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Commutative Zeropotent Semigroups

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Various examples of commutative semigroups $S(+)$ such that $S + S = S$ and $2x + y = 2x$ are collected.
Jsou sesbírány rozmanité příklady komutativních pologrup $S(+)$ takových, že $S + S = S$ a $2x + y = 2x$.

1. Introduction

Throughout the paper, the word "semigroup" will always mean a commutative semigroup. Unless specified explicitly, the associative and commutative binary operation of a semigroup will be denoted additively, i.e., by the symbol $+$.

Let S be a semigroup. An element $w \in S$ is called an absorbing element of S if $w + x = w$ for every $x = S$. There exists at most one absorbing element in S and, if it exists, it will be denoted by the symbol o_S (or only o). This fact will also be expressed by $o \in S$.

If A, B are subsets of S, then $A + B = \{a + b;\ a \in A,\ b \in B\}$. A non-empty subset I of S is an ideal if $I + S \subseteq I$.

Lemma 1.1.
(i) A one-element subset $\{w\}$ of S is an ideal iff $w = o_S$.
(ii) If I is an ideal of S, then the relation $(I \times I) \cup \mathrm{id}_S$ is a congruence of S.
(iii) If $o \in S$ and r is a congruence of S, then the set $\{a; (a, o) \in r\}$ is an ideal of S.

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Lemma 1.2. *The following conditions are equivalent for a semigroup S:*

(i) $|S + S| = 1$.

(ii) $o \in S$ *and* $S + S = o$.

(iii) $x + y = u + v$ *for all* $x, y, u, v \in S$.

(iv) $x + y = x + z$ *for all* $x, y, z \in S$

A semigroup S satisfying the equivalent conditions of the foregoing lemma will be called a *za-semigroup.*

Lemma 1.3. *The following conditions are equivalent for a semigroup S:*

(i) $o \in S$ *and* $2x = o$ *for every* $x \in S$.

(ii) $2x + y = 2x$ *for all* $x, y \in S$

A semigroup S satisfying the equivalent conditions of the foregoing lemma will be called *zeropotent* (or a *zp-semigroup*).

Lemma 1.4. *Every za-semigroup is zp-semigroup.*

A zp-semigroup will be called *zs-semigroup* if $S = S + S$.

2. The ordering $\dashv_S$

In this section, let S be a semigroup such that $o \in S$; we put $R = S \setminus \{o\}$.

For every $a \in S$, let $(\mathrm{Ann}_S(a) =)\ \mathrm{Ann}(a) = \{x \in S;\ a + x = o\}$ and $(\mathrm{Anh}_S(a) =)\ \mathrm{Anh}(a) = \{x \in S;\ a + x + S = o\}$. Further, $\mathrm{Ann}(S) = \bigcap \mathrm{Ann}(a)$, $a \in S$.

Lemma 2.1.

(i) For every $a \in S$, *both* $\mathrm{Ann}(a)$ *and* $\mathrm{Anh}(a)$ *are ideals of* S

(ii) $\mathrm{Ann}(S) = \{x \in S;\ S + x = o\}$.

(iii) $\mathrm{Ann}(a) \subseteq \mathrm{Anh}(a) = \{x \in S;\ a + x \in \mathrm{Ann}(S)\}$.

Now, define a relation $\dashv (= \dashv_S)$ on S by $a \dashv b$ iff $\mathrm{Ann}(a) \subseteq \mathrm{Ann}(b)$.

Lemma 2.2.

(i) $\dashv$ *is reflexive and transitive (i.e.,* $\dashv$ *is quasiordering).*

(ii) $a \dashv b$ *implies* $a + x \dashv b + x$ *for every* $x \in S$ *(i.e.,* $\dashv$ *is compatible).*

(iii) $x \dashv o$ *for every* $x \in S$

(iv) $a \in \mathrm{Ann}(S)$ *iff* $\mathrm{Ann}(a) = S$.

Furthermore, define relations $\pi (= \pi_S)$ and $\varrho (= \varrho_S)$ on S by $(a, b) \in \pi$ iff $\mathrm{Ann}(a) = \mathrm{Ann}(b)$ and $(c, d) \in \varrho$ iff $\mathrm{Anh}(c) = \mathrm{Anh}(d)$.

Lemma 2.3.

(i) Both $\pi = \mathrm{Ker}(\dashv)$ *and r are congruences of* S *and* $\pi \subseteq \varrho$.

(ii) $\varrho/\pi = \pi_T$, $T = S/\pi$.

Lemma 2.4. *The following conditions are equivalent:*

(i) $\dashv$ *is antisymmetric.*

(ii) $\dashv$ *is an ordering*

(iii) $\pi = \mathrm{id}_S$.

If these equivalent conditions are satisfied, then S will be called *separable.* The semigroup S will be called *semiseparable* iff $\mathrm{Ann}(S) = o$.

Lemma 2.5.

(i) $T = S/\pi$ *is separable iff* $\pi = \varrho$.

(ii) *If S is semiseparable, then T is separable.*

(iii) *If S is separable, then S is semiseparable.*

Lemma 2.6. *The following conditions are equivalent:*

(i) *If $a, b, c \in R$ are such that $a + b \neq o$, $a \dashv c$, $b \dashv c$, then $a + b \dashv c$.*

(ii) *If $a, b, c \in R$ are such that $a + b \neq o$ and $\mathrm{Ann}(a) \cup \mathrm{Ann}(b) \subseteq \mathrm{Ann}(c)$, then $\mathrm{Ann}(a + b) \subseteq \mathrm{Ann}(c)$.*

A semigroup satisfying the equivalent conditions of foregoing lemma will be called *upwards-regular.*

Lemma 2.7. *Assume that S is separable. Then the following conditions are equivalent:*

(i) *S is upwards-regular.*

(ii) *If $a, b \in R$ are such that $a + b \neq o$, then $a + b = \sup(a,b)$ in $(S, \dashv)$ (and $(R, \dashv)$).*

Lemma 2.8. *The following conditions are equivalent:*

(i) *If $a, b, c \in R$ are such that $a + b \neq o$, $b + c \neq o$, $c + a \neq o$, then $a + b + c \neq o$.*

(ii) *If $a, b \in R$ are such that $a + b \neq o$, then $\mathrm{Ann}(a) \cup \mathrm{Ann}(b) = \mathrm{Ann}(a + b)$.*

If the equivalent conditions of foregoing lemma are satisfied, then S will be called *strongly upwards-regular.*

Lemma 2.9. *If S is strongly upwards-regular, then S is upwards-regular.*

In the sequel, let $(\tau_S =)$ $\tau = \{(a,b) \in S \times S;\ a + b \neq o\}$ and $(\sigma_S =)$ $\sigma = \{(a,b) \in S \times S;\ a + b = o\} = (S \times S)\setminus\tau$. Finally, define a relation $(\nu_S =)$ ν on S by $(a, b) \in \nu$ iff $c \dashv a$ and $c \dashv b$ for at least one $c \in S$. Clearly, the relations τ, σ, ν are symmetric and ν is reflexive.

Lemma 2.10. *Assume that S is zeropotent. Then:*

(i) *$a \in \mathrm{Ann}(a)$ for every $a \in S$.*

(ii) *If $a \dashv b$, then $(a, b) \in \sigma$ and $\{a,b\} \subseteq \mathrm{Ann}(a) \cap \mathrm{Ann}(b)$.*

(iii) $\pi \cup \nu \subseteq \sigma$.

(iv) *τ is irreflexive and σ is reflexive.*

If S is zeropotent and $\sigma_S = \nu_S$ (see 2.10 (iii)), then we say that S is *balanced.*

3. Nil-semigroups

In this section, let S be a semigroup with $o \in S$.

An element $a \in S$ is said to be *nilpotent (of index at most m)* iff $ma = o$ for a positive integer m. Let $N_m(S)$ denote the set of nilpotent elements of index at most m and $N(S)$ the set of nilpotent elements.

Lemma 3.1.

(i) $N_m(S)$ *is an ideal of* S *for every positive integer* m.

(ii) $N(S)$ *is an ideal of* S.

(iii) $\{o\} \subseteq N_1(S) \subseteq N_2(S) \subseteq \dots$ *and* $N(S) = \bigcup N_m(S)$, $m \geq 1$.

The semigroup S is said to be a *nil-semigroup (of index at most m)* iff $N(S) = S(N_m(S) = S)$.

Lemma 3.2.

(i) S *is a nil-semigroup of index at most* 1 *iff* $S = o$.

(ii) S *is a nil-semigroup of index at most* 2 *iff* S *is a zp-semigroup.*

Lemma 3.3. $N(T) = o_T$, *where* $T = S/N(S)$.

The semigroup S is said to be *nilpotent (of index at most m)* iff $a_1 + \dots + a_m = o$ for all $a_1, \dots, a_m \in S$.

Lemma 3.4.

(i) *If* S *is nilpotent of index at most* $m \geq 1$, *then* S *is a nil-semigroup of index at most* m.

(ii) S *is nilpotent of index at most* 1 *iff* $S = o$.

(iii) S *is nilpotent of index at most* 2 *iff* S *is a za-semigroup.*

Lemma 3.5. *If* S *is a finitely generated nil-semigroup, then* S *is finite and nilpotent.*

4. The ordering $\preceq_S$

In this section, let S be a nil-semigroup. Define a relation $\preceq (\preceq_S)$ on S by $a \preceq b$ iff $b \in (S + a) \cup \{a\}$.

Lemma 4.1. *Let* $a, b \in S$ *be such that* $a = a + b$. *Then* $a = o$.

Proof. We have $a = a + b = a + 2b = a + 3b = \dots = a + mb$. But b is nilpotent. □

Lemma 4.2.

(i) $\preceq$ *is a compatible ordering of* S.

(ii) o *is the greatest element of* $(S, \preceq)$.

(iii) *If* $|S| \geq 2$, *then* $S \setminus (S + S)$ *is the set of minimal element of* $(S, \preceq)$.

Proof.

(i) Clearly, $\preceq$ is reflexive, transitive and compatible. Now, if $a \preceq b \preceq a$, $a \neq b$, then $a = b + c$, $b = a + d$, and so $a = a + e$, $e = c + d$. By 4.1, $a = o$. Then $b = o$ too, and hence $a = b$, a contradiction.

(ii) Easy.

(iii) Easy.

□

Corollary 4.3. *If $|S| \geq 2$ and $S + S = S$, then the ordered set $(S \preceq)$ has no minimal elements. In particular, S is infinite and not finitely generated.*

Lemma 4.4. $\mathrm{Ann}(S)\setminus\{o\}$ *is the set of maximal elements of the ordered set $(R, \preceq)$, $R = S\setminus\{o\}$*

Proof. Easy (use 4.1). □

Corollary 4.5. *If $|S| \geq 2$ and S is semiseparable, then the ordered set $(R, \preceq)$ has no maximal elements.*

Lemma 4.6. *If $|S| \geq 3$, then the ordered set $(S, \preceq)$ does not have smallest element.*

Proof. Use 4.1. □

Lemma 4.7. *The followinng conditions are equivalent:*

(i) If $a, b, c, d, e \in R$ are such that $a + b \neq o$ and $a + d = c = b + e$, then $c = a + b$ or $c = a + b + f$ for some $f \in S$.

(ii) If $a, b, c \in R$ are such that $a + b \neq o$, $a \preceq c$ and $b \preceq c$, then $a + b \preceq c$.

(iii) If $a, b \in R$ are such that $a + b \neq o$, then $a + b = \sup(a, b)$ in $(S, \preceq)$ (and $(R, \preceq)$).

If equivalent conditions of 4.7 are satisfied, then S will be called *downwards-regular.*

Lemma 4.8. *If $a \preceq b$, then $a \dashv b$.*

The semigroup S will be called *decent* if the relations $\preceq_S$ and $\dashv_S$ coincide (i.e., if $a \dashv_S b$ implies $a \preceq_S b$).

Lemma 4.9. *Assume that S is decent. Then:*

(i) S is separable

(ii) S is downwards-regular iff it is upwards-regular.

Define a relation μ $(= \mu_S)$ on S by $(a, b) \in \mu$ iff $c \preceq a$ and $c \preceq b$ for at least one $c \in S$ (i.e., $a, b \in (S + c) \cup \{c\}$). Clearly, μ is reflexive and symmetric.

Lemma 4.10.

(i) $\mu \subseteq \nu$.

(ii) If S is zeropotent, then $\mu \subseteq \nu \subseteq \sigma$.

If S is zeropotent and $\sigma_S = \mu_S$ (see 4.10 (ii)), then we shall say that S is *strongly balanced.*

Lemma 4.11. *Assume that S is decent. Then:*

(i) $\mu = \nu$.

(ii) If S is zeropotent, then S is balanced iff it is strongly balanced.

5. Ordered sets of special type

5.1. Let $(R, \preceq)$ be a non-empty ordered set together with an irreflexive and symmetric relation $\tau (= \tau_R)$ defined on R. For $a, b \in R$, we put $a \vee b = \sup(a, b)$, provided that this supremum exists in $(R, \preceq)$. Now, we will assume that the following condition is satisfied:

(Z0) If $a, b \in R$ are such that $(a, b) \in \tau$, then $a \vee b$ exists.

For $a \in R$, let $t(a) = \{x \in R; (a, x) \in \tau\}$. Consider the following condition:

(Z1) If $(a, b) \in \tau$ and $(c, a \vee b) \in \tau$, then $(a, c) \in \tau$ and $(b, a \vee c) \in \tau$.

Lemma 5.2. *Assume that (Z1) is true.*

(i) If $a, b, c \in R$ *are such that* $(a, b) \in \tau$ *and* $(c, a \vee b) \in \tau$, *then* (a, b), (a, c), $(b, c) \in \tau$ *and* $(a, b \vee c)$, $(b, a \vee c)$, $(c, a \vee b) \in \tau$.

(ii) If $a, b \in R$ *are such that* $a \leq b$, *then* $(a, b) \notin \tau$.

(iii) If $(a, b) \in \tau$, *then* $a \neq a \vee b \neq b$.

Consider some more conditions:

(Z2) For every $a \in R$ there exist $b, c \in R$ such that $(b, c) \in \tau$ and $a = b \vee c$

(Z3) For every $a \in R$ there exists at least one $b \in R$ with $(a, b) \in \tau$ (i.e., $t(a) \neq \emptyset$).

(Z4) For all $a, b \in R$, $a \neq b$, $(a, b) \notin \tau$, there exists at least one $c \in R$ such that either $(a, c) \in \tau$, $(b, c) \notin \tau$ or $(a, c) \notin \tau$, $(b, c) \in \tau$ (i.e., $t(a) \neq t(b)$).

(Z5) If $a \leq b$, $a \neq b$ then there exists at least one $c \in R$ such that $(a, c) \in \tau$ and $b = a \vee c$.

(Z6) If $a, b \in R$ are such that $a \leq b$, then $t(b) \subseteq t(a)$.

(Z7) If $a, b \in R$ are such that $(a, b) \notin \tau$ and $t(b) \subseteq t(a)$, then $a \leq b$.

(Z8) If $a, b, c \in R$ are such that $(a, b) \in \tau$ and $t(c) \subseteq t(a) \cap t(b)$, then $t(c) \subseteq t(a \vee b)$.

(Z9) If $a, b \in R$ are such that $(a, b) \in \tau$, then $t(a) \cap t(b) = t(a \vee b)$.

(Z10) If $a, b, c \in R$ are pair-wise different such that $(a, b) \in \tau$ and $a \vee d = c = b \vee e$ for some $d, e \in R$, $(a, d) \in \tau$, $(b, e) \in \tau$, then there exists $f \in R$ such that $(a \vee b, f) \in \tau$ and $c = a \vee b \vee f$.

(Z11) If $a, b \in R$ are such that $\emptyset \neq t(a) \neq t(b) \neq \emptyset$ and $(a, b) \notin \tau$, then there exists $c \in R$ such that $t(a) \cup t(b) \subseteq t(c)$.

(Z12) If $a, b \in R$ are such that $a \neq b$ and $(a, b) \notin \tau$, then there exist $c, d, e \in R$ such that $(c, d) \in \tau$, $(c, e) \in \tau$, $a = c \vee d$, $b = c \vee e$

5.3. Let $(R, \leq)$ be a non-empty ordered set. Define a relation τ on R by $(a,b) \in \tau$ iff the infimum $a \wedge b = \inf(a,b)$ does not exist in $(R, \leq)$. Clearly, τ is irreflexive and symmetric.

5.4. Let $T(= (T, \wedge, \vee))$ be a distributive lattice with a smallest element 0_T and a greatest element 1_T such that $|T| \geq 3$. Consider the basic order $\leq$ defined on T and also the ordered set $(R, \leq)$, where $R = T \backslash \{0_T, 1_T\}$. Define τ on R by $(a,b) \in \tau$ iff $a \wedge b = 0_T$ (see 5.3). Clearly, τ is irreflexive and symmetric. Now, assume that the following condition is satisfied:

(Y0) If $a, b \in R$ and $a \wedge b = 0_T$, then $a \vee b \neq 1_T$ (and hence $a \vee b \in R$).

Next, consider some more conditions:

(Y2) For every $a \in R$ there exist $b, c \in R$ such that $b \wedge c = 0$ and $a = b \vee c$

(Y3) For every $a \in R$ there exists at least one $b \in R$ with $a \wedge b = 0$.

(Y4) For all $a, b \in R$, $a \neq b$, $a \wedge b \neq 0$, there exists at least one $c \in R$ such that either $a \wedge c = 0 \neq b \wedge c$ or $a \wedge c \neq 0 = b \wedge c$.

(Y5) For all $a, b \in R$, $a \leq b$, $a \neq b$, there exists at least one $c \in R$ such that $a \wedge c = 0$ and $b = a \vee c$.

(Y7) If $a, b \in R$ are such that $a \wedge b \neq 0$ and $a \not\leq b$, then there exists at least one $c \in R$ with $a \wedge c = 0 \neq b \wedge c$.

(Y12) If $a, b \in R$ are such that $a \neq b$ and $a \wedge b \neq 0$, then there exist $c, d, e \in R$ such that $c \wedge d = 0 = c \wedge e$, $a = c \vee d$, $b = c \vee e$.

Lemma 5.5.

(i) The conditions (Z0), (Z1), (Z6), (Z8), (Z9), (Z10), (Z11) are satisfied.

(ii) If $i \in \{2,3,4,5,7,12\}$, then (Zi) is equivalent to (Yi).

Example 5.6. Let $\mathfrak{a}$ be an uncountable cardinal. Put $\mathfrak{T} = \{A \subseteq \mathfrak{a};\ |A| \leq \aleph_0\} \cup$ $\cup \{\mathfrak{a}\}$. Then $\mathfrak{T}$ is a sublattice of the lattice of all subsets of $\mathfrak{a}$ and r is a congruence of $\mathfrak{T}$, where $(A, B) \in r$ iff $|(A \cup B) \backslash (A \cap B)| < \aleph_0$. Now, $T = \mathfrak{T}/r$ is an (infinite) distributive lattice, $0_T = \emptyset/r$, $1_T = \mathfrak{a}/r$ and we consider the ordered set $R = T \backslash \{0_T, 1_T\}$ together with the irreflexive and symmetric relation τ. If $(a,b) \in \tau$, then $a \wedge b = 0_T \notin R$ and $1_T \neq a \vee b \in R$. Moreover, it is easy to check that all the conditions (Z0), ..., (Z12) are satisfied (use 5.5).

6. One sort of examples of zs-semigroups

Let $(R, \leq)$ be a nonempty ordered set together with an irreflexive and symmetric relation τ such that the conditions (Z0), (Z1) and (Z2) are satisfied. Let o be an element not belonging to R and $S = R \cup \{o\}$. We extend the ordering $\leq$ to S setting $a \leq o$ for every $a \in S$. Now, define an addition on S by $a + b = a \vee b$ if $(a,b) \in \tau$ (see (Z0)) and $a + b = o$ otherwise.

Proposition 6.1. *S $(= S(+))$ is a zs-semigroup.*

Proof. Since τ is symmetric, the operation $+$ is commutative. Further, $(x, o) \notin \tau$ for every $x \in S$, hence $x + o = o$ and o is an absorbing element. Since τ is irreflexive, we have $x + x = o$ for every $x \in S$. The equality $S = S + S$ follows from (Z2). It remains to show that $S(+)$ is associative.

Let $a, b, c \in S$. If $o \in \{a,b,c\}$, then $(a + b) + c = o = a + (b + c)$, and so we assume that $a, b, c \in R$.

If $(a, b) \notin \tau$ and $(b, c) \notin \tau$, then $a + b = o = b + c$, and so $(a + b) + c = o = a + (b + c)$.

If $(a, b) \notin \tau$ and $(b, c) \in \tau$, then $a + b = o$, $b + c = b \vee c$, $(a, b \vee c) \notin \tau$ by (Z1) and $(a + b) + c = o = a + (b + c)$.

If $(a, b) \in \tau$ and $(b, c) \notin \tau$, then $a + b = a \vee b$, $b + c = o$, $(c, a \vee b) \notin \tau$ by (Z1) and $(a + b) + c = o = a + (b + c)$.

If $(a, b) \in \tau$ and $(b, c) \in \tau$, then $a + b = a \vee b$, $b + c = b \vee c$. Now, if $(a, b \vee c) \notin \tau$, then $(c, a \vee b) \notin \tau$ by (Z1) and $(a + b) + c = o = a + (b + c)$. Similarly, if $(c, a \vee b) \notin \tau$. Finally, if $(a, b \vee c) \in \tau$ and $(c, a \vee b) \in \tau$, then $(a + b) + c = (a \vee b) + c = (a \vee b) \vee c = \sup(a, b, c) = a \vee (b \vee c) = a + (b \vee c) = a + (b + c)$. □

Lemma 6.2.
(i) $\operatorname{Ann}(a) = S \setminus t(a)$ *for every* $a \in R$.
(ii) $\operatorname{Ann}(o) = S$.

Lemma 6.3. $\operatorname{Ann}(S) = \{a \in R;\ t(a) = \emptyset\} \cup \{o\}$.

Lemma 6.4. *The semigroup S is semiseparable iff (Z3) is true.*

Lemma 6.5. *If* $a, b \in R$, *then* $(a, b) \in \pi$ *iff* $t(a) = t(b)$

Lemma 6.6. *The semigroup is separable iff the conditions (Z3) and (Z4) are satisfied.*

Lemma 6.7. *Let* $a, b \in R$, $a \neq b$. *Then* $a \preceq b$ *iff* $b = a \vee c$ *for some* $c \in R$ *such that* $(a, c) \in \tau$.

Lemma 6.8. *If* $a, b \in S$ *are such that* $a \preceq b$, *then* $a \leq b$.

Lemma 6.9. *The relations* $\preceq$ *and* $\leq$ *coincide iff the condition (Z5) is satisfied.*

Lemma 6.10. *Let* $a, b \in R$. *Then:*
(i) $a \dashv b$ *iff* $t(b) \subseteq t(a)$.
(ii) $o \dashv a$ *iff* $t(a) = \emptyset$.
(iii) $a \dashv o$.

Lemma 6.11. *If* $a, b, c \in R$ *are such that* $a \leq b$ *and* (a, b), $(b, c) \in \tau$, *then* $a + c \leq b + c$.

Lemma 6.12. *The ordering* $\leq$ *of S is compatible with the addition iff* $\leq$ *is contained in* $\dashv$ *and this is equivalent to the condition (Z6).*

Lemma 6.13. *The relations $\leq$ and $\dashv$ coincide iff the conditions (Z3), (Z6) and (Z7) are satisfied.*

Lemma 6.14. *The relations $\preceq$, $\leq$ and $\dashv$ coincide (i.e., S is decent) iff the conditions (Z3), (Z5), (Z5) and (Z7) are satisfied.*

Lemma 6.15. *The semigroup S is upwards-regular iff (Z8) is true.*

Lemma 6.16. *The semigroup S is strongly upwards-regular iff (Z9) is true.*

Lemma 6.17. *The semigroup S is downwards-regular iff (Z10) is true.*

Lemma 6.18. *The semigroup S is (strongly) balanced iff (Z11) ((Z12)) is true.*

In the sequel, the semigroup S $(= S(+))$ will be denoted by $\mathscr{Z}(R, \leq, \tau, o)$.

7. A few consequences

Proposition 7.1. *Let S be a non-trivial separable upwards-regular zs-semigroup. Put $R = S\backslash\{o\}$, denote by $\leq$ the restriction of the ordering $\dashv_S$ to R (see 2.4) and define a relation τ_R on R by $(a,b) \in \tau_R$ iff $a + b \neq o$. Then:*

(i) $(R, \leq)$ is an infinite ordcered set.

(ii) τ_R is irreflexive and symmetric.

(iii) If $(a,b) \in \tau_R$, then $a + b = a \vee b = \sup(a,b)$ in $(R, \leq)$.

(iv) The conditions (Z0), (Z1), (Z2), (Z3), (Z4), (Z6), (Z7) and (Z8) are satisfied.

(v) The condition (Z5) is satisfied iff S is decent.

(vi) The condition (Z9) is satisfied iff S is strongly upwards-regular.

(vii) The condition (Z10) is satisfied iff S is downwards-regular.

(viii) The condition (Z11) ((Z12)) is satisfied iff S is (strongly) balanced.

Proof. See 2.4, 2.6, 2.7, 4.3 and 6. □

Corollary 7.2. *The following conditions are equivalent for a groupoid S:*

(i) S is a non-trivial separable upwards-regular zs-semigroup.

(ii) $o \in S$, $|S| \geq 2$ and there exist an ordering $\leq$ and an irreflexive and symmetric relation τ defined on $R = S\backslash\{o\}$ such that the conditions (Z0), (Z1), (Z2), (Z3), (Z4), (Z6), (Z7) and (Z8) are satisfied and $S = \mathscr{Z}(R, \leq, \tau, o)$ (then $\leq$ is $\dashv_S$ restricted to R, τ is τ_S restricted to R, $a + b = \sup(a,b)$ for $(a,b) \in \tau$ and $a + b = o$ otherwise).

Proposition 7.3. *Let S be a non-trivial downwards-regular zs-semigroup. Put $R = S\backslash\{o\}$, denote by $\leq$ the restriction of the ordering $\preceq_S$ to R (see 4.2) and define a relation τ_R on R by $(a,b) \in \tau_R$ iff $a + b \neq o$. Then:*

(i) $(R, \leq)$ is an infinite ordered set.

(ii) τ_R is irreflexive and symmetric.

(iii) *If* $(a,b) \in \tau_R$, *then* $a + b = a \vee b = \sup(a,b)$ *in* $(R, \leq)$.
(iv) *The conditions (Z0), (Z1), (Z2), (Z5), (Z6) and (Z10) are satisfied.*
(v) *The condition (Z3) is satisfied iff S is semiseparable.*
(vi) *The conditions (Z3) and (Z4) are satisfied iff S is separable.*
(vii) *The conditions (Z3) and (Z7) are satisfied iff S is decent.*
(viii) *The condition (Z8) ((Z9)) is satisfied iff S (strongly) upwards-regular.*
(ix) *The condition (Z11) ((Z12)) is satisfied iff S (strongly) balanced.*

Proof. See 4.2, 4.3, 4.7 and 6. □

Corollary 7.4. *The following conditions are equivalent for a groupoid S:*
(i) *S is a non-trivial downwards-regular zs-semigroup.*
(ii) $o \in S$, $|S| \geq 2$ *and there exist an ordering* $\leq$ *and an irreflexive and symmetric relation* τ *defined on* $R = S \setminus \{o\}$ *such that the conditions (Z0), (Z1), (Z2), (Z5), (Z6) and (Z10) are satisfied and* $S = \mathscr{Z}(R, \leq, \tau, o)$ *(then* $\leq$ *is* $\preceq_S$ *restricted to R,* τ *is* τ_S *restricted to R,* $a + b = \sup(a,b)$ *for* $(a,b) \in \tau$ *and* $a + b = o$ *otherwise).*

8. Particular examples of zs-semigroups

Example 8.1. Let I be a infinite set, $|I| = \mathfrak{a}$, and $\mathfrak{I}$ the set of infinite subset of I. Define an operation $\oplus$ on $\mathfrak{I}$ by $A \oplus B = A \cup B$ if $A \cap B = \emptyset$ and $A \oplus B = I$ otherwise.

Proposition 8.2. $\mathfrak{I}$ $(= \mathfrak{I}(\oplus))$ *is a zs-semigroup, where* $o_{\mathfrak{I}} = I$.

Lemma 8.3.
(i) $\mathfrak{A} = \operatorname{Ann}(\mathfrak{I})$ *is the set of cofinite subsets of I.*
(ii) $\pi_{\mathfrak{I}} = \varrho_{\mathfrak{I}} = (\mathfrak{A} \times \mathfrak{A}) \cup \mathrm{id}_{\mathfrak{I}}$

Corollary 8.4. $\mathfrak{I}$ *is not separable.*

Lemma 8.5. $A \dashv_{\mathfrak{I}} B$ *iff either* $A \subseteq B$ *or B is a cofinite subset of I (i.e.,* $B \in \mathfrak{A}$*).*

Lemma 8.6. $A \preceq_{\mathfrak{I}} B$ *iff either* $A = B$ *or* $B = I$ *or* $A \subseteq B$ *and* $B \setminus A$ *is infinite.*

Corollary 8.7.
(i) *If* $A \preceq_{\mathfrak{I}} B$, *then* $A \subseteq B$. *The converse is not true.*
(ii) *If* $A \subseteq B$, *then* $A \dashv_{\mathfrak{I}} B$. *The converse is not true.*

Proposition 8.8. $\mathfrak{I}$ *is upwards-regular but neither strongly upwards-regular nor downwards-regular.*

Lemma 8.9.
(i) $(A,B) \in \sigma_{\mathfrak{I}}$ *iff either* $A \cap B \neq \emptyset$ *or* $A \cup B = I$.
(ii) $(A,B) \in \nu_{\mathfrak{I}}$ *iff* $(A,B) \in \mu_{\mathfrak{I}}$ *and iff* $A \cap B$ *is infinite.*

Corollary 8.10. $\mu_{\mathfrak{I}} = \nu_{\mathfrak{I}}$ *and* $\mathfrak{I}$ *is not balanced.*

Let $\mathfrak{b}$ be an infinite cardinal such that $\mathfrak{b} \leq \mathfrak{a}$. Put

$$\mathfrak{I}_{\mathfrak{b}} = \{A \in \mathfrak{I};\ |A| \leq \mathfrak{b}\} \cup \{I\}.$$

Proposition 8.11. *For every* $\mathfrak{b} \leq \mathfrak{a}$ *is* $\mathfrak{I}_{\mathfrak{b}}$ *a subsemigroup of* $\mathfrak{I}$. $\mathfrak{I}_{\mathfrak{b}}$ *is also a non-trivial zs-semigroup, upwards-regular, but neither downwards-regular nor balanced.*

Proposition 8.12. *If* $\mathfrak{b} < \mathfrak{a}$, *then* $\mathfrak{I}_{\mathfrak{b}}$ *is separable, strongly upwards-regular and the relations* $\subseteq$ *and* $\dashv_{\mathfrak{I}_{\mathfrak{b}}}$ *coincide. Moreover, the automorphism group* $\mathrm{Aut}(\mathfrak{I}_{\mathfrak{b}})$ *of* $\mathfrak{I}_{\mathfrak{b}}$ *operates transitively on* $\mathfrak{I}_{\mathfrak{b}} \setminus \{I\}$.

Let $\mathfrak{K}$ be a (non-principal) maximal ideal of the Boolean algebra of all subsets of I such that $A \in \mathfrak{K}$ for every $A \subseteq I$, $|A| < \mathfrak{a}$. Put $\mathfrak{L} = \{B \in \mathfrak{K};\ |B| = \mathfrak{a}\} \cup \{I\}$.

Proposition 8.13. $\mathfrak{L}$ *is a subsemigroup of* $\mathfrak{I}$ *and* $\mathfrak{L}$ *is a non-trivial separable zs-semigroup. Moreover, the automorphism group* $\mathrm{Aut}(\mathfrak{L})$ *of* $\mathfrak{L}$ *operates transitively on* $\mathfrak{L} \setminus \{\mathfrak{I}\} = \{B \in \mathfrak{K};\ |B| = \mathfrak{a}\}$.

Proof. Take $A, B \in \mathfrak{L}$, $A \neq I \neq B$. Then $A' = I \setminus A \notin \mathfrak{K}$, $B' = I \setminus B \notin \mathfrak{K}$ and $A' \cap B' \notin \mathfrak{K}$. Since $A \cup B \in \mathfrak{L}$, we have $|A' \cap B'| = \mathfrak{a}$. Consequently, $A' \cap B' = C_1 \cup C_2$, $C_1 \cap C_2 = \emptyset$, $|C_1| = \mathfrak{a} = |C_2|$. Since $A' \cap B' \notin \mathfrak{K}$, we may assume that $C_1 \notin \mathfrak{K}$. Then $C_2 \subseteq C_1' \in \mathfrak{K}$ and $C_2 \in \mathfrak{K}$. Further, $D_1 = A' \setminus C_1 \subseteq C_1' \in \mathfrak{K}$, $D_1 \in \mathfrak{K}$ and $D_2 = B' \setminus C_1 \in \mathfrak{K}$. On the other hand, $C_2 \subseteq D_1 \cap D_2$, and so $D_1, D_2 \in \mathfrak{L}$. Clearly, there is a permutation p of I such that $p(A) = B$, $p(D_1) = D_2$ and $p \mid C_1 = \mathrm{id}$. Now, define a transformation f of the Boolean algebra of subsets of I by $f(E) = p(E)$, for every $E \subseteq I$. Then f is a permutation of the Boolean algebra and $f(A) = B$. It remains to show that f is an automorphism of $\mathfrak{L}(\oplus)$.

If $L \in \mathfrak{L}$, $L \neq I$, then $L = L_1 \cup L_2 \cup L_3$, $L_1 = L \cap A$, $L_2 = L \cap C_1$, $L_3 = L \cap D_1$, and $f(L) = p(L_1) \cup p(L_2) \cup p(L_3) \subseteq B \cup L_2 \cup D_2 \in \mathfrak{K}$. Thus $f(L) = p(L) \in \mathfrak{L}$. Quite similarly, $f^{-1}(L) \in \mathfrak{L}$. It follows that $f \mid \mathfrak{L}$ is a permutation of $\mathfrak{L}$. The rest is clear. □

Example 8.14. Define another operation $\boxplus$ on $\mathfrak{I}$ (see example 8.1) by $A \boxplus B = A \cup B$ if $A \cap B$ is finite and $A \boxplus B = I$ otherwise.

Proposition 8.15. $\mathfrak{I}$ $(= \mathfrak{I}(\boxplus))$ *is a zs-semigroup, where* $o_{\mathfrak{I}} = I$.

Lemma 8.16.

(i) $\mathfrak{A} = \mathrm{Ann}(\mathfrak{I})$ *is the set of cofinite subsets of* I.
(ii) $\pi_{\mathfrak{I}} = (\mathfrak{A} \times \mathfrak{A}) \cup \mathrm{id}_{\mathfrak{I}}$.
(iii) $(A, B) \in \varrho_{\mathfrak{I}}$ *iff* $(A \cup B) \setminus (A \cap B)$ *is finite.*

Corollary 8.17. $\mathfrak{I}$ *is not separable.*

Lemma 8.18. $A \dashv_{\mathfrak{I}} B$ *iff either* $A \subseteq B$ *or* $A \setminus B$ *is finite and* $B \setminus A$ *is infinite or* B *is a cofinite subset of* I *(i.e.,* $B \in \mathfrak{A}$*).*

Lemma 8.19. *$A \preceq_{\mathfrak{I}} B$ iff either $A = B$ or $B = I$ or $A \subseteq B$ and $B \setminus A$ is infinite.*

Corollary 8.20.
(i) If $A \preceq_{\mathfrak{I}} B$, then $A \subseteq B$. The converse is not true.
(ii) If $A \subseteq B$, then $A \dashv_{\mathfrak{I}} B$. The converse is not true.

Proposition 8.21. *$\mathfrak{I}$ is neither upwards- nor downwards-regular.*

Lemma 8.22.
(i) $(A, B) \in \sigma_{\mathfrak{I}}$ iff either $A \cap B$ is infinite or $A \cup B = I$.
(ii) $(A, B) = \nu_{\mathfrak{I}}$ iff $(A, B) \in \mu_{\mathfrak{I}}$ and iff $A \cap B$ is infinite.

Corollary 8.23. *$\nu_{\mathfrak{I}} = \nu_{\mathfrak{I}}$ and $\mathfrak{I}$ is not balanced.*

Proposition 8.24. *ϱ is a congruence of the semigroup $\mathfrak{I}$, the factor $\mathfrak{J} = \mathfrak{I}/\varrho$ is a non-trivial zs-semigroup and $\mathfrak{J}$ is separable, upwards-regular, downwards-regular and decent. $\mathfrak{J}$ is neither strongly upwards-regular nor balanced.*

Proposition 8.25. *If $\mathfrak{a} = \aleph_0$, then the automorphism group $\mathrm{Aut}(\mathfrak{J})$ of $\mathfrak{J}$ operates transitively on $\mathfrak{J} \setminus \{o_{\mathfrak{J}}\}$.*

Proposition 8.26. *Assume that $\mathfrak{a} \geq \aleph_1$ and put $\mathfrak{K} = \{A \in \mathfrak{I};\ |A| = \aleph_0\} \cup \{I\}$. Then*
(i) $\mathfrak{K}$ is a subsemigroup of $\mathfrak{J}$.
(ii) $\mathfrak{K}$ is a non-trivial zs-semigroup.
(iii) If $A, B \in \mathfrak{K}$, then $(A, B) \in \pi_{\mathfrak{K}}$ iff $(A, B) \in r$ (i.e., $(A \cup B) \setminus (A \cap B)$ is finite).

Proposition 8.27. *Assume that $\mathfrak{a} \geq \aleph_1$ and put $\mathfrak{L} = \mathfrak{K}/\pi_{\mathfrak{K}}$ (see Proposition 8.26). Then*
(i) $\mathfrak{L}$ is a non-trivial zs-semigroup.
(ii) $\mathfrak{L}$ is separable, strongly upwards-regular, downwards-regular, decent and strongly balanced.
(iii) $\mathrm{Aut}(\mathfrak{L})$ operates transitively on $\mathfrak{L} \setminus \{o_{\mathfrak{L}}\}$.

Remark 8.28. The semigroup $\mathfrak{L}$ is identical with the semiroup constructed by means of Example 5.6 and Proposition 6.1.