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Quasitrivial Semimodules II

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The paper continues the investigation of semimodules. Main emphasis is laid on minimal (i.e., every proper subsemimodule has just one element) and congruence-simple semimodules.

1. Auxiliary results (A)

This paper is a continuation of [1] and we use the same notation. Let ${}_SM$ be a (left) semimodule. For arbitrary elements $x, w \in M$, put $((x : w)_{S,M} =)(x : w) = \{s \in S \mid sx = w\}$. Similarly, if $x \in M$ and $A \subseteq M$, put $(x : A)_l = \{s \in S \mid sx \in A\}$.

1.1 Lemma. (i) $(x : w_1) + (x : w_2) \subseteq (x : w_1 + w_2)$.
(ii) *If* $w_1 \neq w_2$ *then* $(x : w_1) \cap (x : w_2) = \emptyset$.

Proof. Obvious. □

1.2 Lemma. *Assume that* $2w = w$. *Then:*
(i) $(x : w) + (x : w) \subseteq (x : w)$.
(ii) $(x : w) \cap (y : w) \subseteq (x + y : w)$.

Proof. Easy to check. □

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1.3 Lemma. *Assume that* $2w = w$ *and* $(M\setminus\{w\}) + M \subseteq M\setminus\{w\}$. *Then:*

(i) $(x:w) \cap (y:w) = (x+y:w)$.

(ii) $(x:w) = (2x:w)$.

(iii) *If* $x + y = x$ *then* $(x:w) \subseteq (y:w)$.

Proof. Easy to check. □

1.4 Lemma. *If* w *is the absorbing element of* $M(+)$ *then* $S + (x:w) \subseteq (x:w)$.

Proof. Easy to check. □

1.5 Lemma. (i) $(sx:w) = \{r \in S \mid rs \in (x:w)\} = (s:(x:w))_l$ *for every* $s \in S$.

(ii) *If* $sw = w$ *then* $s(x:w) \subseteq (x:w)$.

(iii) *If* $Sw = w$ *(i.e.,* $w \in P({}_SM)$*) then* $2w = w$ *and* $((x:w) + (x:w)) \cup \cup S(x:w) \subseteq (x:w)$.

Proof. Easy to check. □

1.6 Lemma. *If* $Sw = w$ *and* $(x:w) \neq \emptyset$ *then* $(x:w)$ *is a subsemimodule of* ${}_SS$ *(i.e., a left ideal of the semiring* S*).*

Proof. See 1.5 (iii). □

1.7 Lemma. *Assume that* $Sw = w$ *(i.e.,* $w \in P({}_SM)$*) and* $(x:w) \neq \emptyset$. *Then:*

(i) $(x:w)y$ *is a subsemimodule of* ${}_SM$ *for every* $y \in M$.

(ii) *If* $|(x:w)y| = 1$ *then* $(x:w)y \subseteq Q({}_SM)$.

(iii) *If* $|(x:w)y| = 1$ *and* $|Q({}_SM)| = 1$ *(e.g., if* ${}_SM$ *is minimal and not quasitrivial) then* $(x:w) \subseteq (y:w)$.

(iv) *If* ${}_SM$ *is minimal, not quasitrivial, and if* $(x:w) \nsubseteq (y:w)$ *then* $(x:w)y = M$.

Proof. (i) By 1.6, $(x:w)$ is a left ideal. Then, of course, $N = (x:w)y$ is a subsemimodule of ${}_SM$.

(ii) $Ssy \subseteq S(x:w)y \subseteq (x:w)y$, and hence $|Ssy| = 1$ for evey $s \in (x:w)$. Then $sy \in Q({}_SM)$.

(iii) Since $|Q({}_SM)| = 1$, we have $Q({}_SM) = \{w\}$. By (ii), $(x:w)y = w$, so that $(x:w) \subseteq (y:w)$.

(iv) It follows from (iii) that $|(x:w)y| \geq 2$. Thus $(x:w)y$ is a non-trivial subsemimodule of ${}_SM$ and $(x:w)y = M$, since M is minimal. □

1.8 Lemma. *Assume that* ${}_SM$ *is minimal, not quasitrivial, and that* $Q({}_SM) \neq \emptyset$. *Then:*

(i) $P({}_SM) = Q({}_SM) = \{w\}$ *for some* $w \in M$.

(ii) *If* $x, y \in M$ *and* $s \in S$ *are such that* $sx = w \neq sy$ *then for every* $z \in M$ *there exists* $r \in S$ *with* $rx = w$ *and* $ry = z$.

Proof. (i) We have $|Q({}_SM)| = 1$ and the rest is clear.

(ii) We have $(x:w)y \nsubseteq (y:w)$ and 1.7(iv) applies. □

2. Auxiliary results (B)

Let ${}_SM$ be a semimodule. For every $w \in M$, define a relation η_w on M by $(x, y) \in \eta_w$ iff $(x : w) = (y : w)$.

2.1 Lemma. (i) *η_w is an equivalence.*
(ii) *If $(x, y) \in \eta_w$ then $(sx, sy) \in \eta_w$ for every $s \in S$.*

Proof. (i) This follows immediately from the definition of η_w.
(ii) By 1.5(i), $(sx : w) = (sy : w)$. □

2.2 Lemma. *Assume that $2w = w$ and $(M \setminus \{w\}) + M \subseteq M \setminus \{w\}$. Then η_w* is a congruence of the semimodule ${}_SM$.

Proof. By 2.1, $\eta = \eta_w$ is an equivalence that is stable with respect to the scalar S-multiplication. By 1.3(i), $(x, y) \in \eta$ implies $(x + z : w) = (x : w) \cap (z : w) = $ $= (y : w) \cap (z : w) = (y + z : w)$ and $(y + z, y + z) \in \eta$ for every $z \in M$. Thus η is a congruence. □

2.3 Lemma. *The following conditions are equivalent:*
(i) $\eta_w = \mathrm{id}_M$.
(ii) *For all $x, y \in M$, $x \neq y$, there exists at least one $s \in S$ such that either $sx \neq sy = w$ or $sy \neq sx = w$.*

Proof. The assertion follows easily from the definition of η_w. □

2.4 Lemma. *The following condition are equivalent:*
(i) $\eta_w = M \times M$.
(ii) *There is a subset I of S such that $Ix = w \notin (S \setminus I)x$ for every $x \in M$.*

Proof. The assertion follows easily from the definition of η_w. □

2.5 Lemma. *Assume that $\eta_w = M \times M$ and put $I = (w : w)$. Then:*
(i) *$(x : w) = I$ for every $x \in M$.*
(ii) *$I = (s : I)_l$ for every $s \in S$.*
(iii) *$rS \subseteq I$ for every $r \in I$.*
(iv) *$tS \subseteq S \setminus I$ for every $t \in S \setminus I$.*
(v) *If $2w = w$ then $I + I \subseteq I$.*
(vi) *If w* is the absorbing element of $M(+)$ then $S + I \subseteq I$.
(vii) *If $Sw = w$ then $I = S$ and $SM = \{w\}$ (i.e., ${}_SM$ is cs-quasitrivial).*

Proof. (i) This follows immediately from the definition of η_w.
(ii) By (i) and 1.5 (i), $I = (sw : w) = (s : (w : w))_l = (s : I)_l$.
(iii) and (iv). Use (ii).
(v) See 1.2 (i).
(vi) See 1.4.
(vii) Obvious. □

2.6 Lemma. *Assume that $_SM$ is congruence-simple and not cs-quasitrivial. If $w \in M$ is such that $Sw = w$ and $(M \setminus \{w\}) + M \subseteq M \setminus \{w\}$ then $\eta_w = \mathrm{id}_M$.*

Proof. By 2.2, η_w is a congruence of $_SM$ and it follows from 2.5(vii) that $\eta_w \neq M \times M$. Since $_SM$ is congruence-simple, we conclude that $\eta_w = \mathrm{id}_M$. □

2.7 Lemma. Let $w \in M$ *be such that* $2w = w$, $(M \setminus \{w\}) + M \subseteq M \setminus \{w\}$ *and* $\eta_w = \mathrm{id}_M$. *Then:*

(i) *If* $x, y \in M$ *then* $x + y = x$ *if and only if* $(x : w) \subseteq (y : w)$.

(ii) *If* $Sw = w$, $x + y \neq x$ *and* $(x : w) \neq \emptyset$ *then* $(x : w)y$ *is a subsemimodule of* $_SM$ *and* $(x : w)y \neq \{w\}$. *If, moreover,* $(x : w)y \not\subseteq Q(_SM)$ *then* $|(x : w)y| \geq 2$.

Proof. (i) If $x + y = x$ then, with respect to 1.3(iii), $(x : w) \subseteq (y : w)$. Conversely, if $(x : w) \subseteq (y : w)$ then $(x : w) = (x : w) \cap (y : w) = (x + y : w)$ by 1.3(i), hence $(x + y, x) \in \eta_w = \mathrm{id}_M$ and $x + y = x$.

(ii) By 1.7(i), $(x : w)y$ is a subsemimodule of $_SM$. Since $(x : w) \not\subseteq (y : w)$ by (i), we get $(x : w)y \neq \{w\}$. Finally, if $(x : w)y \not\subseteq Q(_SM)$ then $|(x : w)y| \geq 2$ by 1.7(ii). □

2.8 Lemma. *Assume that* $2w = w$ *and* $(M \setminus \{w\}) + M \subseteq M \setminus \{w\}$. *Then:*

(i) $(2x, x) \in \eta_w$ *for every* $x \in M$.

(ii) *If* $\eta_w = \mathrm{id}_M$ *then* $M(+)$ *is idempotent.*

Proof. See 1.3(ii). □

3. Minimal semimodules (A)

Throughout this section, let $_SM$ be a minimal semimodule that is not quasitrivial (cf. [1, 4.1]) and such that $Q(_SM) \neq \emptyset$ (i.e., $_SM$ is not strictly minimal).

3.1 Lemma. $Q(_SM) = P(_SM) = \{w\}$ *for some* $w \in M$ *and* $Sw = w$.

Proof. $Q(_SM)$ is a proper subsemimodule of $_SM$ and the rest is clear. □

3.2 Lemma. *If* $w = M + x$ *for at least one* $x \in M$, $x \neq w$, *then* $w \in M + y$ *for every* $y \in M$.

Proof. We have $w = x + u$ for some $u \in M$. Since $x \neq w$, we also have $Sx = M$, and hence $y = sx$ for some $s \in S$. Now, $w = sw = sx + su = y + su$. □

3.3 Lemma. (i) $\varrho = ((M + w) \times (M + w)) \cup \mathrm{id}_M$ *is a congruence of* $_SM$.

(ii) *If* $\varrho = id_M$ *then* $M + w = w$ *and* w *is the absorbing element of* $M(+)$.

(iii) *If* $\varrho = M \times M$ *then* $M + w = M$ *and* w *is the neutral element of* $M(+)$.

Proof. It is easy. □

3.4 Corollary. *If $_SM$ is congruence-simple then w is either absorbing or neutral in $M(+)$.* □

3.5 Lemma. *Either $M(+)$ is idempotent or w is the only idempotent element of $M(+)$.*

Proof. The set of idempotent element of $M(+)$ is a subsemimodule of $_SM$. □

3.6 Lemma. *Assume that $M(+)$ is idempotent. Then just one of the following two cases holds:*
(1) *w is the absorbing element of $M(+)$.*
(2) *$M \setminus \{w\}$ is an ideal of $M(+)$.*

Proof. If (2) is not true then $w \in M + x$ for at least one $x \in M \setminus \{w\}$. Now, by 3.2, for every $y \in M$ there is $u \in M$ with $w = y + u$ and we get $w = y + u = 2y + u = y + w$. Thus (1) is satisfied. □

3.7 Lemma. $\eta_w \neq M \times M$.

Proof. Use 2.5(vii). □

3.8 Corollary. *If $_SM$ is congruence-simple then and $M \setminus \{w\}$ is an ideal of $M(+)$ then $\eta_w = \mathrm{id}_M$.* □

3.9 Lemma. *Assume that $w = 0$ is neutral in $M(+)$. Then just one of the following two cases holds:*
(1) *$_SM$ is a module (i.e., $M(+)$ is a group).*
(2) *$M \setminus \{w\}$ is an ideal of $M(+)$.*

Proof. Put $N = \{x \in M \mid 0 \in M + x\}$ (i.e., N is the set of invertible elements of $M(+)$). One checks readily that N is a submodule of $_SM$. If $N = M$ then M is a module and if $N = 0$ then (2) is true. □

3.10 Lemma. *Assume that $M \setminus \{w\}$ is an ideal of $M(+)$ and $\eta_w = \mathrm{id}_M$* (see 3.6, 3.8 and 3.9). *Then $M(+)$ is idempotent and if $x, y \in M$ are such that $x + y \neq x$ then $(x : w)y = M$.*

Proof. Use 2.8, 2.7(ii) and the minimality of $_SM$. □

3.11 Lemma. *Assume that $_SM$ is congruence-simple, not a module, and that w is not absorbing in $M(+)$. Then:*
(i) $\eta_w = \mathrm{id}_M$.
(ii) *$w = 0$ is neutral in $M(+)$.*
(iii) *$M \setminus \{0\}$ is an ideal of $M(+)$.*
(iv) *$M(+)$ is idempotent.*
(v) *If $x, y \in M$ are such that $x + y \neq x$ then for every $z \in M$ there is $s \in S$ with $sx = 0$ and $sy = z$.*

Proof. (i) Use 3.4, 3.8, 3.9 and 3.10. □

3.12 Lemma. *If $o \in M$ is absorbing in $M(+)$, $w = 0$ is neutral in $M(+)$ and $o \neq w$ then $So = M$ and $tM = 0$ for at least one $t \in S$.*

Proof. Since ${}_SM$ is minimal and $o \notin Q({}_SM) = \{w\}$, we have $So = M$. Now, $to = 0$ for some $t \in S$, and hence $tx = tx + 0 = tx + to = t(x + o) = to = 0$ for every $x \in M$. □

3.13 Lemma. ([2, 3.13]) *Consider the situation from 3.11 and, moreover, assume that $M = \{a_1, \ldots, a_m\}$ is finite. Then:*

(i) *$M(+)$ contains an absorbing element o.*

(ii) *$So = M$ and $tM = 0$ for at least one $t \in S$.*

(iii) *For all $u, v \in M$ there exists at least one $s \in S$ such that, for every $x \in M$, $sx = 0$ if $x + u = u$ and $sx = v$ if $x + u \neq u$.*

Proof. (i) It suffices to put $o = a_1 + \ldots a_m$.

(ii) See 3.12.

(iii) If $u = o$ then $x + u = u$ for every $x \in M$ and we put $s = t$ (see (ii)). Hence, assume $u \neq o = v$ and put $L = \{x \in M \mid x + u \neq u\}$. The set L is finite and non-empty, and (by 3.11 (v)) for every $x \in L$ there is $s_x \in S$ with $s_x u = 0$ and $s_x x = o$. If $s = \sum_{x \in L} s_x$ then $sx = o$ for every $x \in L$. Moreover, $su = 0$ and if $y + u = u$ then $sy = sy + 0 = sy + su = s(y + u) = su = 0$. Finally, if v is arbitrary then, by (ii), there is $q \in S$ with $v = qo$. Now, if $x + u \neq u$ then $qsx = qo = v$, and if $y + u = u$ then $qsy = q0 = 0$.

3.14 Lemma. *Assume that M is finite and $w = o$ is absorbing in $M(+)$. Then $tM = o$ for at least one $t \in S$.*

Proof. If $x \in M$, $x \neq o$, then $Sx = M$, and hence $t_x x = o$ for some $t_x \in S$. Put $t = \sum_{x \in M} t_x$. Then $tM = o$. □

3.15 Lemma. *Assume that M is finite, congruence-simple and not a module. Then $tM = w$ for at least one $t \in S$.*

Proof. Combine 3.4, 3.11, 3.13 and 3.14. □

4. Minimal semimodules (B)

4.1 Lemma. *Let ${}_SM$ be a semimodule. Define a mapping $\varphi : S \to \mathrm{End}(M(+))$ by $\varphi(s)(x) = sx$ for all $s \in S$ and $x \in M$. Then:*

(i) *φ is a homomorphism of semirings.*

(ii) *$\ker(\varphi)$ is a congruence of the semiring S.*

(iii) *$\ker(\varphi) = S \times S$ if and only if the semimodule ${}_SM$ is quasitrivial.*

(iv) *$\ker(\varphi) = \mathrm{id}_S$ if and only if for all $r, s \in S$, $r \neq s$, there exists at least one $x \in M$ with $rx \neq sx$.*

(v) *If the semiring S is congruence-simple and the semimodule ${}_SM$ is not quasitrivial then φ is an injective homomorphism.*

(vi) *If $s \in S$ then $\varphi(s)$ is a constant endomorphism of $M(+)$ if and only if $|sM| = 1$.*

Proof. Easy to check directly. □

For a semimodule ${}_SM$, put $\mathrm{Ann}({}_SM) = \{s \in S \mid |sM| = 1\}$.

4.2 Lemma. *Put $A = \mathrm{Ann}({}_SM)$. Then:*

(i) *Either $A = \emptyset$ or A is an ideal of the semiring S (i.e., $(A + A) \cup SA \cup \cup AS \subseteq A$).*

(ii) *For every $s \in A$ there is $w_s \in M$ with $sM = w_s$.*

(iii) *$w_{r+s} = w_r + w_s$ for all $r, s \in A$.*

(iv) *$w_{ts} = tw_s$ and $w_{st} = w_s$ for all $s \in A$ and $t \in S$.*

(v) *If $A \neq \emptyset$ then $AM = \{w_s \mid s \in A\}$ is a subsemimodule of ${}_SM$.*

(vi) *$sx = stx$ for all $s \in A$, $t \in S$ and $x \in M$.*

(vii) *If $s \in A$ and $q \in S$ are such that $qw_s = w_s$ then $sx = qsx$ for every $x \in M$.*

(viii) *If $s \in A$ is such that $Sw_s = w_s$ then $AM = \{w_s\}$ (i.e., $w_r = w_s$ for every $r \in A$).*

(ix) *If $s \in A$ is such that $w_s = 0_M$ is neutral in $M(+)$ then $tx = (s + t)x$ for every $t \in S$ and $x \in M$.*

(x) *If $s \in A$ is such that $w_s = o_M$ is absorbing in $M(+)$ then $o_M = sx = (s + t)x$ for every $t \in S$ and $x \in M$.*

Proof. Easy to check directly. □

4.3 Lemma. *Assume that* $\ker(\varphi) = \mathrm{id}_S$ *(e.g., S congruence-simple and ${}_SM$ not quasitrivial). Then:*

(i) *Every element from $A = \mathrm{Ann}({}_SM)$ is left multiplictively absorbing in S.*

(ii) *If $s \in A$ and $t \in S$ are such that $tw_s = w_s$ when $ts = s$.*

(iii) *If $s \in A$ is such that $Sw_s = w_s$ then s is multiplicatively absorbing in S and $A = \{s\}$.*

(iv) *If $s \in A$ is such that w_s is neutral in $M(+)$ then s is additively neutral in S.*

(v) *If $s \in A$ is such that w_s is absorbing in $M(+)$ then s is additively absorbing in S.*

Proof. Use 4.1 and 4.2(vi), (vii), (viii), (ix) and (x). □

4.4 Proposition. *Assume that S is a congruence-simple semiring and ${}_SM$ is a finite non-quasitrivial minimal congruence-simple semimodule with $Q({}_SM) \neq \emptyset$. Then:*

(i) *$Q({}_SM) = P({}_SM) = \{w\}$ for some $w \in M$ and $Sw = w$.*

(ii) *Either $w = 0_M$ is neutral in $M(+)$ or $w = o_M$ is absorbing in $M(+)$.*

(iii) *If $w = 0_M$ is neutral in $M(+)$ then S contains an additively neutral and multiplicatively absorbing element 0_S and $0_SM = 0_M$.*

(iv) *If* $w = o_M$ *is absorbing in* $M(+)$ *then* S *contains a bi-absorbing element* o_S *and* $o_S M = o_M$.

Proof. (i) See 3.1.

(ii) See 3.4.

(iii) We have $\ker(\varphi) = \mathrm{id}_S$ by 4.1(v). Furthermore, $A = \mathrm{Ann}({}_S M) \neq \emptyset$ by 3.13(ii). Now, by 4.3(iii) and 4.3(iv), we have $A = \{0_S\}$, where 0_S is additively neutral and multiplicatively absorbing.

(iv) Again, $\ker(\varphi) = \mathrm{id}_S$. Furthermore, $A \neq \emptyset$ by 3.14 and, by 4.3(iii) and 4.3(v), we have $A = \{o_S\}$, where o_S is bi-absorbing. □

4.5 Lemma. *Let* ${}_S M$ *be a semimodule such that* $M = Sx$ *for some* $x \in M$. *Then:*

(i) *If* S *has a left multiplicatively absorbing element then* $\mathrm{Ann}({}_S M) \neq \emptyset$.

(ii) *If* S *has a right multiplicatively absorbing element then* $P({}_S M) \neq \emptyset$.

(iii) *If* S *has the additively neutral element then* $M(+)$ *has the neutral element.*

(iv) *If* S *has the additively absorbing element then* $M(+)$ *has the absorbing element.*

Proof. The mapping $s \mapsto sx$ is a homomorphism of ${}_S S$ onto ${}_S M$. □

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