is a finitely generated S-module whose additive group is not torsion, then $pM \neq M$ for almost all prime numbers p.

Je-li $R \subseteq S$ celistvé rozšíření komutativních okruhů, kde R je konečně generovaný, a je-li M konečně generovaný S-modul, jehož aditivní grupa není torzní, pak $pM \neq M$ pro skoro všechna prvočísla p.

1. Introduction

Throughout this note, all rings are notrivial, associative, commutative and with unit element. All modules are left and unitary.

A ring R is said o be uniform if $Ra \cap Rb \neq 0$ for all $a, b \in R$, $a \neq 0 \neq b$.

Let $\mathscr{P}$ be a set of prime numbers. An abelian group A is said to be a $\mathscr{P}$-group if it is torsion and $pa \neq 0$ for every nonzero element $a \in A$ and every prime number p such that $p \notin \mathscr{P}$.

An abelian group A is sai to be a free-by-$\mathscr{P}$-group if it contains a free subgroup E suchh that A/E is a $\mathscr{P}$-group.

Lemma 1.1. *Let A be a free-by-$\mathscr{P}$-group such that $pA = A$ for a prime $p \notin \mathscr{P}$. Then A is a $\mathscr{P}$-group.*

Proof. If $u \in A$ is such that $pu \in E$, then $p(u + E) = 0$ in A/E, and hence $u \in E$. Thus $pE = E \cap pA = E$ and $E = 0$.

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2. $\mathscr{A}$-by-$\mathscr{B}$-modules

Let R be a ring and $\mathscr{A}$, $\mathscr{B}$ two classes of modules satisfying the following four conditions:

(A1) Both $\mathscr{A}$, $\mathscr{B}$ are abstract and all zero modules are in $\mathscr{A} \cap \mathscr{B}$;

(A2) $\mathscr{A}$ is closed under direct sums of countably many summands;

(A3) $B \in \mathscr{B}$, provided that there is a sequence $0 = B_0 \subseteq B_1 \subseteq B_2 \dots$ of submodules of B such that $\bigcup B_i = B$, and $B_{i+1}/B_i \in \mathscr{B}$ for every $i \geq 0$;

(A4) All modules from $\mathscr{A}$ are projective.

Now, denote by $\mathscr{C}$ the class of modules C containing a submodule $A \subseteq C$ such that $A \in \mathscr{A}$ and $C/A \in \mathscr{B}$. The modules from $\mathscr{C}$ will be called $\mathscr{A}$-by-$\mathscr{B}$-modules in the sequel.

Lemma 2.1. *Let M be a module possessing a sequence $0 = M_0 \subseteq M_1 \subseteq \subseteq M_2 \subseteq \dots$ of submodules such that $\bigcup M_i = M$ and $M_{i+1}/M_i \in \mathscr{C}$ for every $i \geq 0$. Then $M \in \mathscr{C}$.*

Proof. For every $i \geq 0$, there exists a submodule N_i of M_{i+1} such that $M_i \subseteq N_i \subseteq M_{i+1}$, $N_i/M_i \in \mathscr{A}$ and $M_{i+1}/N_i \in \mathscr{B}$. Since N_i/M_i are projective modules by (A4), there are submodules K_i of N_i such that $M_i \cap K_0 = 0$ and $M_i + K_i = N_i$. Then $K_i \simeq N_i/M_i \in \mathscr{A}$.

For every $k \geq 0$, put $L_k = \sum K_i, 0 \leq i \leq k$. We have $N_0 \subseteq N_1 \subseteq N_2 \subseteq \dots$ and hence $L_k \subseteq N_k \subseteq M_{k+1}$. Further $K_{k+1} \cap L_k \subseteq K_{k+1} \cap M_{k+1} = 0$, and so $L_{k+1} = L_k + K_{k+1} = L_k \oplus K_{k+1}$ is a direct sum. We have $L_0 \subseteq L_1 \subseteq L_2 \subseteq \dots$, $L = \bigcup L_k = \sum K_i$ and $L = K_0 \oplus K_1 \oplus K_2 \oplus \dots$ is a direct sum (we use induction). Since $K_i \in \mathscr{A}$ for every $i \geq 0$, we have $L \in \mathscr{A}$ by (A2).

Let $k \geq 0$ and $m > k + 1$. We have $K_{k+1} \oplus \dots \oplus K_{m-1} \subseteq M_m$, $M_m \cap \cap K_m = 0$, $M_m \cap (K_{k+1} \oplus \dots \oplus K_m) = K_{k+1} \oplus \dots \oplus K_{m-1}$, and therefore $M_{k+1} \cap (K_{k+1} \oplus \dots \oplus K_m) \subseteq K_{k+1} \oplus \dots \oplus K_{m-1}$, since $M_{k+1} \subseteq M_m$. Now, by induction, $M_{k+1} \cap (K_{k+1} \oplus \dots \oplus K_m) \subseteq M_{k+1} \cap K_{k+1} = 0$. It follows easily that $M_{k+1} \cap (K_{k+1} \oplus K_{k+2} \oplus \dots) = 0$ and $M_{k+1} \cap L = K_0 \oplus K_1 \oplus \oplus \dots \oplus K_k$. Of course, $K_0 \oplus \dots \oplus K_{k-1} \subseteq M_k$, and so $M_k + (M_{k+1} \cap L) = M_k + K_k = N_k$. Consequently, we get the isomorphism $(M_{k+1} + L)/(M_k + + L) \simeq M_{k+1}/(M_k + (M_{k+1} \cap L)) = M_{k+1}/N_k \in \mathscr{B}$.

Finally, M/L is the union of the sequence of submodules $0 = (M_0 + L)/L \subseteq \subseteq (M_1 + L)/L \subseteq (M_2 + L)/L \subseteq \dots$, where $((M_{i+1} + L)/L)/((M_i + L)/L) \simeq \simeq (M_{i+1} + L)/(M_i + L) \simeq M_{i+1}/N_i \in \mathscr{B}$ and we get $M/L \in \mathscr{B}$ by (A3). Thus $M \in \mathscr{C}$. □

3. $\mathscr{A}$-by-$\mathscr{B}_I$-modules (A)

In this section, let R be a uniform noetherian ring. Further, let $\mathscr{A}$ and $\mathscr{B}_I$, I any nonzero ideal of R, be classes of modules such that the conditions (A1), (A2), (A3), (A4) are satisfied ad, moreover, the following three conditions are also true:

(A5) ${}_R R \in \mathscr{A}$;

(A6) For every nonzero ideal I, the factor module ${}_R R/I$ is an $\mathscr{A}$-by-$\mathscr{B}_I$-module.

(A7) $\mathscr{B}_I \subseteq \mathscr{B}_J$ whenever I and J are non-zero ideals such that $J \subseteq I$.

Proposition 3.1. *Let $n \geq 0$ and let $P = R[x_1, \ldots, x_n]$ denote their polynomial ring in n (commuting) indeterminates $x_1, \ldots, x_n$ over the ring R. If ${}_P M$ is a finitely generated P-module, then there exists a non-zero ideal I of R such that the corresponding R-module ${}_R M$ is an $\mathscr{A}$-by-$\mathscr{B}_I$-module.*

Proof. It is divided into two parts.

(i) Assume that ${}_P M$ is a cyclic P-module. In fact, assume that ${}_P M = {}_P P/K$ for an ideal K of the ring P. Now, we will proceed by induction on n. The first case $n = 0$ is clear from (A1), (A5) and (A6). Hence, assume $n \geq 1$ and put $S = R[x_1, \ldots, x_{n-1}] \subseteq P$, $x = x_n$. Then $P = S[x]$ and every element from P can be viewed as a polynomial in the single indeterminate x over the ring S.

Put $L_0 = 0$ and $L_k = \{a_0 + a_1 x + \ldots + a_{k-1}x^{k-1} \mid a_i \in S\}$ for every $k \geq 1$. Clearly, L_k are S-submodules of ${}_S P$ and $L_0 \subseteq L_1 \subseteq L_2 \subseteq \ldots$, $\bigcup L_k = P$. Now, if $K_k = K + L_k$, $k \geq 0$, then K_k are S-submodules of ${}_S P$, $K_0 \subseteq K_1 \subseteq K_2 \subseteq \ldots$ and, again, $\bigcup K_k = P$. Moreover, $K_{k+1}/K_k \simeq L_{k+1}/(L_k + (K \cap L_{k+1}))$ is an isomorphism of the S-modules for every $k \geq 0$.

Put $J_0 = 0$, $J_1 = S \cap K$ and $J_k = \{a \in S \mid a_0 + a_1 x + \ldots + a_{k-2}x^{k-2} + {} + \times\, ax^{k-1} \in K$ for some $a_0, \ldots, a_{k-2} \in S\}$ for every $k \geq 2$. Again, J_k are S-submodules of ${}_S S$, i.e., ideals of S, and we have $(K \cap L_{k+1}) + L_k = J_{k+1}x^k + L_k$ and $Sx^k \cap L_k = 0$ for every $k \geq 0$. Consequently, $K_{k+1}/K_k \simeq L_{k+1}/(L_k + J_{k+1}x^k) \simeq {} \simeq (Sx^k + L_k)/(J_{k+1}x^k + L_k) \simeq Sx^k/J_{k+1}x^k \simeq S/J_{k+1}$ are S-module isomorphisms for every $k \geq 0$.

Since $xK \subseteq K$, we have $J_0 \subseteq J_1 \subseteq J_2 \subseteq \ldots$. But S is a noetherian ring, and therefore $J_m = J_{m+1} = \ldots$ for some $m \geq 0$. This means that each among the S-factormodules K_{k+1}/K_k, $k \geq 0$, is S-isomorphic to at least one of the cyclic S-modules ${}_S S/J_0, \ldots, {}_S S/J_m$.

By induction hypothesis, for every j, $0 \leq j \leq m$, there exists a nonzero ideal I_j of R such that ${}_R S/J_j$ is an $\mathscr{A}$-by-$\mathscr{B}_{I_j}$-module. Since R is uniform, we have $I = {} = I_0 \cap \ldots \cap I_m \neq 0$ and it follows from (A7) that all the R-modules ${}_R S/J_j$ are $\mathscr{A}$-by-$\mathscr{B}_I$-modules. Consequently, the same is true for the R-modules ${}_R K_{k+1}/{}_R K_k$, $K \geq 0$.

Finally, $0 = {}_R K_0/{}_R K \subseteq {}_R K_1/{}_R K \subseteq {}_R K_2/{}_R K \subseteq \ldots$, $\bigcup {}_R K_k/{}_R K = {}_R P/{}_R K \simeq {}_R M$ and ${}_R M$ is an $\mathscr{A}$-by-$\mathscr{B}_I$-module by 2.1.

(ii) Now the general case. The P-module ${}_P M$ is finitely generated and we have ${}_P M = Pv_1 + \ldots + Pv_m$, $m \geq 1$. Let ${}_P M_0 = 0$ and ${}_P M_k = Pv_1 + \ldots + Pv_k$, $k \geq 1$. Then all the factors ${}_P M_1/{}_P M_0$, ${}_P M_2/{}_P M_1, \ldots, {}_P M_m/{}_P M_{m-1}$ are cyclic P-modules and ${}_P M_m = {}_P M$. Using (i) for these cyclic P-modules, the uniformity of R and 2.1, our result easily follows. □

Corollary 3.2. *Let R be a subring of a ring S such that $S = R[T]$ for a finite subset T. If ${}_SM$ is a finitely generated S–module, then there exists a nonzero ideal I of R such that the corresponding R–module ${}_RM$ is an $\mathscr{A}$-by-$\mathscr{B}_I$-module.*

4. $\mathscr{A}$-by-$\mathscr{B}_I$-modules (B)

Let R be a noetherian domain. Let $\mathscr{A}$ denote the class of free R–modules and, for every nonzero ideal I of R, let $\mathscr{B}_I$ denote the class of R–modules ${}_RM$ such that for every $u \in M$ there is a positive integer $m(u)$ with $I^{m(u)} \cdot u = 0$.

Lemma 4.1. *All the conditions* (A1), ..., (A7) *are satisfied.*

Proof. Easy to see. □

Proposition 4.2. *Let R be a subring of a ring S such that $S = R[T]$ for a finite set T.* If *${}_SM$ is a finitely generated S–module, then there exist a free R–submodule ${}_RE$ of the R–module ${}_RM$ and a non-zero ideal I of R sch that for every $u \in M$ there exists a positive integer $m(u)$ with $I^{m(u)} \cdot u \subseteq E$.*

Proof. Combine 4.1 and 3.2. □

Remark 4.3. *The preceding result is, in fact, a generalization of a partial version of a well known result due to P. Hall (see [1] for more details).*

5. Finitely generated rings

Throughout this section, let R be a finitely generated ring.

Proposition 5.1. *Let R_M be a finitely generated R–module. Then there exists a finite set $\mathscr{P}$ of primes such that the additive group $M(+)$ is a free-by-$\mathscr{P}$-group.*

Proof. The result is clear if $nR = 0$ for an integer $n \geq 2$. If not, then the prime subring of R is a copy of the ring of integers and we use 4.2. □

Proposition 5.2. *Let R_M be a finitely generated R–module such that $pM = M$ for infinitely many prime p. Then there existss a finite set $\mathscr{P}$ of primes such that $M(+)$ is a $\mathscr{P}$–group.*

Proof. By 5.1, there are a free subgroup E of $M(+)$ and a finite set $\mathscr{P}$ of primes such that M/E is a $\mathscr{P}$–group. Clearly, $pM = M$ for a prime p such that $p \notin \mathscr{P}$ and 1.1 applies. □

Theorem 5.3. *Let $R \subseteq S$ be an integral extension of rings and let ${}_SM$ be a finitely generated S–module such that $M(+)$ is not torsion. Then $pM \neq M$ for almost all prime numbers p.*

Proof. We have $M = Su_1 + \ldots + S_{u_n}$, $n \geq 1$, and we put $N = R_{u_1} + \ldots + R_{u_n}$. Clearly, $N(+)$ is not torsion. By 5.1, there are a nonzero free subgroup $E(+)$ of $N(+)$ and a finite set $\mathscr{P}$ of primes such that $(N/E)(+)$ is a $\mathscr{P}$-group. We are going to show that $pM \neq M$ for every prime $p \neq \mathscr{P}$.

Assume, on the contrary, that $pM = M$. Then $u_i = pv_i$ for some $v_i \in M$ and, since M is generated by the set $\{u_1, \ldots, u_n\}$, there is a finite subset V of S such that $\{v_1, \ldots, v_n\} \subseteq K = Tu_1 + \ldots + Tu_n$, $T = R[V] \subseteq S$. We have $K = pK$ and pK is a finitely generated R-module. The same is true for ${}_RL = {}_RK/pN$.

Denote by A the torsion part of $L(+)$. Then A is a submodule of ${}_RL$and, since ${}_RA$ is noetherian, $A(+)$ is of finite exponent. Then the group $L(+)$ splits, and hence $pA = A$ and $A(+)$ has no elements of order p. It follows easily that $pN = N$ and N is torsion by 1.1, a contradiction. □

References

[1] HALL, P., 'On the finiteness of certain soluble groups', *Proc. London Math. Soc.* 9 (1959) 595 – 622.