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Dalibor Pražák

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Fourier Series and the Colombeau Algebra on the Unit Circle

DALIBOR PRAŽÁK

Praha

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We give a simple construction of Colombeau algebra in the setting of 2π-periodic functions. The canonical embedding is provided by the Fourier series expansion. We develop the basic theory and discuss several examples of products.

1. Introduction

The algebra of generalized functions $\mathscr{G}$ was introduced in [Co]. $\mathscr{G}$ (also called Colombeau algebra after its author) generalizes the space of distributions. Elements of $\mathscr{G}$ can be differentiated, but also arbitrarily multiplied with each other. Moreover, the Leibniz rule holds in the usual form.

The so-called Schwartz' impossibility result [Sch1] shows that natural assumptions on any algebra of functions containing distributions lead to contradiction. Such an conclusion is avoided in $\mathscr{G}$ – rather surprisingly – by dropping the requirement of the pointwise character of the multiplication. In other words

$$(fg)(x) = f(x)g(x) \tag{1}$$

need not hold in $\mathscr{G}$ even for continuous functions f, g. However, (1) holds true if f, g are infinitely smooth; this is one of the central achievements of the Colombeau theory.

Charles University, Faculty of Mathematics and Physics, Department of Math. Analysis, Sokolovská 83, 186 75 Praha 8, Czech Republic

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e-mail: prazak@karlin.mff.cuni.cz

The construction of $\mathscr{G}$ can be found in many books; e.g. [Co], [Ob], [Ro]. In particular, Oberguggenberger [Ob] discusses in detail the relation of $\mathscr{G}$ to other approaches to the multiplication of distributions; also many applications to various partial differential equations are given. Rosinger's book [Ro] adopts a far more general point of view on the problem of construction of differential algebras.

Following Colombeau's original work, there appeared a number of variants of the original definition of $\mathscr{G}$. Some of them are discussed in [Ob]. See also more recent papers [Bo], [Je], or [Sh].

It seems to us, however, that the basic ideas of the construction are usually obscured by a host of technically complicated definitions. The key observation that hides behind the construction of $\mathscr{G}$ is the simple and well-known fact that, roughly speaking, the rate of convergence of a smooth approximation improves with the regularity of the approximated function.

To emphasize this, we aim in this paper to construct Colombeau algebras in a simple setting of 2π-periodic functions. The main advantage is the elementary approach and simple definitions. The canonical embedding of $\mathscr{D}'(\mathbb{T})$ into $\mathscr{G}$ is defined via the Fourier series. We thus remain in the setting that is familiar to most readers. Moreover, the special structure and properties of trigonometric functions greatly simplify the calculations of concrete examples.

The paper is organized in the following way. Section 2 describes the standard function spaces. We review some distibution theory; we also discuss "pointwise" product (1) and its generalizations. The variant of Schwartz' impossibility result is presented. In Section 3 we collect the necessary preliminaries regarding the Fourier series. These results are well-known, but we prove most of them for the sake of completeness.

The main content of the paper follows. Section 4 brings the construction of the Colombeau algebra on the unit circle. We define the canonical embedding ι, the relation of association $\approx$, and show their basic properties. Last Section 5 is devoted to several examples that demonstrate the nonlinear properties of $\mathscr{G}$. All these are variants of previous constructions, but, up to our knowledge, are new in this particular setting of 2π-periodic functions.

2. Functions and distributions

By $\mathbb{T}$ we denote the unit circle $\mathbb{R}/2\pi$. The elements of $C(\mathbb{T})$ *of* $C(\mathbb{T})$, $C^k(\mathbb{T})$, $C^\infty(\mathbb{T})$ are naturally identified with the 2π-periodic continuous (resp. k-time continuously differentiable resp. infinitely differentiable) functions on $\mathbb{R}$.

Similarly $L^p(\mathbb{T})$ are just 2π-periodic functions from $L^p_{loc}(\mathbb{R})$. With this convention

$$\int_{\mathbb{T}} f(x)\,dx$$

is computed as an integral over arbitrary interval of length 2π.

Distributions $\mathscr{D}'(\mathbb{T})$ are the linear continuous functionals on $\mathscr{D}(\mathbb{T}) := C^\infty(\mathbb{T})$, considered with the topology generated by

$$\|\varphi\|_k = \sup_{\mathbb{T}} \left|\tfrac{d^k}{dx^k}\varphi\right|, \qquad k = 0, 1, \ldots \tag{2}$$

The duality between $\mathscr{D}'(\mathbb{T})$ and $\mathscr{D}(\mathbb{T})$ is denoted $\langle , \rangle$. Occcasionally we add a subscript to indicate the variable over which the pairing is taken. δ_a stands for the Dirac measure in $a \in \mathbb{T}$, defined by

$$\langle \delta_a, \varphi \rangle = \langle \delta_a(x), \varphi(x) \rangle_x := \varphi(a).$$

To each $f \in L^1(\mathbb{T})$ there corresponds a distribution T_f, given by

$$\langle T_f, \varphi \rangle := \int_{\mathbb{T}} f(x)\varphi(x)\,dx. \tag{3}$$

One says that T_f is a regular distribution with the density f. The derivative of the distribution T is defined by

$$\langle \tfrac{d^k}{dx^k} T, \varphi \rangle := (-1)^k \langle T, \tfrac{d^k}{dx^k}\varphi \rangle.$$

One verifies easily that $\frac{d^k}{dx^k} T \in \mathscr{D}'(\mathbb{T})$. A well-known deeper result from the theory of distributions (see e.g. [Sch2]) says that every distribution is a derivative of a continuous function. More precisely, we have the following theorem.

Theorem 1. *Given $T \in \mathscr{D}'(\mathbb{T})$, there exist $c \in \mathbb{R}$, $k \in \mathbb{N}$ and $F \in C(\mathbb{T})$ such that $T = c + \frac{d^k}{dx^k} T_F$.*

For example

$$\delta_0 = \frac{1}{2\pi} - \left(\tfrac{d}{dx}\right)^2 T_w$$

where

$$w(x) = \frac{x^2}{4\pi} - \frac{x}{2}, \qquad x \in [0, 2\pi).$$

Remark. Concerning the spaces considered so far, only $C^k(\mathbb{T})$ and $L^\infty(\mathbb{T})$ can be equipped with the pointwise product (1), yielding algebras. In other circumstances this product can be used with certain limitations. For example, Hölder's inequality asserts that $fg \in L^1$ provided that $f \in L^p$, $g \in L^q$ where $1/p + 1/q \leq 1$ (with the convention that the result is defined almost everywhere.) This is the common way to interpret the nonlinearities in the PDE theory.

A natural generalization of (1) defines $\omega \cdot T = T \cdot \omega \in \mathscr{D}'(\mathbb{T})$ as the product of $\omega \in \mathscr{D}(\mathbb{T})$ and $T \in \mathscr{D}'(\mathbb{T})$ by

$$\langle \omega \cdot T, \varphi \rangle = \langle T \cdot \omega, \varphi \rangle := \langle T, \omega\varphi \rangle \qquad \forall \varphi \in \mathscr{D}(\mathbb{T}). \tag{4}$$

Unfortunately, the Schwartz' impossibility argument applies here. Consider "cosecant" $T_{\mathrm{csc}} \in \mathscr{D}'(\mathbb{T})$ given by

$$\langle T_{\mathrm{csc}}, \varphi \rangle = \lim_{\varepsilon \to 0+} \int_{\mathbb{T}_\varepsilon} \frac{\varphi(x)}{\sin x} dx,$$

where $\mathbb{T}_\varepsilon = (-\pi + \varepsilon, -\varepsilon) \cup (\varepsilon, \pi - \varepsilon)$. Integration by parts shows that

$$\begin{aligned}\langle T_{\mathrm{csc}}, \varphi \rangle = & \lim_{\varepsilon \to 0+} (\varphi(-\varepsilon) - \varphi(\varepsilon)) \ln \left|\tan \frac{\varepsilon}{2}\right| \\ & + \lim_{\varepsilon \to 0+} (\varphi(\pi - \varepsilon) - \varphi(-\pi + \varepsilon) \ln \left|\tan \frac{\pi - \varepsilon}{2}\right| \\ & - \lim_{\varepsilon \to 0+} \int_{\mathbb{T}_\varepsilon} \ln \left|\tan \frac{x}{2}\right| \varphi'(x)\, dx.\end{aligned}$$

The first two limits are zero since $\varphi(x + \varepsilon) - \varphi(x - \varepsilon) = \mathcal{O}(\varepsilon)$. We see that T_{csc} is a distributional derivative of $\ln |\tan \frac{x}{2}| \in L^1(\mathbb{T})$; in particular it is a distribution.

The product $\sin \cdot T_{\mathrm{css}}$ makes sense according to (4). One has

$$\langle \sin \cdot T_{\mathrm{csc}}, \varphi \rangle = \lim_{\varepsilon \to 0+} \int_{\mathbb{T}_\varepsilon} \frac{(\sin x)\varphi(x)}{\sin x} dx = \int_{\mathbb{T}} \varphi(x)\, dx = \langle 1, \varphi \rangle. \tag{5}$$

In other words, $\sin \cdot T_{\mathrm{csc}} = 1$ as expected. Hence

$$\delta_0 \cdot (\sin \cdot T_{\mathrm{csc}}) = \delta_0 \cdot 1 = \delta_0.$$

On the other hand,

$$\langle \delta_0 \cdot \sin, \varphi \rangle = \langle \delta_0, \sin \cdot \varphi \rangle = \sin(0)\varphi(0) = 0, \tag{6}$$

hence

$$(\delta_0 \cdot \sin) \cdot T_{\mathrm{csc}} = 0 \cdot T_{\mathrm{csc}} = 0.$$

This shows that the product (4), though it seems very natural, is not even associative.

3. Fourier series

For $f \in L^1(\mathbb{T})$ the Fourier coefficients are defined by

$$a_k := \frac{1}{\pi} \int_{\mathbb{T}} f(x) \cos kx\, dx, \qquad k \geq 0,$$

$$b_k := \frac{1}{\pi} \int_{\mathbb{T}} f(x) \sin kx\, dx, \qquad k \geq 1.$$

The Fourier series is given via its n-th partial sum

$$\mathcal{F}_{f,n}(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx + b_k \sin kx. \tag{7}$$

Equivalently we can write

$$\mathcal{F}_{f,n}(x) = \frac{1}{\pi} \int_{\mathbb{T}} f(t) D_n(x-t)\, dt,$$

with the so-called Dirichlet kernel

$$D_n(x) = \frac{\sin\left(n + \frac{1}{2}\right)x}{2 \sin \frac{x}{2}}.$$

The fol owing approximation properties of $\mathcal{F}_{f,n}$ are well-known.

Theorem 2. *1. If $f \in L^2(\mathbb{T})$, then $\mathcal{F}_{f,n} \to f$ in $L^2(\mathbb{T})$.*

2. Assume f has bounded variation, i.e., f can be written as a difference of two monotone functions. Then

$$\mathcal{F}_{f,n}(x) \to \frac{1}{2}[f(x+) + f(x-)]$$

for any $x \in \mathbb{T}$. In particular, $\mathcal{F}_{f,n}(x) \to f(x)$ if f is continuous at x.

Proof. See e.g. [Ko]. $\square$

It is a well-known fact that the rate of convergence $\mathcal{F}_{f,n} \to f$ improves as f becomes smoother. This fact is crucial for the presented construction of Colombeau algebras. We will need the following (not necessarily optimal) lemma.

Remark. In what follows we write $a_n = \mathcal{O}(n^K)$ with the usual meaning: $(\exists c > 0)(\exists n_0 \in \mathbb{N})(\forall n \geq n_0)\, |a_n| \leq cn^K$. We mostly drop the symbol $n \to \infty$.

Lemma 1. *Let $f \in C^{N+1}(\mathbb{T})$, $N \geq 0$. Then*

$$\sup_{x \in \mathbb{T}} |\mathcal{F}_{f,n}(x) - f(x)| = \mathcal{O}(n^{-N}).$$

Proof. Integration by parts yields

$$a_k = \pm k^{\ (N+1)} \int_{\mathbb{T}} f^{(N+1)}(x) \begin{Bmatrix} \cos kx \\ \sin kx \end{Bmatrix} dx.$$

Similarly for b_k. The boundedness of $f^{(N+1)}$ gives $|a_k| + |b_k| \leq ck^{\ (N+1)}$. The smoothness of f guarantees $\mathcal{F}_{f,n}(x) \to f(x)$ everywhere, and one gets

$$|\mathcal{F}_{f,n}(x) - f(x)| = \left| \sum_{k-n+1}^{\infty} a_k \cos kx + b_k \sin kx \right| \leq c \sum_{k=n+1}^{\infty} \frac{1}{k^{N+1}} = \mathcal{O}(n^{\ N}).$$

$\square$

If T is a distribution, we define $\mathscr{F}_{T,n}$ in a formal analogy to (7) by

$$a_k = \frac{1}{\pi}\langle T(x), \cos kx\rangle_x,$$

$$b_k = \frac{1}{\pi}\langle T(x), \sin kx\rangle_x.$$

One also has[1]

$$\mathscr{F}_{T,n}(x) = \frac{1}{\pi}\langle T(y), D_n(x-y)\rangle_y.$$

We observe that

$$\tfrac{d^k}{dx^k}\mathscr{F}_{T,n}(x) = \mathscr{F}_{\frac{d^k}{dx^k}T,n}(x) \tag{8}$$

– this follows easily from the definition of the distributional derivative. Finally, the following result will be needed.

Lemma 2. *Let $T \in \mathscr{D}'(\mathbb{T})$. Then $\mathscr{F}_{T,n} \to T$ in $\mathscr{D}'(\mathbb{T})$.*

Proof. One has

$$\int_{\mathbb{T}} \mathscr{F}_{T,n}(x)\,\varphi(x)\,dx = \frac{1}{\pi}\int_{\mathbb{T}} \langle T(y), D_n(x-y)\rangle_y\,\varphi(x)\,dx$$
$$= \langle T(y), \frac{1}{\pi}\int_{\mathbb{T}} D_n(x-y)\,\varphi(x)\,dx\rangle_y.$$

For the proof that the duality $\langle,\rangle$ commutes with the integral see e.g. [Vl]. The conclusion follows since the last integral converges to φ in $\mathscr{D}(\mathbb{T})$, i.e. in the seminorms (2), cf. Lemma 1. □

4. Colombeau algebra on the unit circle

The main idea is that the elements of $\mathscr{G}$ are represented by sequences of infinitely smooth functions; two elements are considered equal if the difference of the representing sequences tends to zero sufficiently fast.

The space of representatives $\mathscr{E}(\mathbb{T})$ consists of functions

$$R = R(n,x) : \mathbb{N} \times \mathbb{T} \to \mathbb{R}$$

such that $R(n,\cdot) \in \mathscr{D}(\mathbb{T})$ for $\forall n \in \mathbb{N}$. $\mathscr{E}(\mathbb{T})$ can be viewed simply as a space of sequences (indexed by $\mathbb{N}$) of C^∞ functions.

Moderate representatives $\mathscr{E}_M(\mathbb{T})$ are given by

$$\mathscr{E}_M(\mathbb{T}) = \{R \in \mathscr{E}(\mathbb{T});\ (\forall k \geq 0)(\exists N > 0) \sup_{x\in\mathbb{T}} |\tfrac{d^k}{dx^k} R(n,x)| = \mathscr{O}(n^N)\}. \tag{9}$$

Obviously $\mathscr{E}_M(\mathbb{T})$ is a linear subspace of $\mathscr{E}(\mathbb{T})$, which is closed under the operation $\frac{d^k}{dx^k}$. Recalling the Leibniz rule

$$\frac{k}{dx^k}\{RS\} = \sum_{l=0}^{k} \binom{k}{l} \left(\tfrac{d}{dx}\right)^l R \left(\tfrac{d}{dx}\right)^{k-1} S, \tag{10}$$

one verifies easily that $\mathscr{E}_M(\mathbb{T})$ is in fact an algebra, i.e. $R, S \in \mathscr{E}_M(\mathbb{T})$ implies $RS \in \mathscr{E}_M(\mathbb{T})$.

The space of negligible representatives $\mathscr{N}(\mathbb{T})$ is defined as

$$\mathscr{N}(\mathbb{T}) = \{R \in \mathscr{E}(\mathbb{T});\ (\forall k \geq 0)(\forall M > 0) \sup_{x \in T} |\tfrac{d^k}{dx^k} R(n, x)| = \mathcal{O}(n^{\;M})\}. \tag{11}$$

Once again, $\mathscr{N}(\mathbb{T})$ is closed under the differentiation w.r. to x. Clearly, $\mathscr{N}(\mathbb{T}) \subset \subset \mathscr{E}_M(\mathbb{T})$; moreover, $\mathscr{N}(\mathbb{T})$ is an ideal in $\mathscr{E}_M(\mathbb{T})$. This means, $\mathscr{N}(\mathbb{T})$ is in algebra and if $R \in \mathscr{E}_M(\mathbb{T})$, $N \in \mathscr{N}(\mathbb{T})$, then $RN \in \mathscr{N}(\mathbb{T})$.

Let us verify the last claim. Given $k \geq 0$, $M > 0$, we know by (9) that $\frac{d^l}{dx^l} R$ is (uniformly w.r. to $x \in \mathbb{T}$) $\mathcal{O}(n^N)$ for some $N > 0$, $l = 0, \ldots, k$. By (11), $\frac{d^l}{dx^l} R$ is $\mathcal{O}(n^{N\;M})$ for $l = 0, \ldots, k$. Hence $\frac{d^k}{dx^k}\{RS\}$ is $\mathcal{O}(n^{-M})$ in virtue of (10).

The Colombeau algebra $\mathscr{G}(\mathbb{T})$ is now defined as the quotient

$$\mathscr{G}(\mathbb{T}) = \mathscr{E}_M(\mathbb{T}) / \mathscr{N}(\mathbb{T}).$$

It is convenient to denote the elements of $\mathscr{G}(\mathbb{T})$ by $[R]$, where $R \in \mathscr{E}_M(\mathbb{T})$ is arbitrary member of the given equivalence class. We call R the representative of $[R]$. Clearly $[R] = [R']$ if and only if $R - R' \in \mathscr{N}(\mathbb{T})$.

The operations of addition, multiplication and differentiation on $\mathscr{G}(\mathbb{T})$ are defined as

$$[R] + [S] := [R + S],$$
$$[R][S] := [RS],$$
$$\frac{d^k}{dx^k}[R] := \left[\frac{d^k}{dx^k} R\right].$$

It follows from the fact that $\mathscr{N}(\mathbb{T})$ is an ideal that the definitions are independent of the choice of the representatives.

For the sake of completeness, let us check it for the product. Assume $[R] = [R']$, $[S] = [R']$. This means $R' = R + N_1$, $S' = S + N_2$, where $N_i \in \mathscr{N}(\mathbb{T})$. We have

$$R'S' = RS + N_1 S + R N_2 + N_1 N_2 = RS + N,$$

where $N \in \mathscr{N}(\mathbb{T})$, hence $[R'S'] = [RS]$.

We remark that the Leibniz rule holds in $\mathscr{G}(\mathbb{T})$, precisely because it holds for the representatives. Let us verify it formally for $k = 1$:

$$\frac{d}{dx}\{[R][S]\} = \left[\frac{d}{dx}(RS)\right] = \left[\frac{d}{dx} R + R \frac{d}{dx} S\right] =$$
$$= \left[\frac{d}{dx} R\right][S] + [R]\left[\frac{d}{dx} S\right] = \frac{d}{dx}[R][S] + [R]\frac{d}{dx}[S].$$

Now we introduce the canonical embedding $\iota : \mathscr{D}'(\mathbb{T}) \to \mathscr{E}_M(\mathbb{T})$. This is usually defined by the means of a suitable mollification; we will use the Fourier series. For $T \in \mathscr{D}'(\mathbb{T})$ we define the canonical representative

$$\iota T(n, x) = \mathscr{F}_{T,n}(x).$$

In particular, if $f \in L^1(\mathbb{T})$, then $\iota f(n,\cdot)$ is just the n-th partial sum of its Fourier series. It is easy to see that

$$\begin{aligned} \iota(aT) &= a(\iota T) \\ \iota(T_1 + T_2) &= \iota T_1 + \iota T_2 \\ \iota\left(\tfrac{d^k}{dx^k} T\right) &= \tfrac{d^k}{dx^k}(\iota T), \end{aligned}$$

cf. (8) above.

It is clear that $\iota T \in \mathscr{E}(\mathbb{T})$, but we want to prove that in fact $\iota T \in \mathscr{E}_M(\mathbb{T})$. If $f \in L^1(\mathbb{T})$, then its Fourier coefficients are bounded. Hence

$$|\tfrac{d^l}{dx^l} \iota f(n, x)| \leq c \sum_{k=1}^{n} k^l = \mathscr{O}(n^{l+1})$$

and we see that ιf is moderate.

For $T \in \mathscr{D}'(\mathbb{T})$ arbitrary, we need the Theorem 1: $T = a + \frac{d^k}{dx^k} T_F$, $F \in C(\mathbb{T})$. Hence by the above $\iota T = a + \frac{d^k}{dx^k} \iota F$, and $\iota F \in \mathscr{E}_M(\mathbb{T})$ as we already know.

The space $\mathscr{D}'(\mathbb{T})$ is embedded in $\mathscr{G}(\mathbb{T})$ by the means of the mapping

$$T \mapsto [\iota T]. \tag{12}$$

We remark that it is injective. Indeed, it is enough to check that $\iota T \in \mathscr{N}(\mathbb{T})$ implies $T = 0$. However, if $\iota T \in \mathscr{N}(\mathbb{T})$, then obviously $\iota T(n, \cdot) \to 0 \in \mathscr{D}'(\mathbb{T})$ as $n \to \infty$, and we conclude thanks to Lemma 2.

We have seen above that (12) is linear, and preserves the derivatives. However, it is not consistent with the product; more precisely,

$$[\iota f][\iota g] = [\iota(fg)] \tag{13}$$

is not true – provided that the functions (distributions) f, g are regular enough so that the product fg can be computed pointwise (1), or in the generalized sense (4). Even the continuity of f, g is enough: see example 5.4 below.

For $\varphi \in \mathscr{D}(\mathbb{T})$ there appears to exist a more natural canonical representative, namely

$$\iota'\varphi(n, x) = \varphi(x), \qquad \forall n \in \mathbb{N}.$$

An important achievement of the Colombeau theory is the following observation:

Lemma 3. *Let $\varphi \in \mathscr{D}(\mathbb{T})$. Then $\iota\varphi - \iota'\varphi \in \mathscr{N}(\mathbb{T})$.*

Proof. Given $k \geq 0$, one has

$$\tfrac{d^k}{dx^k}\left\{\imath\varphi(n,x) - \imath'\varphi(n,x)\right\} = \mathscr{F}_{\frac{d^k}{dx^k}\varphi,n}(x) - \tfrac{d^k}{dx^k}\varphi(x),$$

which is uniformly $\mathcal{O}(n^{-M})$ for any M by Lemma 1. □

As a consequence, $\mathscr{D}(\mathbb{T})$ can be regarded a subalgebra of $\mathscr{G}(\mathbb{T})$, in particular, we have the following.

Theorem 3. *Let $f, g \in \mathscr{D}(\mathbb{T})$. Then* (13) *holds.*

Proof. By the previous lemma, and an obvious identity $(\imath' f)(\imath' g) = \imath'(fg)$ one has

$$[\imath f][\imath g] = [\imath' f][\imath' g] = [(\imath' f)(\imath' g)] = [\imath'(fg)] = [\imath(fg)]. \qquad \square$$

The canonical embedding brings the elements of $\mathscr{D}'(\mathbb{T})$ in $\mathscr{G}(\mathbb{T})$. The concept of association plays the opposite rôle. We say that $[R] \in \mathscr{G}(\mathbb{T})$ is associated to a distribution $T \in \mathscr{D}'(\mathbb{T})$, if

$$\lim_{n\to\infty} \int_{\mathbb{T}} R(n,x)\varphi(x)\,dx = \langle T, \varphi\rangle, \qquad \forall \varphi \in \mathscr{D}(\mathbb{T}). \tag{14}$$

The definition is independent of the choice of the representative R. We write $[R] \approx T$. We say that $[R], [S] \in \mathscr{G}$ are associated, if $[R - S] \approx 0 \in \mathscr{D}'(\mathbb{T})$, and we write $[R] \approx [S]$. Basic properties of association are given in the following:

Theorem 4. *1. If $[R_i] \in \mathscr{G}(\mathbb{T})$, $T_i \in \mathscr{D}'(\mathbb{T})$ and $[R_i] \approx T_i$, $i = 1, 2$, then $[R_1 + R_2] \approx T_1 + T_2$, $a[R_1] \approx aT_1$.*

2. If $[R] \in \mathscr{G}(\mathbb{T})$, $T \in \mathscr{D}'(\mathbb{T})$ and $[R] \approx T$, then

$$\tfrac{d^k}{dx^k}[R] \approx \tfrac{d^k}{dx^k} T. \tag{15}$$

3. For every $T \in \mathscr{D}'(\mathbb{T})$ one has $[\imath T] \approx T$.

Proof. 1. Obvious.

2. By (14) we have

$$\begin{aligned}\int_{\mathbb{T}} \tfrac{d^k}{dx^k} R(n,x)\varphi(x)\,dx &= (-1)^k \int_{\mathbb{T}} R(n,x)\tfrac{d^k}{dx^k}\varphi(x)\,dx \\ &\to (-1)^k \langle T, \tfrac{d^k}{dx^k}\varphi\rangle \\ &= \langle \tfrac{d^k}{dx^k} T, \varphi\rangle.\end{aligned}$$

3. Follows immediately from Lemma 2. □

One can say that $\imath(\mathscr{D}'(\mathbb{T}))$ equipped with $\approx$ as an equivalence relation is a representation of $\mathscr{D}'(\mathbb{T})$ in $\mathscr{G}(\mathbb{T})$. However, $\approx$ is not compatible with nonlinearities, i.e. $[R_1] \approx [R_2]$ does not imply $[R_1][S] = [R_2][S]$. We also remark that not every $F \in \mathscr{G}(\mathbb{T})$ is associated to some distribution. See example 5.5 below.

5. Examples

The last section is devoted to computing several products in $\mathscr{G}(\mathbb{T})$. Our main focus is to emphasize the difference between the nonlinearities computed in $\mathscr{G}(\mathbb{T})$ and in the pointwise sense (1), (4). We consider various levels of regularity of f, g.

5.1 Sinus times Dirac

We have already shown above in (6) that $\delta_0 \cdot \sin = 0$ according to the definition (4). We claim that the result is not zero in $\mathscr{G}$; more precisely,

$$[\imath(\sin)][\imath\delta_0] \neq 0. \tag{16}$$

The corresponding canonical representatives read

$$\begin{aligned} R_1(n,x) &= \mathscr{F}_{\sin,n}(x) = \sin x, \qquad (\forall n \geq 1) \\ R_2(n,x) &= \mathscr{F}_{\delta_0,n}(x) = \frac{1}{\pi}\langle\delta_0(y), D_n(x-y)\rangle_y = \frac{1}{\pi}D_n(x) \\ &= \frac{\sin(n+\frac{1}{2})x}{2\pi\sin\frac{x}{2}}. \end{aligned}$$

Thus we have (using $\sin x = 2\sin x/2 \cos x/2$)

$$R_1(n,x)R_2(n,x) = \frac{1}{\pi}\cos x \sin\left(n+\tfrac{1}{2}\right)x.$$

This is not an element of $\mathscr{N}(\mathbb{T})$, and we conclude (16).

On the other hand, by Riemann-Lebesgue lemma

$$\lim_{n\to\infty}\int_{\mathbb{T}} \cos x \sin\left[\left(n+\tfrac{1}{2}\right)x\right]\varphi(x)\,dx = 0 \qquad \forall\varphi\in\mathscr{D}(\mathbb{T}).$$

Hence $[R_1R_2] \approx 0$; the equality in (16) holds in the sense of association.

5.2 Sinus times cosecant

The second example shows that neither (5), i.e. $\sin\cdot T_{\csc} = 1$, holds in $\mathscr{G}$. We claim that

$$[\imath\sin][\imath T_{\csc}] \neq [1]. \tag{17}$$

The Fourier coefficient of $T_{\csc}$ read $a_k = 0$, $b_{2k} = 0$ and

$$b_{2k+1} = \frac{1}{\pi}\int_0^{2\pi}\frac{\sin(2k+1)x}{\sin x}dx = \frac{2}{\pi}, \qquad \forall k \geq 0.$$

Hence the canonical representative $\imath T_{\csc}$ is

$$R_1(2n+1,x) = \frac{2}{\pi}\sum_{k=0}^{n}\sin(2k+1)x;$$

the product on the left-hand side of (5.2) is represented by

$$R(2n+1,x) = \frac{2}{\pi}\sin x \sum_{k=0}^{n} \sin(2k+1)x.$$

Note that $\sin(2k+1)\frac{\pi}{2} = (-1)^k$, hence $R(2n+1,\frac{\pi}{2}) \not\to 1$.

On the other hand, by Lemma 2, $R_1(2n+1,\cdot) \to T_{\csc}$, and $\sin \in \mathscr{D}(\mathbb{T})$. So it is not difficult to conclude that $[\imath \sin][\imath T_{\csc}] \approx 1$.

These two examples demonstrate how the Schwartz' impossibility argument is avoided in $\mathscr{G}$ – simply (5), (6) do not hold, and the contradiction does not arise.

5.3 Heaviside function

The products of discontinuous functions belong to standard counterexamples in the Colombeau theory. We set

$$h(x) := \begin{cases} 0, & x \in (-0,\pi) \\ 1, & x \in (0,\pi) \end{cases}$$

Thinking of h as an element of $L^\infty(\mathbb{T})$, the pointwise product gives $hh = h$. Yet this again is not true in $\mathscr{G}(\mathbb{T})$. We claim that

$$[\imath h][\imath h] \neq [\imath h]. \tag{18}$$

The canonical representative $\imath h$ is

$$R(n,x) = \mathscr{F}_{h,n}(x) = \frac{1}{2} + \sum_{k=1}^{n} \frac{1}{\pi k}(1-(-1)^k)\sin kx.$$

Thus $R(n,0)R(n,0) = \frac{1}{4} \neq \frac{1}{2} = R(n,0)$ for each n, and we conclude (18).

On the other hand, by part 1 of Theorem 2, $R(n,\cdot) \to h$ in $L^2(\mathbb{T})$. By Hölder's inequality, $R(n,\cdot)R(n,\cdot) \to hh = h$ in $L^1(\mathbb{T})$, and we obtain $[\imath h][\imath h] \approx [\imath h]$.

5.4 Continuous functions

The next example shows that (13) need not hold even for continuous functions. Set

$$f(x) := \begin{cases} 0, & x \in [-\pi,0) \\ \sin x, & x \in [0,\pi) \end{cases}$$

$$g(x) := \begin{cases} \sin x, & x \in [-\pi,0) \\ 0, & x \in [0,\pi) \end{cases}$$

Clearly $f,g \in C(\mathbb{T})$ and $fg = 0$. Evaluating the Fourier coefficients, one finds the canonical representatives $R = \imath f$, $S = \imath g$ respectively:

$$R(n,x) = \frac{1}{\pi} + \frac{1}{2}\sin x - \sum_{k=2}^{n} \frac{(-1)^k+1}{\pi(k^2-1)}\cos kx,$$

$$S(n,x) = \frac{1}{\pi} - \frac{1}{2}\sin x - \sum_{k\ 2}^{n} \frac{(-1)^k+1}{\pi(k^2-1)}\cos kx.$$

By Theorem 2, $R(n,x) \to f(x)$, $S(n,x) \to g(x)$. In particular

$$R(n,0) = S(n,0) = \frac{1}{\pi} - \sum_{k=2}^{n} \frac{(-1)^k + 1}{\pi(k^2 - 1)} \to 0.$$

It is easy to see that rate of convergence is $1/n$. Hence $RS = \mathcal{O}(n^{-2})$ at best, i.e. $RS \notin \mathcal{N}(\mathbb{T})$. We conclude that

$$[\iota f][\iota g] \neq 0. \tag{19}$$

On the other hand, as $R(n,x) \to f(x)$, $X(n,x) \to g(x)$ uniformly, the above equality is preserved at least in the sense of association.

5.5 Dirac times Dirac

As the last example, we multiply two singular distributions with disjoint supports. Not only that the result is not zero, it is even not associated to any distribution. — This however, is connected to the simplicity of our setting, in particular, weak localization properties of the Dirichlet kernel.

We want to compute

$$[\iota\delta_0][\iota\delta_\pi]. \tag{20}$$

The representatives $\iota\delta_0$, $\iota\delta_\pi$ are $\frac{1}{\pi}D_n(x)$, $\frac{1}{\pi}D_n(x,\pi)$ respectively. So the representative of (20) is

$$\begin{aligned} R(n,x) &= \frac{1}{\pi^2} D_n(x) D_n(x+\pi) \\ &= \frac{\sin\left(n+\frac{1}{2}\right)x \cdot \sin\left(n+\frac{1}{2}\right)(x+\pi)}{4\pi^2 \sin\frac{x}{2}\sin\left(\frac{x}{2}+\frac{\pi}{2}\right)}. \end{aligned}$$

Using the formulas $\sin(y + n\pi) = (-1)^n \sin y$ and $\sin\left(y + \frac{\pi}{2}\right) = \cos y$, we continue

$$\begin{aligned} &= \frac{(-1)^n}{4\pi^2} \cdot \frac{\sin\left(n+\frac{1}{2}\right)x \cdot \cos\left(n+\frac{1}{2}\right)x}{\sin\frac{x}{2}\cos\frac{x}{2}} \\ &= \frac{(-1)^n}{4\pi^2} \frac{\sin(2n+1)x}{\sin x} = \frac{(-1)^n}{2\pi^2} D_n(2x). \end{aligned}$$

Since $D_n(2x) \to \frac{1}{2}(\delta_0 + \delta_\pi)$ in $\mathcal{D}'(\mathbb{T})$, see the lemma below, we conclude that (20) is not associated to any distribution; in particular it is not zero.

Lemma 4. *For any $\varphi \in \mathcal{D}(\mathbb{T})$ one has*

$$\lim_{n\to\infty} \int_{\mathbb{T}} D_n(2x)\varphi(x)\,dx = \tfrac{\pi}{2}[\varphi(0) + \varphi(\pi)].$$

Proof. By a simple substitution and using the 2π-periodicity of D_n we have

$$\begin{aligned}\int_0^{2\pi} D_n(2x)\,\varphi(x)\,dx &= \tfrac{1}{2}\int_0^{4\pi} D_n(y)\,\varphi\left(\tfrac{y}{2}\right)dy\\ &= \tfrac{1}{2}\int_0^{2\pi} D_n(y)\,\varphi\left(\tfrac{y}{2}\right)dy + \tfrac{1}{2}\int_{2\pi}^{4\pi} D_n(y)\,\varphi\left(\tfrac{y}{2}\right)dy\\ &= \int_0^{2\pi} D_n(y)\,\psi(y)\,dy,\end{aligned}$$

where $\psi(y) = \frac{1}{2}\left[\varphi\left(\frac{y}{2}\right) + \varphi\left(\frac{y}{2} + \pi\right)\right]$. Note that $\psi \in \mathscr{D}(\mathbb{T})$; and since $D_n \to \pi\delta_0$ in $\mathscr{D}'(\mathbb{T})$, the proof is finished. □

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