Quasitrivial Semimodules I

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In the paper, quasitrivial (i.e., every element has just one scalar multiple) semimodules and semirings are investigated. In particular, minimal (i.e., every proper subsemimodule has just one element) and congruence-simple quasitrivial semimodules are characterized.

1. Preliminaries (A)

Let S be a semigroup (whose operation is denoted multiplicatively). Put $A = \{a \in S \mid |Sa| = 1\}$ and $B = \{a \in S \mid Sa = \{a\}\}$(obviously $B \subseteq A$).

1.1 Lemma. *Assume that $A \neq \emptyset$. Then:*

(i) *A is an ideal (and hence a subsemigroup) of the semigroup S.*

(ii) *There is an endomorphism f of the semigroup A such that $f^2 = f$ and $xa = f(a)$ for all $a \in A$ and $x \in S$.*

(iii) *$B = f(A)$, B is an ideal of S and B is just the set of right absorbing elements of the semigroup S.*

(iv) *$f = \mathrm{id}_A$ (or, equivalently, $A = B$), provided that f is an injective or projective transformation of A.*

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Proof. First, we have $a^2 = f(a)$ and $xf(a) = xa^2 = xa \cdot a = f(a)$ for all $a \in A$ and $x \in S$. Consequently, $f^2(a) = f(a) \in A$ and A is a left ideal of S. Moreover, $xay = f(a)y$ for all $a \in A$, $x, y \in S$, and we see that $ay \in A$. Thus A is an ideal of the semigroup S. The remaining assertions are easy. □

1.2 Lemma. (i) *$A = S$ if and only if the semigroup S satisfies the equation $xz = yz$.*

(ii) *$(A =)B = S$ if and only if S satisfies the equation $xy = y$ (i.e., S is a semigroup of right absorbing (left neutral, resp.) elements).*

(iii) *$A = S$ and $|B| = 1$ if and only if $|SS| = 1$ (i.e., S a semigroup with constant multiplication) or if and only if S satisfies $xy = zx$ (or $xy = uv$).*

Proof. It is easy. □

1.3 Lemma. *Assume that S is ideal-simple and $A \neq \emptyset$. Then either $B = S$* (see 1.2(ii)), *or $|S| = 2$ and $|SS| = 1$, or $A = B$ and $|A| = 1$.*

Proof. By 1.1(i), (iii), both sets A and B are ideals of S. If $|A| = 1$ then $A = B$. On the other hand, if $|A| \geq 2$ then $A = S$, since S is ideal-simple. Now, either $|B| = 1$ or $B = S$ and the rest is clear from 1.2. □

1.4 Lemma. *Assume that S is congruence-simple and $A \neq \emptyset$. Then either $|S| = 2$ and $B = S$, or $|S| = 2$* and *$|SS| = 1$, or $A = B$ and $|A| = 1$.*

Proof. S is ideal-simple and 1.3 applies. □

2. Preliminaries (B)

Let M be a left semimodule over a non-trivial semiring S.

2.1 Lemma. *The set ${}_S\langle x\rangle = \{mx, sx, mx + sx \mid m \geq 1,\ s \in S\}$ is the subsemimodule generated by the one-element set $\{x\}, x \in M$.*

Proof. It is easy. □

2.2 Lemma. *Let $x \in M$. Then:*

(i) *Sx is a subsemimodule of ${}_SM$.*

(ii) *$Sx \subseteq {}_S\langle x\rangle$.*

(iii) *$Sx = {}_S\langle x\rangle$ if and only if $x = sx$ for some $s \in S$.*

(iv) *The mapping $s \mapsto sx$ is a homomorphism of the semimodule ${}_SS$ into ${}_SM$.*

(v) *If ${}_SS$ is a congruence-simple semimodule then either $s \mapsto sx$ is injective (and then ${}_SS \simeq {}_SSx$) or $|Sx| = 1$.*

Proof. It is easy. □

2.3 Lemma. *If $S(\cdot)$ is a monoid with neutral element 1_S and ${}_SM$ is unitary then $Sx = {}_S\langle x\rangle$ for every $x \in M$.*

Proof. Use 2.2(iii). □

2.4 Lemma. *The following conditions are equivalent for $x \in M$:*

(i) *The one-element set $\{x\}$ is a subsemimodule of ${}_SM$.*

(ii) ${}_S\langle x\rangle = \{x\}$.

(iii) $Sx = \{x\} = \{2x\}$.

(iv) $Sx = \{x\}$.

Proof. If $Sx = x$ then $x + x = sx + sx = (s + s)x = x$. The rest is clear. □

Put $P({}_SM) = \{x \in M \mid Sx = \{x\}\}$(see 2.4) and $Q({}_SM) = \{x \in M \mid |Sx| = 1\}$.

2.5 Lemma. (i) $P(M) \subseteq Q(M)$.

(ii) *Either $P(M) = \emptyset$ or $P(M)$ is a subsemimodule of ${}_SM$.*

(iii) *Either $Q(M) = \emptyset$ or $Q(M)$ is a subsemimodule of ${}_SM$.*

Proof. Easy to check. □

2.6 Lemma. $P(M) = SQ(M)$.

Proof. If $r, s \in S$ and $x \in Q(M)$ then $r(sx) = (rs)x = sx$. □

The semimodule M is said to be *minimal* if it is non-trivial and every proper subsemimodule of ${}_SM$ contains just one element. If, moreover, ${}_SM$ has no one-element subsemimodules then M is called *strictly minimal.*

2.7 Lemma. *Let M be minimal. Then:*

(i) *$Sx = M$ for every $x \in M \setminus Q(M)$.*

(ii) *Either $Q(M) = M$, or $|Q(M)| = 1$, or $Q(M) = \emptyset$.*

(iii) *Either $Q(M) = M$ or ${}_SM$ is a homomorphic image of the semimodule ${}_SS$.*

Proof. (i) Since $x \notin Q(M)$, we have $|Sx| \geq 2$. But Sx is a subsemimodule and M is minimal. Consequently, $Sx = M$.

(ii) If $Q(M) \neq \emptyset$ then $Q(M)$ is a subsemimodule of M.

(iii) Combine (i) and 2.2(iv). □

2.8 Lemma. *Let M be strictly minimal. Then:*

(i) $Q(M) = \emptyset$.

(ii) *$Sx = M$ for every $x \in M$.*

(iii) *${}_SM$ is a homomorphic image of ${}_SS$.*

Proof. Follows immediately from 2.7. □

2.9 Lemma. *Assume that M is minimal and $Q(M) \neq \emptyset$. Then just one of the following three cases takes place:*

(1) *$(Q(M) =) P(M) = M$ (then $sx = x$ for all $s \in S$ and $x \in M$);*

(2) *$Q(M) = M$ and $|P(M)| = 1$ (then $|SM| = 1$);*

(3) *$Q(M) = P(M)$ and $|Q(M)| = 1$,*

Proof. It is easy (use 2.7). $\square$

2.10 Proposition. *The following conditions are equivalent:*

(i) $Q(M) = M$.

(ii) $rx = sx$ *for all* $r, s \in S$ *and* $x \in M$.

(iii) *There is a (uniquely determined) endomorphism* f_M *of* $M(+)$ *such that* $f_M^2 = f_M = 2f_M$ *and* $sx = f_M(x)$ *for all* $s \in S$ *and* $x \in M$.

If these conditions are satisfied then $P(M) = f_M(M)$ *and* f_M *is an endomorphism of* ${}_SM$.

Proof. Obviously, (i) is equivalent to (ii) and (iii) implies (ii).
(ii) implies (iii). Put $f_M = \{(x,sx) \mid s \in S,\ x \in M\}$. Then f_M is a correctly defined endomorphism of $M(+)$ and $f_M(sx) = f_M^2(x) = sf_M(x) = s(sx) = s^2x = f_M(x)$, $f_M(x) + f_M(x) = sx + sx = (s + s)x = f_M(x)$ for all $x \in M$, $s \in S$. $\square$

If the equivalent conditions of 2.10 are satisfied then the semimodule ${}_SM$ is called *quasitrivial.* If, moreover, $f_M = \mathrm{id}_M$ (i.e., $sx = x$ for all $s \in S$ and $x \in M$) then ${}_SM$ is called *id-quasistrivial.* If f_M is constant (i.e., $|SM| = 1$) then M is called *cs-quasitrivial.* The semiring S is called *left (id-, cs-)quasitrivial* if so is the (left) semimodule ${}_SS$.

2.11 Lemma. *The following conditions are equivalent:*

(i) *M is id-quasitrivial.*

(ii) $sx = x$ *for all* $s \in S$ *and* $x \in M$.

(iii) $P(M) = M$.

(iv) *M is quasitrivial and f_M is injective.*

(v) *M is quasitrivial and f_M is projective.*

Moreover, if these conditions are satisfied then:

(1) *$M(+)$ is idempotent (i.e., $M(+)$ is a semilattice).*

(2) *Every subsemigroup of $M(+)$ is a subsemimodule of ${}_SM$.*

(3) *Every congruence of $M(+)$ is a congruence of ${}_SM$.*

Proof. Obvious ((1) follows from 2.10(iii)). $\square$

2.12 Lemma. *The following conditions are equivalent:*

(i) *M is cs-quasitrivial.*

(ii) $sx = ry$ *for all* $r, s \in S$ *and* $x, y \in M$.

(iii) $Q(M) = M$ *and* $|P(M)| = 1$.

Moreover, if these conditions are satisfied then:

(1) *There is $w \in M$ such that $SM = \{w\}$ and $2w = w$.*

(2) *Every subsemigroup of $M(+)$ containing w is a subsemimodule of ${}_SM$.*

(3) *Every congruence of $M(+)$ is a congruence of ${}_SM$.*

Proof. It is easy. $\square$

2.13 Lemma. *Let M be a quasitrivial semimodule such that $|N| = 1$ whenever N is a cs-quasitrivial subsemimodule of M. Then M is id-quasitrivial.*

Proof. For every $a \in M$, $N_a = \{x \in M \mid f_M(x) = f_M(a)\}$ is a cs-quasitrivial subsemimodule of $M, a \in N_a$ and $f_M(a) \in N_a$. □

2.14 Lemma. *Let ϱ be a congruence of a minimal semimodule M, $\varrho \neq M \times M$. Then the factorsemimodule M/ϱ is minimal.*

Proof. Easy to see. □

2.15 Lemma. *Let ϱ be a congruence of the (left) semimodule ${}_SS$ such that ${}_SS/\varrho$ is quasitrivial. Then ϱ is a congruence of the semiring S.*

Proof. For all $r, s, t \in S, \pi(rt) = r\pi(t) = s\pi(t) = \pi(st)$, where π is the canonical projection of ${}_SS$ onto ${}_SS/\varrho$, and hence $(rt, st) \in \varrho$. □

2.16 Corollary. *Let S be a congruence-simple semiring. If ϱ is a congruence of the semimodule ${}_SS$ such that ${}_SS/\varrho$ is quasitrivial then either $\varrho = \mathrm{id}_S$ (and S is left quasitrivial) or $\varrho = S \times S$* (and *${}_SS/\varrho$ is trivial).* □

2.17 Corollary. *Let S be a congruence-simple semiring such that ${}_SS/\varrho$ is quasitrivial whenever ϱ is a congruence of ${}_SS$ and $\mathrm{id}_S \neq \varrho \neq S \times S$. Then the (left) semimodule ${}_SS$ is congruence-simple.* □

2.18 Corollary. *Let S be a congruence-simple semiring. If $f: {}_SS \to {}_SM$ is a homomorphism of semimodules such that $\mathrm{Im}(f)$ is quasitrivial (e.g., ${}_SM$ quasitrivial) then either $|\mathrm{Im}(f)| = 1$ or f is injective, ${}_SS \simeq {}_S\mathrm{Im}(f)$ and S is left quasitrivial.* □

2.19 Corollary. *Let S be a congruence-simple semiring and ${}_SM$ be a semimodule. If $a \in M$ is such that the subsemimodule Sa is quasitrivial then either $|Sa| = 1$ and $a \in Q(M)$ or ${}_SSa \simeq {}_SS$ and S is left quasitrivial.* □

2.20 Corollary. *Let S be a congruence-simple semiring that is not left quasitrivial. If ϱ is a maximal congruence of the (left) semimodule ${}_SS$ (such a congruence exists, provided that ${}_SS$ is finitely generated) then ${}_SS/\varrho$ is a non-quasitrivial congruence-simple semimodule.* □

3. A few examples

3.1 Let f be an endomorphism of a commutative semigroup $M(+)$ such that $f^2 = f = 2f$. Define an S-scalar multiplication on M by $sx = f(x)$ for all $s \in S$ and $x \in M$.

3.1.1 Lemma. *${}_SM = M(+, f, S)$ is a quasitrivial left S-semimodule.*

Proof. Easy to check. □

3.1.2 Lemma. (i) $P(M) = f(M)$ *and* $f = f_M$.
(ii) $\ker(f)$ *is a congruence of* ${}_SM$.
(iii) f *is an endomorphism of* ${}_SM$.
(iv) M *is id-quasitrivial iff* $f = \mathrm{id}_M$.
(iv) M *is cs-quasitrivial iff* f *is constant.*

Proof. Obvious. □

3.1.3 Lemma. *Let* $x \in M$ *be an arbitrary element. Then:*
(i) ${}_S\langle x\rangle = \{f(x), mx, mx + f(x) \mid m \geq 1\}$.
(ii) *If* $2x = x$ *then* ${}_S\langle x\rangle = \{f(x), x, x + f(x)\}$ *and* $|{}_S\langle x\rangle| \leq 3$.
(iii) *If* $f(x) + x = f(x)$ *then* ${}_S\langle x\rangle = \{f(x), mx \mid m \geq 1\}$.
(iv) $Sx = \{f(x)\}$.

Proof. Easy to check. □

3.2 Define an id-quasitrivial semimodule $Q_{1,S}$ in the following way: $Q_{1,S}(+) = \{0,1\}, 0 + 0 = 0 + 1 = 1 + 0 = 0, 1 + 1 = 1$, is the 2-element semilattice and $s0 = 0$, $s1 = 1$ for all $s \in S$.

3.2.1 Lemma. *The semimodule* $Q_{1,S}$ *is a homomorphic image of the semimodule* ${}_SS$ *if and only if there are two non-empty subsets* A, B *of* S *such that* $A \cup B = S$, $A \cap B = \emptyset$, $(S + A) \cup SA \subseteq A$ *and* $(B + B) \cup SB \subseteq B$.

Proof. Let $\varphi : {}_SS \to Q_{1,S}$ be a projective homomorphism of semimodules. It is enough to put $A = \varphi^{-1}(0)$ and $B = \varphi^{-1}(1)$. Conversely, setting $\varrho = (A \times A) \cup \cup (B \times B)$, we get a congruence of ${}_SS$ such that ${}_SS/\varrho \simeq Q_{1,S}$. □

3.3 Define a cs-quasitrivial semimodule $Q_{2,S}$ in the following way: $Q_{2,S}(+) = \{0,1\}, 0 + 0 = 0 + 1 = 1 + 0 = 0, 1 + 1 = 1$, is the 2-element semilattice and $s0 = s1 = 0$ for all $s \in S$.

3.3.1 Lemma. *The semimodule* $Q_{2,S}$ *is a homomorphic image of the semimodule* ${}_SS$ *if and only if there are two non-empty subsets* A, B *of* S *such that* $A \cup B = S$, $A \cap B = \emptyset$, $(S + A) \cup SS \subseteq A$ *and* $B + B \subseteq B$.

Proof. Similar to that of 3.2.1. □

3.4 Define a cs-quasitrivial semimodule $Q_{3,S}$ in the following way: $Q_{3,S}(+) = \{0,1\}, 0 + 0 = 0 + 1 = 1 + 0 = 0, 1 + 1 = 1$, is the 2-element semilattice and $s0 = s1 = 1$ for all $s \in S$.

3.4.1 Lemma. *The semimodule* $Q_{3,S}$ *is a homomorphic image of the semimodule* ${}_SS$ *if and only if there are two non-empty subsets* A, B *of* S *such that* $A \cup B = S$, $A \cap B = \emptyset$, $S + A \subseteq A$ *and* $(B + B) \cup SS \subseteq B$.

Proof. Similar to that of 3.2.1. □

3.5 Define a cs-quasitrivial semimodule $Q_{4,S}$ in the following way: $Q_{4,S}(+) = \{0,1\}, 0 + 0 = 0 + 1 = 1 + 0 = 1 + 1 = 0$, and $s0 = s1 = 0$ for all $s \in S$.

3.5.1 Lemma. *The semimodule $Q_{4,S}$ is a homomorphic image of the semimodule ${}_SS$ if and only if there are two non-empty subsets A, B of S such that $A \cup B = S$, $A \cap B = \emptyset$ and $(S + S) \cup SS \subseteq A$ (equivalently, $(S + S) \cup SS \neq S$).*

Proof. Similar to that of 3.2.1. □

3.6 Let $p \geq 2$ be a prime integer. Define a cs-quasitrivial semimodule $Z_{p,S}$ in the following way: $Z_{p,S}(+) = \mathbb{Z}_p(+)$ is the cyclic group of integers modulo p and $sx = 0$ for all $s \in S$.

4. Minimal and congruence-simple quasitrivial semimodules

4.1 Proposition. *The following conditions are equivalent for a quasitrivial semimodule ${}_SM$:*

(i) *M is minimal.*
(ii) *M is congruence-simple.*
(iii) *At least (and then just) one of the following five cases takes place:*
 (1) $M \simeq Q_{1,S}$;
 (2) $M \simeq Q_{2,S}$;
 (3) $M \simeq Q_{3,S}$;
 (4) $M \simeq Q_{4,S}$;
 (5) $M \simeq Z_{p,S}$ *for a prime integer $p \geq 2$.*

Proof. (i) implies (iii). Assume first that M is id-quasitrivial. It follows from 2.11(iv),(v) that $M(+)$ is a minimal semilattice. But then $|M| = 2$ and ${}_SM \simeq Q_{1,S}$.

Next, assume that M is not id-quasitrivial. Then $P(M) \neq M$ and, since M is minimal, we have $|P(M)| = 1$. Consequently, M is cs-quasitrivial by 2.12 and there is $w \in M$ with $SM = \{w\} = \{w + w\}$. Put $N = w + M$. Then N is a subsemimodule of M, and hence either $N = \{w\}$ or $N = M$.

Now, assume $N = \{w\}$, so that w is that w is the absorbing element of $M(+)$. If $v + v = w$ for some $v \in M \setminus \{w\}$ then $K = \{w,v\}$ is a two-element subsemimodule of M, and therefore $K = M$ and $M \simeq Q_{4,S}$. Consequently, assume that $x + x \neq w$ for every $x \in M \setminus \{w\}$. Now, take $u \in M \setminus \{w\}$ and put $L = \{mu \mid m \geq 1\}$. If $w \in L$ then there is $n \geq 3$ with $nu = w$ and $(n - 1)u \neq w$, and consequently $2(n - 1)u = w$, a contradiction. Thus $w \notin L$. The set L is a subsemigroup of $M(+)$. If L is infinite then $L(+)$ is a copy of the additive semigroup of positive integers, $L + L \neq L$ and $(L + L) \cup \{w\}$ is a proper non-trivial subsemimodule of M, a contradiction. It follows that L is a finite cyclic semigroup. Then it contains at least one idempotent element, say z. Now, $J = \{w,z\}$ is a two-element subsemimodule of $M, J = M$ and we see that ${}_SM \simeq Q_{2,S}$.

Assume, finally, that $N = M$. Since $2w = w$, it is easy to see that $w = 0$ is the neutral element of $M(+)$. Put $I = \{x \in M \mid 0 \in M + x\}$. Then $0 \in I$ and I is a subsemimodule of M. If $I = 0$ then $G = M \setminus \{0\}$ is a subsemigroup of $M(+)$. Proceeding similarly as above, we conclude that $G(+)$ contains an idempotent element z. Again, $\{0,z\}$ is a two-element subsemimodule of $M, M = \{0,z\}$. and ${}_SM \simeq Q_{3,S}$. On the other hand, if $I = M$ then $M(+)$ is a group, M is a module and every subgroup of $M(+)$ is a submodule of M. Since M is minimal, we get $M(+) \simeq \mathbb{Z}_p(+)$ and ${}_SM \simeq Z_{p,S}$.

(ii) implies (iii). By 2.10, we have ${}_SM = M(+, f_M, S)$, where $f_M(x) = sx$ for all $s \in S$ and $x \in M$, $\ker(f_M)$ is a congruence of ${}_SM$, and hence either $\ker(f_M) = \mathrm{id}_M$ or $\ker(f_M) = M \times M$.

First, assume that $\ker(f_M) = \mathrm{id}_M$. Then f_M is injective and 2.11 implies that M is id-quasitrivial. By 2.11(1),(3), $M(+)$ is a congruence-simple semilattice. Then $|M| = 2$ and ${}_SM \simeq Q_{1,S}$.

Next, assume that $\ker(f_M) = M \times M$. Then M is cs-quasitrivial and $M(+)$ is a congruence-simple (commutative) semigroup by 2.12(3). If $|M| = 2$ then M is minimal. If $|M| \geq 3$ then $M(+) \simeq \mathbb{Z}_p(+)$ for a prime integer $p \geq 2$ and M is minimal, too.

(iii) implies (i) and (ii). Easy. □

5. Quasitrivial semirings

5.1 Proposition. *Let S be a left quasitrivial semiring. Then:*

(i) *There is a (uniquely determined) endomorphism f of both the semiring S and the semimodule ${}_SS$ such that $f^2 = f = 2f$ and $rs = f(s)$ for all $r, s \in S$.*

(ii) $\ker(f)$ *is a congruence of both the semiring S and the semimodule ${}_SS$.*

Proof. We have $f(rs) = f^2(s) = f(s) = f(r)f(s)$ for all $r, s \in S$. The rest follows from 2.10. □

5.2 Let f be an endomorphism of a commutative semigroup $S(+)$ such that $f^2 = f = 2f$. Define a multiplication on S by $rs = f(s)$ for all $r, s \in S$.

5.2.1 Lemma. $S = S(+,\cdot)$ *is a left quasitrivial semiring.*

Proof. Easy to check. □

5.3 Define a left id-quasitrivial semiring $\mathbb{K}_1$ in the following way: $\mathbb{K}_1(+) = \{0,1\}, 0 + 0 = 0 + 1 = 1 + 0 = 0, 1 + 1 = 1, rs = s$ for all $r, s \in \mathbb{K}_1$.

5.4 Define a (left) cs-quasitrivial semiring $\mathbb{K}_2$ in the following way: $\mathbb{K}_2(+) = \{0,1\}, 0 + 0 = 0 + 1 = 1 + 0 = 0, 1 + 1 = 1, rs = 0$ for all $r, s \in \mathbb{K}_2$.

5.5 Define a (left) cs-quasitrivial semiring $\mathbb{K}_3$ in the following way: $\mathbb{K}_3(+) = \{0,1\}, 0 + 0 = 0 + 1 = 1 + 0 = 0, 1 + 1 = 1, rs = 1$ for all, $r, s \in \mathbb{K}_3$.

5.6 Define a (left) cs-quasitrivial semiring $\mathbb{K}_4$ in the following way: $\mathbb{K}_4(+) = \{0,1\}, 0 + 0 = 0 + 1 = 1 + 0 = 1 + 1 = 0$, $rs = 0$ for all $r, s \in \mathbb{K}_4$.

5.7 Proposition. *The two-element semirings $\mathbb{K}_1$, $\mathbb{K}_2$, $\mathbb{K}_3$, $\mathbb{K}_4$ and zero multiplication rings of finite prime order are (up to isomorphism) the only congruence-simple left quasitrivial semirings. These semirings are pair-wise non-isomorphic.*

Proof. Let S be a congruence-simple left quasitrivial semiring. By 5.1(ii), $\ker(f)$ is a congruence of the semiring, and so either $\ker(f) = \mathrm{id}_S$ or $\ker(f) = S \times S$. In the first case, S is id-quasitrivial, so that S is additively idempotent and $rs = s$ for all $r, s \in S$ and in the latter case, there is $w \in S$ with $SS = w$. Now, every congruence of $S(+)$ is a congruence of S and the rest is clear. □

5.8 Proposition. *Let S be a left quasitrivial semiring. Then:*
(i) *$SM \subseteq Q(M)$ for every semimodule ${}_SM$.*
(ii) *Every minimal semimodule ${}_SM$ is quasitrivial.*

Proof. If $s \in S$ and $x \in M$ then $|Ss| = 1$, hence $|Ssx| = 1$ and $sx \in Q(M)$. Consequently, $SM \subseteq Q(M)$. Moreover, if $|SM| = 1$ then M is apparently quasitrivial. On the other hand, if M is minimal and $|SM| \geq 2$ then $SM = S$, hence $Q(M) = M$ and M is quaitrivial, too. □

5.9 Proposition. *A congruence-simple semiring S is finite, provided that there exists at least one non-quasitrivial finite (left) S-semimodule.*

Proof. Let ${}_SM$ be finite and not quasitrivial. The endomorphism semiring E of the additive semigroup $M(+)$ is finite and the mapping $\varphi : S \to E$, defined by $\varphi(s)(x) = sx$ for all $s \in S$, $x \in M$, is a homomorphism of semirings. If $\ker(\varphi) = S \times S$ then ${}_SM$ is quasitrivial, a contradiction. On the other hand, if $\ker(\varphi) \neq S \times S$ then $\ker(\varphi) = \mathrm{id}_S$ (since S is congruence-simple) and S is finite. □

5.10 Proposition. *Let S be a finite semiring. Then every minimal S-semimodule M is finite.*

Proof. If M is quasitrivial then M is finite by 4.1. If M is not quasitrivial then $M = Sa$ for some $a \in M$ and M is finite, too. □

6. Further results

In this section, we are going to generalize some results from [1].

6.1 Lemma. *Let ${}_SM$ be a semimodule such that $SM = M$. If ϱ is a congruence of ${}_SM$ such that M/ϱ is cs-quasitrivial then $\varrho = M \times M$.*

Proof. Since M/ϱ is cs-quasitrivial, we have $|SM/\varrho| = 1$, and so $M = SM$ is contained in a block of ϱ. Then $\varrho = M \times M$. $\square$

6.2 Lemma. (cf. [1, 3.12(i)]) *Let $_SM$ be a minimal semimodule and let ϱ be a congruence of $_SM$ such that $_SM/\varrho$ is quasitrivial. Then either $\varrho = \mathrm{id}_M$ (and $_SM$ is quasitrivial) or $\varrho = M \times M$ (and M/ϱ is trivial).*

Proof. Put $N = M/\varrho$. According to 2.14, N is either trivial or minimal. If N is trivial then $\varrho = M \times M$, and hence we assume that $\varrho \neq M \times M$. Then N is minimal and quasitrivial. Furthermore, if M is quasitrivial then $\varrho = \mathrm{id}_M$ by 4.1. On the other hand, if M is not quasitrivial then $|S_x| \geq 2$ for some $x \in M$, and hence $SM = M$, since M is minimal. Then also $SN = N$ and it follows from 4.1 that $N \simeq Q_{1,S}$ is id-quasitrivial. Thus both blocks A, B of ϱ are subsemimodules. Since M is minimal, $|A| = 1 = |B|$, and hence $\varrho = id_M$. $\square$

6.3 Lemma. *Let $_SM$ be a minimal semimodule that is not quasitrivial. Then there is at least one congruence ϱ of M such that the factor-semimodule $_SM/\varrho$ is minimal, congruence-simple and not quasitrivial.*

Proof. Since M is minimal and not quasitrivial, we have $M = {}_s\langle x \rangle$ for any $x \in M \setminus Q(M)$. Thus $_SM$ has at least one (proper) maximal congruence ϱ. Now, $_SN = {}_SM/\varrho$ is congruence-simple and it is minimal by 2.14. According to 6.2, N is not quasitrivial. $\square$

6.4 Lemma. (cf. [1, 3.12(ii)]) *Let $_SM$ be a congruence-simple semimodule containing a non-trivial cs-quasitrivial subsemimodule N. Then $N = M$ and M is cs-quasitrivial and minimal.*

Proof. There is $w \in N$ with $SN = \{w\}$. Define a relation ϱ on M by $(x, y) \in \varrho$ iff $(x + N) \cap (y + N) \neq \emptyset$. Since N is a subsemimodule of M, it follows easily that ϱ is a congruence of $_SM$. Moreover, if $a, b \in N$ then $a + b \in (a + N) \cap \cap (b + N)$, and so N is contained in a block of ϱ. Consequently, $\varrho \neq \mathrm{id}_M$ and $\varrho = M \times M$, since M is congruence-simple. Now, $(x, w) \in \varrho$ for every $x \in M$ and we get $x + a = w + b$ for some $a, b \in N$. Then $sx + w = sx + sa = = sw + sb = w + w = w$ for every $s \in S$, and so $SM + w = w$. Put $A = \{a + M \mid Sa = w\}$. Clearly, A is a subsemimodule of $M, Sa = w$ and $N \subseteq A$. If $x \in M$ and $a \in A$ then $S(x + a) = Sx + w = w$, $x + a \in A$ and $M + A \subseteq A$. From this, it follows easily that $\eta = (A \times A) \cup \mathrm{id}_M$ is a congruence of $_SM$. Then $\eta = M \times M$, $A = M$ and M is cs-quasitrivial. By 4.1, M is minimal, and therefore $N = M$. $\square$

6.5 Lemma. *Let $_SM$ be a congruence-simple semimodule containing a non-trivial id-quasitrivial subsemimodule N. Then $N = P(M) \simeq Q_{1,S}$.*

Proof. For every $a \in P(M)$, put $I_a = a + M$. Then $(I_a + M) \cup SI_a \subseteq I_a$, and so $\tau_a = (I_a \times I_a) \cup \mathrm{id}_M$ is a congruence of $_SM$. If $\tau_a = \mathrm{id}_M$ then $I_a = \{a\}$ and a is the absorbing element of $M(+)$. If $\tau_a = M \times M$ then $I_a = M$ and a is the neutral element of $M(+)$. We have shown that every element from $P(M)$ is either absorbing or neutral in $P(M)$. Consequently, $|P(M)| = 2$ and $N = P(M) \simeq Q_{1,S}$. $\square$

6.6 Proposition. *Let ${}_SM$ e a congruence-simple semimodule such that M contains a non-trivial quasitrivial subsemimodule, but M is not quasitrivial. Then $P(M) = Q(M) \simeq Q_{1,S}$.*

Proof. According to the assumption, we have $|Q(M)| \geq 2$. Combining 6.4 and 2.13, we conclude that $Q(M)$ is id-quasitrivial and the rest follows from 6.5. $\square$

6.7 Remark. Let ${}_SM$ be a congruence-simple semimodule as in 6.6. Then $P(M) = Q(M) = \{0,o\}$, where 0 is additively neutral, $S0 = 0$, o is additively absorbing and $So = o$ (see the proof of 6.5).

(i) First, we show that $M(+)$ is idempotent.
Indeed, define a relation ϱ on M by $(x,y) \in \varrho$ iff $2^i x = y + u$ and $2^i y = x + v$ for a non-negative integer i and some $u, v \in M$. One checks easily that ϱ is a congruence of ${}_SM$, $(z, 2z) \in \varrho$ for every $z \in M$ and $(0, o) \notin \varrho$. Then $\varrho = \mathrm{id}_M$ and $M(+)$ is a semilattice.

(ii) Next, we show that $G + M \subseteq G$, where $G = M \setminus \{0\}$.
Indeed, the set $\{a \in M \mid 0 \in a + M\}$ is the group of invertible elements of $M(+)$ and, since $M(+)$ is a semilattice, the group equals 0.

(iii) Put $J = \{a \in M \mid 0 \notin Sa\}$. Then $o \in J$, $0 \notin J$ and $(J + M) \cup SJ \subseteq J$. Consequently, $(J \times J) \cup \mathrm{id}_M$ is a congruence of M, and therefore $|J| = 1$ and $J = \{o\}$. Thus $0 \in Sa$ for every $a \in H = M \setminus \{o\}$.

(iv) We have $Q(M) = \{0,o\}$. Henceforth, if $a \in K = M \setminus \{0,o\}$ then $|Sa| \geq 2$. In particular, $Sa \neq \{0\}$ and $Sa \neq \{o\}$.

(v) Assume that $o \notin Sa$ for some $a \in G$. By (iii) and (iv), $0 \in Sa$ and $|Sa| \geq 2$. Moreover, $Q(Sa) = \{0\}$ and Sa is not quasitrivial. Finally, if M is finite and $|Sa|$ minimal then Sa is a minimal semimodule.

(vi) Let $a \in M$ be such that $Sa = \{0,o\}$. Then $a \in K$ and $S = A \cup B$, where $A = \{s \in S \mid sa = o\}$, $B = \{r \in S \mid ra = 0\}$, $A \neq \emptyset \neq B$, $A \cap B = \emptyset$, $(A + S) \cup SA \subseteq A$ and $(B + B) \cup SB \subseteq B$. But then $\xi = (A \times A) \cup$ $\cup (B \times B)$ is a congruence of the semiring S and $S/\xi \simeq \mathbb{K}_1$ (see 5.3). In particular, if S is congruence-simple then $|A| = 1 = |B|, |S| = 2$ and $S \simeq \mathbb{K}_1$.

(vii) Assume that $\mathbb{K}_1$ is not a homomorphic image of S. Then $Sa \neq \{0,o\}$ for every $a \in M$. By (iii) and (iv), $0 \in Sb$ and $|Sb| \geq 2$ for every $b \in K$. Moreover, if $o \in Sb$ then $|Sb| \geq 3$.

(viii) Assume that M is finite and $o \in Sa$ for every $a \in K$. If $K = \{a_1, \ldots, a_m\}$, $m \geq 1$, and $t_i a_i = o$, $i = 1, \ldots, m$, then $tG = \{o\}$, where $t = t_1 + \ldots t_m$. Moreover, $tM = \{0,o\}$.

(ix) Assume that $o \in Sa$ for every $a \in K$ and put $L = \{b \in K \mid Sb \neq \{0,o\}\}$. In view of (iii), $\{0,o\} \subseteq Sb$ and $|Sb| \geq 3$ for every $b \in L$. Moreover, by (vii), if $\mathbb{K}_1$ is not a homomorphic image of the semiring S then $L = K$.

(x) Let N be a subsemimodule of ${}_SM$ such that $\{0,o\} \subseteq N$.
First, let λ be a congruence of ${}_SN$ such that $\lambda \neq N \times N$ and $\bar{N} = {}_SN/\lambda$ is quasitrivial. Then $(0,o) \notin \lambda$ and, for every $a \in N \cap K$, the subsemimodule Sa is contained in a block of λ. According to (iii), $0 \in Sa$, and so Sa and 0 are contained in the same block of λ. Thus $|\bar{N}| = 2$ and $\bar{N} \simeq Q_{1,S}$.
Let σ be a congruence of ${}_SN$ maximal with respect to $(0,o) \notin \sigma$. Then σ is a maximal congruence and ${}_SN/\sigma$ is congruence-simple.

(xi) Assume that M is finite with minimal $|M|$ and that $o \in Sa$ for every $a \in K$. If $L = \emptyset$ then $Sa = \{0,o\}$ and $SM = \{0,o\}$. Now, assume that $L \neq \emptyset$ and take $b \in L$ with minimal $|Sb|$. Consider a congruence σ of Sb maximal with respect to $(0,o) \notin \sigma$. Then ${}_ST = {}_SSb/\sigma$ is a congruence-simple semimodule. If ${}_ST$ is quasitrivial then ${}_ST \simeq Q_{1,S}$, and so $\mathbb{K}_1$ is a homomorphic image of S. On the other hand, if ${}_ST$ is not quasitrivial then $|T| = |M|$ due to the minimality of $|M|$ (and the fact that $|Q({}_ST)| \geq 2$). That is, $\sigma = \mathrm{id}_{Sb}$ and $Sb = M$. Consequently, $Sc = M$ for every $c \in L$.

(xii) If $\mathbb{K}_1$ is not a homomorphic image of S, M is finite with minimal $|M|$ and $o \in Sa$ for every $a \in K$ then $Sa = M$.

6.8 Remark. Let ${}_SM$ be a non-quasitrivial finite semimodule with minimal $|M|$ (of course, such a semimodule exists iff there is at least one finite non-quasitrivial semimodule).

(i) Clearly, M is a minimal semimodule if and only if $|Q(M)| \leq 1$.

(ii) We have $Q(M) \neq M$ and every proper subsemimodule of ${}_SM$ is contained in $Q(M)$. Consequently, $M = {}_S\langle x\rangle$ for every $x \in M \setminus Q(M)$.

(iii) If $\varrho \neq \mathrm{id}_M$ is a congruence of ${}_SM$ then ${}_SM/\varrho$ is quasitrivial.

(iv) Assume that S is a congruence-simple semiring that is not left quasitrivial. If $a \in M$ is such that $Sa \subseteq Q(M)$ then Sa is quasitrivial and it follows from 2.16 that $a \in Q(M)$. In particular, $Sb = M$ for every $b \in M \setminus Q(M)$ (use (ii)). Using this and (iii), we conclude easily that ${}_SM$ is congruence-simple semimodule. Consequently (see 6.6), if $|Q(M)| \geq 2$ then $P(M) = Q(M) \simeq Q_{1,S}$.

7. A few consequences

7.1 Proposition. *Let S be a finite congruence-simple semiring that is not left quasitrivial. Consider a maximal congruence ϱ of the left semimodule ${}_SS$ and put ${}_SM = {}_SS/\varrho$. Then:*

(i) *${}_SM$ is a finite congruence-simple semimodule that is not quasitrivial.*

(ii) *$s/\varrho \in Q({}_SM)$ if and only if Ss is contained in a block of ϱ.*

(iii) $|tM| = 1$ *for every* $t \in Q(S_S)$.
(iv) *If* $Q(S_S) \neq \emptyset$ *then* $|P({}_SM)| \leq 1$.

Proof. Easy (use 2.20). $\square$

7.2 Proposition. *Assume that there exists at least one non-trivial finite semimodule* ${}_SM$ *with* $|Q({}_SM)| \leq 1$. *Then there exists at least one non-quasitrivial finite minimal semimodule.*

Proof. Take $a \in M \setminus Q(M)$ with minimal $|Sa|$. Then ${}_SSa$ is our semimodule. $\square$

7.3 Proposition. *Assume that there exists a non-quasitrivial finite congruence-simple semimodule* (cf. 6.8(iv)). *Then at least one of the following two cases takes place:*

(1) *There exists a non-quasitrivial finite minimal semimodule.*
(2) *There exists a non-quasitrivial finite congruence-simple additively idempotent semimodule* ${}_SM$ *such that* $Q({}_SM) = \{0,o\}$ *and:*
  (2a) o *is the absorbing element of* $M(+)$ *and* $So = o$;
  (2b) 0 *is the neutral element of* $M(+)$ *and* $S0 = 0$;
  (2c) *for every* $a \in M \setminus \{0,o\}$, *either* $Sa = \{0,o\}$ *or* $Sa = M$;
  (2d) *if* $\mathbb{K}_1$ *is not a homomorphism image of the semiring* S *then* $S_a = M$ *for every* $a \in M \setminus \{0,o\}$.

Proof. Suppose that every finite minimal semimodule is quasitrivial and let ${}_SM$ be a non-quasitrivial finite congruence-simple semimodule. According to our assumption and 7.2, we have $|Q({}_SM)| \geq 2$. Now, by 6.6 and 6.7, $P({}_SM) = Q({}_SM) = \{0,o\}$, where o is absorbing and 0 neutral in $M(+)$, and so $So = o$, $S0 = 0$. By 6.7(i), $M(+)$ is idempotent. By 6.7(iii),(v), we have $\{0,o\} \subseteq Sa$ for every $a \in M \setminus \{0,o\}$. Furthermore, we can assume that $|M|$ is minimal. Then the rest follows from 6.7(xi),(xii). $\square$

7.4 Proposition. *Let* S *be a semiring containing at least one left multiplicatively absorbing element (equivalently,* $Q(S_S) \neq \emptyset$*) and such that there exists a non-quasitrivial finite congruence-simple semimodule. Then there exists a non-quasitrivial finite minimal semimodule.*

Proof. Let, on the contrary, ${}_SM$ be as in 7.3(2). Take $a \in M \setminus \{0,o\}$ and consider a congruence ϱ of ${}_SS$ with $Sa \simeq {}_SS/\varrho$ (see 2.2). Let $t \in S$ be left multiplicatively absorbing. For every $s \in S$ we have $ts = t$, and hence $t \cdot s/\varrho = ts/\varrho = t/\varrho$. It follows that $|tSa| = 1$, however $to = o \neq 0 = t0$, a contradiction. $\square$

7.5 Proposition. *Let* S *be a finite congruence-simple semiring that is not left quasitrivial and contains at least one left multiplicatively absorbing element. Then there exists at least one non-quasitrivial finite minimal semimodule.*

Proof. Combine 7.1(i) and 7.4. $\square$

7.6 Remark. Consider the situation from 7.5. Let $t \in S$ be left multiplicatively absorbing and let ${}_SM$ be a finite minimal semimodule that is not quasitrivial. Then $|Sa| \geq 2$ for some $a \in M$ and ${}_SM \simeq {}_SS/\varrho$ for a congruence ϱ of ${}_SS$ (see 2.2 and 2.7). Consequently, $|tM| = 1$ and there is $w \in M$ with $tM = \{w\}$.We have $tw = w$ and $w + w = tw + tw = t(w + w) = w$.

(i) If t is additively neutral (absorbing, resp.) in S then, by 2.7 (i), w is neutral (absorbing, resp.) in $M(+)$.

(ii) If t is multiplicatively absorbing in S then $Sw = w$ and $w \in P({}_SM)$.

7.7 Corollary. ([1, 3.10]) *Let S be a finite congruence-simple semiring such that S contains an element 0_S which is both additively neutral and multiplicatively absorbing. If $rs \neq 0_S$ for some $r, s \in S$ (i.e., S is not left quasitrivial) then there exists a finite minimal congruence-simple semimodule ${}_SM$ such that ${}_SM$ is not quasitrivial and contains an additively neutral element 0_M with $S0_M = 0_M$ and $0_SM = 0_M$.* □

Reference

[1] Zumbrägel, J., *Classification of finite congruence-simple semirings with zero* (preprint).