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Quasitrivial Semimodules II

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The paper continues the investigation of semimodules. Main emphasis is laid on minimal (i.e., every proper subsemimodule has just one element) and congruence-simple semimodules.

1. Auxiliary results (A)

This paper is a continuation of [1] and we use the same notation. Let $sM$ be a (left) semimodule. For arbitrary elements $x, w \in M$, put $((x : w)_{S,M} =) (x : w) = \{ s \in S \mid sx = w \}$. Similarly, if $x \in M$ and $A \subseteq M$, put $(x : A)_l = \{ s \in S \mid sx \in A \}$.

1.1 Lemma. (i) $(x : w_1) + (x : w_2) \subseteq (x : w_1 + w_2)$.

(ii) If $w_1 \neq w_2$ then $(x : w_1) \cap (x : w_2) = \emptyset$.

Proof. Obvious.

1.2 Lemma. Assume that $2w = w$. Then:

(i) $(x : w) + (x : w) \subseteq (x : w)$.

(ii) $(x : w) \cap (y : w) \subseteq (x + y : w)$.

Proof. Easy to check.
1.3 Lemma. Assume that $2w = w$ and $(M \setminus \{w\}) + M \subseteq M \setminus \{w\}$. Then:

(i) $(x : w) \cap (y : w) = (x + y : w)$.
(ii) $(x : w) = (2x : w)$.
(iii) If $x + y = x$ then $(x : w) \subseteq (y : w)$.

Proof. Easy to check.

1.4 Lemma. If $w$ is the absorbing element of $M(+) \text{ then } S + (x : w) \subseteq (x : w)$.

Proof. Easy to check.

1.5 Lemma. (i) $(sx : w) = \{ r \in S \mid rs \in (x : w) \} = (s : (x : w))$ for every $s \in S$.
(ii) If $sw = w$ then $s(x : w) \subseteq (x : w)$.
(iii) If $Sw = w$ (i.e., $w \in P(sM)$) then $2w = w$ and $(x : w) \cup S(x : w) \subseteq (x : w)$.

Proof. Easy to check.

1.6 Lemma. If $Sw = w$ and $(x : w) \neq \emptyset$ then $(x : w)$ is a subsemimodule of $sS$ (i.e., a left ideal of the semiring $S$).

Proof. See 1.5 (iii).

1.7 Lemma. Assume that $Sw = w$ (i.e., $w \in P(sM)$) and $(x : w) \neq \emptyset$. Then:

(i) $(x : w)y$ is a subsemimodule of $sM$ for every $y \in M$.
(ii) If $|(x : w)y| = 1$ then $(x : w)y \subseteq Q(sM)$.
(iii) If $|(x : w)y| = 1$ and $|Q(sM)| = 1$ (e.g., if $sM$ is minimal and not quasitrivial) then $(x : w) \subseteq (y : w)$.
(iv) If $sM$ is minimal, not quasitrivial, and if $(x : w) \not\subseteq (y : w)$ then $(x : w)y = M$.

Proof. (i) By 1.6, $(x : w)$ is a left ideal. Then, of course, $N = (x : w)y$ is a subsemimodule of $sM$.
(ii) $Ssy \subseteq S(x : w)y \subseteq (x : w)y$, and hence $|Ssy| = 1$ for every $s \in (x : w)$. Then $sy \in Q(sM)$.
(iii) Since $|Q(sM)| = 1$, we have $Q(sM) = \{w\}$. By (ii), $(x : w)y = w$, so that $(x : w) \subseteq (y : w)$.
(iv) It follows from (iii) that $|(x : w)y| \geq 2$. Thus $(x : w)y$ is a non-trivial subsemimodule of $sM$ and $(x : w)y = M$, since $M$ is minimal.

1.8 Lemma. Assume that $sM$ is minimal, not quasitrivial, and that $Q(sM) \neq \emptyset$.

Then:

(i) $P(sM) = Q(sM) = \{w\}$ for some $w \in M$.
(ii) If $x, y \in M$ and $s \in S$ are such that $sx = w \neq sy$ then for every $z \in M$ there exists $r \in S$ with $rx = w$ and $ry = z$.

Proof. (i) We have $|Q(sM)| = 1$ and the rest is clear.
(ii) We have $(x : w)y \not\subseteq (y : w)$ and 1.7(iv) applies.
2. Auxiliary results (B)

Let $sM$ be a semimodule. For every $w \in M$, define a relation $\eta_w$ on $M$ by $(x, y) \in \eta_w$ iff $(x : w) = (y : w)$.

2.1 Lemma. (i) $\eta_w$ is an equivalence.
(ii) If $(x, y) \in \eta_w$ then $(sx, sy) \in \eta_w$ for every $s \in S$.

Proof. (i) This follows immediately from the definition of $\eta_w$.
(ii) By 1.5(i), $(sx : w) = (sy : w)$. □

2.2 Lemma. Assume that $2w = w$ and $(M \setminus \{w\}) + M \subseteq M \setminus \{w\}$. Then $\eta_w$ is a congruence of the semimodule $sM$.

Proof. By 2.1, $\eta = \eta_w$ is an equivalence that is stable with respect to the scalar $S$-multiplication. By 1.3(i), $(x, y) \in \eta$ implies $(x + z : w) = (x : w) \cap (z : w) = (y : w) \cap (z : w) = (y + z : w)$ and $(y + z, y + z) \in \eta$ for every $z \in M$. Thus $\eta$ is a congruence. □

2.3 Lemma. The following conditions are equivalent:
(i) $\eta_w = \text{id}_M$.
(ii) For all $x, y \in M$, $x \neq y$, there exists at least one $s \in S$ such that either $sx \neq sy = w$ or $sy \neq sx = w$.

Proof. The assertion follows easily from the definition of $\eta_w$. □

2.4 Lemma. The following condition are equivalent:
(i) $\eta_w = M \times M$.
(ii) There is a subset $I$ of $S$ such that $Ix = w \notin (S \setminus I)x$ for every $x \in M$.

Proof. The assertion follows easily from the definition of $\eta_w$. □

2.5 Lemma. Assume that $\eta_w = M \times M$ and put $I = (w : w)$. Then:
(i) $(x : w) = I$ for every $x \in M$.
(ii) $I = (s : I)_s$ for every $s \in S$.
(iii) $rS \subseteq I$ for every $r \in I$.
(iv) $tS \subseteq S \setminus I$ for every $t \in S \setminus I$.
(v) If $2w = w$ then $I + I \subseteq I$.
(vi) If $w$ is the absorbing element of $M (+)$ then $S + I \subseteq I$.
(vii) If $Sw = w$ then $I = S$ and $SM = \{w\}$ (i.e., $sM$ is cs-quasitrivial).

Proof. (i) This follows immediately from the definition of $\eta_w$.
(ii) By (i) and 1.5(i), $I = (sw : w) = (s : (w : w))_s = (s : I)_s$.
(iii) and (iv). Use (ii).
(v) See 1.2(i).
(vi) See 1.4.
(vii) Obvious. □
2.6 Lemma. Assume that $S\mathcal{M}$ is congruence-simple and not cs-quasitrivial. If $w \in M$ is such that $Sw = w$ and $(M \setminus \{w\}) + M \subseteq M \setminus \{w\}$ then $\eta_w = \text{id}_M$.

Proof. By 2.2, $\eta_w$ is a congruence of $S\mathcal{M}$ and it follows from 2.5(vii) that $\eta_w \neq M \times M$. Since $S\mathcal{M}$ is congruence-simple, we conclude that $\eta_w = \text{id}_M$. □

2.7 Lemma. Let $w \in M$ be such that $2w = w$, $(M \setminus \{w\}) + M = M \setminus \{w\}$ and $\eta_w = \text{id}_M$. Then:

(i) If $x, y \in M$ then $x + y = x$ if and only if $(x : w) \subseteq (y : w)$.
(ii) If $Sw = w$, $x + y \neq x$ and $(x : w) \neq \emptyset$ then $(x : w)y$ is a subsemimodule of $S\mathcal{M}$ and $(x : w)y \neq \{w\}$. If, moreover, $(x : w)y \not\subseteq Q(\mathcal{S}M)$ then $|(x : w)y| \geq 2$.

Proof. (i) If $x + y = x$ then, with respect to 1.3(iii), $(x : w) \subseteq (y : w)$. Conversely, if $(x : w) \subseteq (y : w)$ then $(x : w) = (x : w) \cap (y : w) = (x + y : w)$ by 1.3(i), hence $(x + y, x) \in \eta_w = \text{id}_M$ and $x + y = x$.

(ii) By 1.7(i), $(x : w)y$ is a subsemimodule of $S\mathcal{M}$. Since $(x : w) \not\subseteq (y : w)$ by (i), we get $(x : w)y \neq \{w\}$. Finally, if $(x : w)y \not\subseteq Q(\mathcal{S}M)$ then $|(x : w)y| \geq 2$ by 1.7(ii). □

2.8 Lemma. Assume that $2w = w$ and $(M \setminus \{w\}) + M \subseteq M \setminus \{w\}$. Then:

(i) $(2x, x) \in \eta_w$ for every $x \in M$.
(ii) If $\eta_w = \text{id}_M$ then $M (+)$ is idempotent.

Proof. See 1.3(ii). □

3. Minimal semimodules (A)

Throughout this section, let $S\mathcal{M}$ be a minimal semimodule that is not quasitrivial (cf. [1, 4.11]) and such that $Q(S\mathcal{M}) \neq \emptyset$ (i.e., $S\mathcal{M}$ is not strictly minimal).

3.1 Lemma. $Q(S\mathcal{M}) = P(S\mathcal{M}) = \{w\}$ for some $w \in M$ and $Sw = w$.

Proof. $Q(S\mathcal{M})$ is a proper subsemimodule of $S\mathcal{M}$ and the rest is clear. □

3.2 Lemma. If $w = M + x$ for at least one $x \in M$, $x \neq w$, then $w \in M + y$ for every $y \in M$.

Proof. We have $w = x + u$ for some $u \in M$. Since $x \neq w$, we also have $Sx = M$, and hence $y = sx$ for some $s \in S$. Now, $w = sw = sx + su = = y + su$. □

3.3 Lemma. (i) $\varrho = ((M + w) \times (M + w)) \cup \text{id}_M$ is a congruence of $S\mathcal{M}$.
(ii) If $\varrho = \text{id}_M$ then $M + w = w$ and $w$ is the absorbing element of $M (+)$.
(iii) If $\varrho = M \times M$ then $M + w = M$ and $w$ is the neutral element of $M (+)$.

Proof. It is easy. □
3.4 Corollary. If $SM$ is congruence-simple then $w$ is either absorbing or neutral in $M(+)$. □

3.5 Lemma. Either $M(+) is idempotent or $w$ is the only idempotent element of $M(+)$.

Proof. The set of idempotent element of $M(+) is a subsemimodule of $SM$. □

3.6 Lemma. Assume that $M(+) is idempotent. Then just one of the following two cases holds:
\begin{enumerate}
\item $w$ is the absorbing element of $M(+)$.\item $M\{w\}$ is an ideal of $M(+)$.\end{enumerate}

Proof. If (2) is not true then $w \in M + x$ for at least one $x \in M\{w\}$. Now, by 3.2, for every $y \in M$ there is $u \in M$ with $w = y + u$ and we get $w = y + u = 2y + u = y + w$. Thus (1) is satisfied. □

3.7 Lemma. $\eta_w \neq M \times M$.

Proof. Use 2.5(vii). □

3.8 Corollary. If $SM$ is congruence-simple then and $M\{w\}$ is an ideal of $M(+) then $\eta_w = \text{id}_M$. □

3.9 Lemma. Assume that $w = 0$ is neutral in $M(+)$. Then just one of the following two cases holds:
\begin{enumerate}
\item $SM$ is a module (i.e., $M(+) is a group).\item $M\{w\}$ is an ideal of $M(+)$.\end{enumerate}

Proof. Put $N = \{x \in M \mid 0 \in M + x\}$ (i.e., $N$ is the set of invertible elements of $M(+)$. One checks readily that $N$ is a submodule of $SM$. If $N = M$ then $M$ is a module and if $N = 0$ then (2) is true. □

3.10 Lemma. Assume that $M\{w\}$ is an ideal of $M(+) and $\eta_w = \text{id}_M$ (see 3.6, 3.8 and 3.9). Then $M(+) is idempotent and if $x, y \in M$ are such that $x + y \neq x then $(x : w)y = M$.

Proof. Use 2.8, 2.7(ii) and the minimality of $SM$. □

3.11 Lemma. Assume that $SM$ is congruence-simple, not a module, and that $w$ is not absorbing in $M(+)$. Then:
\begin{enumerate}
\item $\eta_w = \text{id}_M$.\item $w = 0$ is neutral in $M(+)$.\item $M\{0\}$ is an ideal of $M(+)$.\item $M(+) is idempotent.\item If $x, y \in M$ are such that $x + y \neq x then for every $z \in M there is $s \in S with $sx = 0$ and $sy = z$.\end{enumerate}

Proof. (i) Use 3.4, 3.8, 3.9 and 3.10. □
3.12 Lemma. If \( o \in M \) is absorbing in \( M(+) \), \( w = 0 \) is neutral in \( M(+) \) and \( o \neq w \) then \( S_0 = M \) and \( tM = 0 \) for at least one \( t \in S \).

Proof. Since \( S_0M \) is minimal and \( o \notin Q(S_0M) = \{ w \} \), we have \( S_0 = M \). Now, \( to = 0 \) for some \( t \in S \), and hence \( tx = tx + 0 = tx + to = t(x + o) = to = 0 \) for every \( x \in M \).

3.13 Lemma. ([2,3.13]) Consider the situation from 3.11 and, moreover, assume that \( M = \{ a_1, \ldots, a_m \} \) is finite. Then:

(i) \( M(+) \) contains an absorbing element \( o \).

(ii) \( S_0 = M \) and \( tM = 0 \) for at least one \( t \in S \).

(iii) For all \( u, v \in M \) there exists at least one \( s \in S \) such that, for every \( x \in M \), \( sx = 0 \) if \( x + u = u \) and \( sx = v \) if \( x + u \neq u \).

Proof. (i) It suffices to put \( o = a_1 + \ldots a_m \).

(ii) See 3.12.

(iii) If \( u = 0 \) then \( x + u = u \) for every \( x \in M \) and we put \( s = t \) (see (ii)).

3.14 Lemma. Assume that \( M \) is finite and \( w = 0 \) is absorbing in \( M(+) \). Then \( tM = 0 \) for at least one \( t \in S \).

Proof. If \( x \in M \), \( x \neq 0 \), then \( Sx = M \), and hence \( t_x x = 0 \) for some \( t_x \in S \). Put \( t = \sum_{x \in M} t_x \). Then \( tM = 0 \).

3.15 Lemma. Assume that \( M \) is finite, congruence-simple and not a module. Then \( tM = w \) for at least one \( t \in S \).


4. Minimal semimodules (B)

4.1 Lemma. Let \( S_0M \) be a semimodule. Define a mapping \( \varphi : S \to \text{End}(M(+)) \) by \( \varphi(s)(x) = sx \) for all \( s \in S \) and \( x \in M \). Then:

(i) \( \varphi \) is a homomorphism of semirings.

(ii) \( \ker(\varphi) \) is a congruence of the semiring \( S \).

(iii) \( \ker(\varphi) = S \times S \) if and only if the semimodule \( S_0M \) is quasitrivial.

(iv) \( \ker(\varphi) = \text{id}_S \) if and only if for all \( r, s \in S \), \( r \neq s \), there exists at least one \( x \in M \) with \( rx \neq sx \).
(v) If the semiring $S$ is congruence-simple and the semimodule $\mathcal{S}M$ is not quasitrivial then $\varphi$ is an injective homomorphism.

(vi) If $s \in S$ then $\varphi(s)$ is a constant endomorphism of $M(\,\cdot\,)$ if and only if $|sM| = 1$.

**Proof.** Easy to check directly. \qed

For a semimodule $\mathcal{S}M$, put $\text{Ann}(\mathcal{S}M) = \{s \in S \mid |sM| = 1\}$.

### 4.2 Lemma

Put $A = \text{Ann}(\mathcal{S}M)$. Then:

1. Either $A = \emptyset$ or $A$ is an ideal of the semiring $S$ (i.e., $(A + A) \cup SA \cup AS \subseteq A$).
2. For every $s \in A$ there is $w_s \in M$ with $sM = w_s$.
3. $w_{r+s} = w_r + w_s$ for all $r, s \in A$.
4. $w_{ts} = tw_s$ and $w_{st} = w_s$ for all $s, t \in S$.
5. If $A \neq \emptyset$ then $AM = \{w_s \mid s \in A\}$ is a subsemimodule of $\mathcal{S}M$.
6. $sx = stx$ for all $s \in A$, $t \in S$ and $x \in M$.
7. If $s \in A$ and $q \in S$ are such that $qw_s = w_s$ then $sx = qsx$ for every $x \in M$.
8. If $s \in A$ is such that $Sw_s = w_s$ then $AM = \{w_s\}$ (i.e., $w_r = w_s$ for every $r \in A$).
9. If $s \in A$ is such that $w_s = 0_M$ is neutral in $M(\,\cdot\,)$ then $tx = (s + t)x$ for every $t \in S$ and $x \in M$.
10. If $s \in A$ is such that $w_s$ is absorbing in $M(\,\cdot\,)$ then $sx = (s + t)x$ for every $t \in S$ and $x \in M$.

**Proof.** Easy to check directly. \qed

### 4.3 Lemma

Assume that $\ker(\varphi) = \text{id}_{\mathcal{S}}$ (i.e., $S$ congruence-simple and $\mathcal{S}M$ not quasitrivial). Then:

1. Every element from $A = \text{Ann}(\mathcal{S}M)$ is left multiplicatively absorbing in $S$.
2. If $s \in A$ and $t \in S$ are such that $tw_s = w_s$ when $ts = s$.
3. If $s \in A$ is such that $Sw_s = w_s$ then $s$ is multiplicatively absorbing in $S$ and $A = \{s\}$.
4. If $s \in A$ is such that $w_s$ is neutral in $M(\,\cdot\,)$ then $s$ is additively neutral in $S$.
5. If $s \in A$ is such that $w_s$ is absorbing in $M(\,\cdot\,)$ then $s$ is additively absorbing in $S$.

**Proof.** Use 4.1 and 4.2(vi), (vii), (viii), (ix) and (x). \qed

### 4.4 Proposition

Assume that $S$ is a congruence-simple semiring and $\mathcal{S}M$ is a finite non-quasitrivial minimal congruence-simple semimodule with $Q(\mathcal{S}M) \neq \emptyset$. Then:

1. $Q(\mathcal{S}M) = P(\mathcal{S}M) = \{w\}$ for some $w \in M$ and $Sw = w$.
2. Either $w = 0_M$ is neutral in $M(\,\cdot\,)$ or $w = o_M$ is absorbing in $M(\,\cdot\,)$.
3. If $w = 0_M$ is neutral in $M(\,\cdot\,)$ then $S$ contains an additively neutral and multiplicatively absorbing element $0_S$ and $0_SM = 0_M$. 

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(iv) If $w = o_M$ is absorbing in $M(\cdot)$ then $S$ contains a bi-absorbing element $o_S$ and $o_SM = o_M$.

**Proof.** (i) See 3.1.

(ii) See 3.4.

(iii) We have $\ker(\varphi) = \text{id}_S$ by 4.1(v). Furthermore, $A = \text{Ann}(S) \neq \emptyset$ by 3.13(ii). Now, by 4.3(iii) and 4.3(iv), we have $A = \{0_S\}$, where $0_S$ is additively neutral and multiplicatively absorbing.

(iv) Again, $\ker(\varphi) = \text{id}_S$. Furthermore, $A \neq \emptyset$ by 3.14 and, by 4.3(iii) and 4.3(v), we have $A = \{0_S\}$, where $0_S$ is bi-absorbing. □

**4.5 Lemma.** Let $S$ be a semimodule such that $M = Sx$ for some $x \in M$. Then:

(i) If $S$ has a left multiplicatively absorbing element then $\text{Ann}(S) \neq \emptyset$.

(ii) If $S$ has a right multiplicatively absorbing element then $P(S) \neq 0$.

(iii) If $S$ has the additively neutral element then $M(\cdot)$ has the neutral element.

(iv) If $S$ has the additively absorbing element then $M(\cdot)$ has the absorbing element.

**Proof.** The mapping $s \mapsto sx$ is a homomorphism of $S$ onto $SM$. □

**References**
