Kybernetika

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Kybernetika, Vol. 48 (2012), No. 2, 287--293

Persistent URL: http://dml.cz/dmlcz/142814

KYBERNETIKA — VOLUME *48* (2012), NUMBER 2, PAGES 287–293

ON A PROBLEM BY SCHWEIZER AND SKLAR

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We give a representation of the class of all n–dimensional copulas such that, for a fixed $m \in \mathbb{N}$, $2 \leq m < n$, all their m–dimensional margins are equal to the independence copula. Such an investigation originated from an open problem posed by Schweizer and Sklar.

Keywords: copulas, distributions with given marginals, Fréchet–Hoeffding bounds, partial mutual independence

Classification: 60E05, 62E10

1. INTRODUCTION

The representation and the construction of n–dimensional distribution functions (=d.f.'s) with given lower dimensional marginal distributions is one of the classical problem in probability theory, due to its relevance to applications. Questions of this kind arise, for example, when one wants to build a multivariate stochastic model and has some idea about the kind of dependence, or knows exactly certain marginal distributions (see, for instance, [1, 2, 3, 7, 9] and the references therein).

In this note, we investigate a special problem of this type, namely we consider the class of all possible joint d.f.'s of a random vector $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ such that: (a) X_i has a continuous d.f. F_i, for each $i \in \{1, 2, \ldots, n\}$; (b) for a given $m \in \mathbb{N}$, $2 \leq m < n$, every sub-vector of m–elements in $\mathbf{X}$ is formed by independent random variables (=r.v.'s). Such a problem has been originally posed by Schweizer and Sklar (see [10, Problem 6.7.3]) in the class of all distribution functions whose univariate margins are uniformly distributed on $[0, 1]$, i.e., in the class of *copulas*.

In fact, in view of Sklar's Theorem [12], copulas are exactly the objects that allow to capture the dependence properties of a random vector. Therefore, in this note, we investigate the above-stated problem in terms of multivariate copulas and its lower-dimensional margins.

The paper is organized as follows. First, in section 2 we define the basic elements that are necessary in order to make the paper self-contained. Then, in section 3, we characterize the dependence structures of the previous type by providing also some upper and lower bounds.

2. PRELIMINARIES

Let n, m be in $\mathbb{N}$, $2 \leq m < n$. We denote by $\mathcal{P}_{n,m}$ the class of all permutations $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n)$ of $(1, 2, \dots, n)$ such that

$$\sigma_1 < \sigma_2 < \cdots < \sigma_m \quad \text{and} \quad \sigma_{m+1} < \sigma_{m+2} < \cdots < \sigma_n.$$

For example, $\mathcal{P}_{3,2} = \{(1,2,3), (1,3,2), (2,3,1)\}$.

We denote by $\mathbf{x} = (x_1, \dots, x_n)$ any point in $\mathbb{R}^n$ and by $\mathbb{I}^n$ the product of n copies of the unit interval $\mathbb{I} = [0, 1]$. For basic definitions and properties about copulas, we refer to [6, 8]. Here we recall that an *n–copula* is a function $C : \mathbb{I}^n \to \mathbb{I}$ satisfying the following properties:

(C1) $C(\mathbf{u}) = 0$ whenever $\mathbf{u} \in \mathbb{I}^n$ has at least one argument equal to 0;

(C2) $C(\mathbf{u}) = u_i$ whenever $\mathbf{u} \in \mathbb{I}^n$ has all the arguments equal to 1 except possibly the ith one, which is equal to u_i;

(C3) C is *n–increasing*, viz., for each n–box $B = \times_{i=1}^{n} [u_i, v_i] \subseteq \mathbb{I}^n$, $u_i \leq v_i$ for any $i \in \{1, 2, \dots, n\}$,
$$V_C(B) = \sum_{\mathbf{z} \in B} \operatorname{sgn}(\mathbf{z}) C(\mathbf{z}) \geq 0,$$
where the sum is taken over all vertices $\mathbf{z}$ in B, $z_i \in \{u_i, v_i\}$ for every i in $\{1, 2, \dots, n\}$, and $\operatorname{sgn}(\mathbf{z}) = -1$, if the number of u_i's among the arguments of $\mathbf{z}$ is odd, and $\operatorname{sgn}(\mathbf{z}) = 1$, otherwise.

We denote by $\mathcal{C}_n$ the set of all n–copulas. For all $C \in \mathcal{C}_n$ and for all $\mathbf{u} \in \mathbb{I}^n$,

$$W_n(\mathbf{u}) \leq C(\mathbf{u}) \leq M_n(\mathbf{u}), \tag{1}$$

where

$$W_n(\mathbf{u}) = \max\left\{ \sum_{i=1}^{n} u_i - n + 1, 0 \right\}, \qquad M_n(\mathbf{u}) = \min\{u_1, u_2, \dots, u_n\}.$$

These inequalities are called *Fréchet–Hoeffding bounds* [7, 8]. Notice that $M_n \in \mathcal{C}_n$, but $W_n \in \mathcal{C}_n$ only for $n = 2$. Another important n–copula is $\Pi_n(\mathbf{u}) = \prod_{i=1}^{n} u_i$, which is associated with independent r.v.'s.

Given $C \in \mathcal{C}_n$, the m–marginals of C, $2 \leq m < n$, are the $\binom{n}{m}$ m–copulas obtained by setting $(n - m)$ of the arguments of C equal to 1. Moreover, we denote by $\mathcal{C}_n(\Pi_m)$ the class of all n–copulas such that all their m–marginals are equal to Π_m.

Here, we present a method for constructing n–copulas, which we shall use in the sequel.

Proposition 2.1. Let $\mathbf{C} = \{C_{\mathbf{t}}\}_{\mathbf{t} \in \mathbb{I}^{m-1}}$ be a family in $\mathcal{C}_{n-m+1}$ indexed by a parameter $\mathbf{t} \in \mathbb{I}^{m-1}$. Then $C : \mathbb{I}^n \to \mathbb{I}$ given by

$$C(\mathbf{u}) = \int_0^{u_1} \dots \int_0^{u_{m-1}} C_{\mathbf{t}}(u_m, \dots, u_n) \, \mathrm{d}t_1 \dots \mathrm{d}t_{m-1}. \tag{2}$$

is in $\mathcal{C}_n$, provided that the above integral exists.

Proof. It is immediate to prove that the function C given by (2) satisfies (C1) and (C2). In order to prove that C is n–increasing, consider the n–box $B = \times_{i=1}^{n}[u_i, v_i]$ in $\mathbb{I}^n$, $u_i \le v_i$ for any $i \in \{1, 2, \dots, n\}$. Then, we have that

$$V_C(B) = \int_{u_1}^{v_1} \dots \int_{u_{m-1}}^{v_{m-1}} V_{C_{\mathbf{t}}}([u_m, v_m] \times \dots \times [u_n, v_n]) \, \mathrm{d}t_1 \dots \mathrm{d}t_{m-1},$$

which is non–negative because, for any $\mathbf{t} \in \mathbb{I}^{m-1}$, $C_{\mathbf{t}}$ belongs to $\mathcal{C}_{n-m+1}$ and hence $\mathbf{t} \mapsto V_{C_{\mathbf{t}}}([u_m, v_m] \times \dots \times [u_n, v_n])$ is non–negative. □

Example 2.2. Let C and $\{C_{\mathbf{t}}\}_{\mathbf{t} \in \mathbb{I}^{m-1}}$ be in $\mathcal{C}_{n-m+1}$ and suppose that $C_{\mathbf{t}} = C$ for every $\mathbf{t} \in \mathbb{I}^{m-1}$. Then elementary integration yields

$$D(\mathbf{u}) = \left(\prod_{i=1}^{m-1} u_i \right) \cdot C(u_m, \dots, u_n),$$

which is the n–d.f. of the random vector $(U_1, U_2, \dots, U_n)$ such that: U_i are uniformly distributed on $\mathbb{I}$, C is the d.f. of $(U_m, U_{m+1}, \dots, U_n)$, Π_{m-1} is the d.f. of $(U_1, \dots, U_{m-1})$, and $(U_1, \dots, U_{m-1})$ and $(U_m, \dots, U_n)$ are independent random vectors.

Example 2.3. Let $\{C_{(t_1,t_2)}\}_{(t_1,t_2) \in \mathbb{I}^2}$ be in $\mathcal{C}_2$ defined by

$$C_{(t_1,t_2)}(u_1, u_2) = \begin{cases} \Pi_2(u_1, u_2), & t_1 \le \frac{1}{2}, \\ M_2(u_1, u_2), & \text{otherwise.} \end{cases}$$

Then, by using Proposition 2.1, we obtain that $C_4 : \mathbb{I}^4 \to \mathbb{I}$ given by

$$\begin{aligned} C_4(u_1, u_2, u_3, u_4) &= \int_0^{u_1} \int_0^{u_2} C_{(t_1,t_2)}(u_3, u_4) \, \mathrm{d}t_1 \, \mathrm{d}t_2 \\ &= \begin{cases} \Pi_4(u_1, u_2, u_3, u_4), & u_1 \le \frac{1}{2}, \\ \frac{u_2 u_3 u_4}{2} + \left(u_1 - \frac{1}{2}\right) u_2 M_2(u_3, u_4), & \text{otherwise,} \end{cases} \end{aligned}$$

is a 4–copula such that all its 2–marginals are equal to Π_2. The copula C_4 can be also obtained by means of a gluing construction [11].

Remark 2.4. If $C \in \mathcal{C}_n$, then, for every permutation $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n)$ of $(1, 2, \dots, n)$ the function $C^\sigma : \mathbb{I}^n \to \mathbb{I}$ given by

$$C^\sigma(u_1, \dots, u_n) = C(u_{\sigma_1}, \dots, u_{\sigma_n})$$

is also in $\mathcal{C}_n$. In particular, if C is the copula given by (2), then, for every permutation $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n)$ of $(1, 2, \dots, n)$, $C^\sigma : \mathbb{I}^n \to \mathbb{I}$ given by

$$C^\sigma(\mathbf{u}) = \int_0^{u_{\sigma_1}} \dots \int_0^{u_{\sigma_{m-1}}} C_{\mathbf{t}}(u_{\sigma_m}, \dots, u_{\sigma_n}) \, \mathrm{d}t_1 \dots \mathrm{d}t_{m-1} \tag{3}$$

is also in $\mathcal{C}_n$.

3. DESCRIPTION OF A SPECIAL CLASS OF COPULAS

Following our approach, the description of the class of all possible joint d.f.'s of a random vector $\mathbf{X} = (X_1, X_2, \dots, X_n)$ such that, for a given $m \in \mathbb{N}$, $2 \le m < n$, every sub-vector of m–elements in $\mathbf{X}$ is formed by independent r.v.'s, is equivalent to the description of the class $\mathcal{C}_n(\Pi_m)$. The elements of such a class are described in the following result.

Theorem 3.1. Let $n, m \in \mathbb{N}$, $2 \le m < n$. The following statements are equivalent:

(a) $C \in \mathcal{C}_n(\Pi_m)$;

(b) for every $\sigma \in \mathcal{P}_{n,m}$, there exists a family $\mathbf{C}^\sigma = \{C^\sigma_\mathbf{t}\}_{\mathbf{t}\in\mathbb{I}^{m-1}}$ in $\mathcal{C}_{n-m+1}$ such that, for every $\mathbf{u} \in \mathbb{I}^n$,

$$C(\mathbf{u}) = \int_0^{u_{\sigma_1}} \dots \int_0^{u_{\sigma_{m-1}}} C^\sigma_\mathbf{t}(u_{\sigma_m}, \dots, u_{\sigma_n})\,\mathrm{d}t_1 \dots \mathrm{d}t_{m-1}. \tag{4}$$

Proof. Let C be in $\mathcal{C}_n(\Pi_m)$. Then, there exist a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a random vector $\mathbf{U} = (U_1, U_2, \dots, U_n)$, U_i uniformly distributed on $\mathbb{I}$ for every $i \in \{1, 2, \dots, n\}$, such that C is the joint d.f. of $\mathbf{U}$. Let $\sigma \in \mathcal{P}_{n,m}$. Then, for each $\mathbf{u} \in \mathbb{I}^n$,

$$\begin{aligned} C(\mathbf{u}) &= \mathbb{P}(U_1 \le u_1, \dots, U_n \le u_n) \\ &= \int_0^{u_{\sigma_1}} \dots \int_0^{u_{\sigma_{m-1}}} F^\sigma_\mathbf{t}(u_{\sigma_m}, \dots, u_{\sigma_n})\,\mathrm{d}t_1 \dots \mathrm{d}t_{m-1}, \end{aligned}$$

where, for every $\mathbf{t} = (t_1, t_2, \dots, t_{m-1}) \in \mathbb{I}^{m-1}$, $F^\sigma_\mathbf{t} : \mathbb{I}^{n-m+1} \to \mathbb{I}$ defined by

$$F^\sigma_\mathbf{t}(u_{\sigma_m}, \dots, u_{\sigma_n}) = \mathbb{P}\left(\bigcap_{i=m}^{n} \{U_{\sigma_i} \le u_{\sigma_i}\} \mid U_{\sigma_1} = t_1, \dots, U_{\sigma_{m-1}} = t_{m-1} \right),$$

is the (conditional) d.f. of $(U_{\sigma_m}, \dots, U_{\sigma_n})$ given $(U_{\sigma_1} = t_1, \dots, U_{\sigma_{m-1}} = t_{m-1})$. The one–dimensional marginals of $F^\sigma_\mathbf{t}$ are uniformly distributed on $\mathbb{I}$, because any subset of m elements in $\{U_1, U_2, \dots, U_n\}$ is composed by independent r.v.'s. Therefore, $F^\sigma_\mathbf{t}$ is a copula and (b) follows.

In the other direction, let $C : \mathbb{I}^n \to \mathbb{I}$ be such that, for every $\sigma \in \mathcal{P}_{n,m}$ there exists a family $\mathbf{C}^\sigma = \{C^\sigma_\mathbf{t}\}_{\mathbf{t}\in\mathbb{I}^{m-1}} \subseteq \mathcal{C}_{n-m+1}$ such that C can be represented in the form (4). Because of Proposition 2.1 (and Remark 2.4), C is a copula. Therefore, we have only to prove that all the m–marginals of C are equal to Π_m. To this end, let C^m be the m–marginal of C obtained by setting equal to 1 the arguments of C with indices $\xi_1 < \xi_2 < \dots < \xi_{n-m}$, viz.

$$C^m(u_1, \dots, u_m) = C(\widetilde{\mathbf{u}}),$$

where $\widetilde{\mathbf{u}} \in \mathbb{I}^n$ is obtained by setting $\widetilde{u}_i = 1$ for $i \in \{\xi_1, \dots, \xi_{n-m}\}$, and $\widetilde{u}_i = u_i$, otherwise. Consider the (unique) permutation $\widehat{\xi} \in \mathcal{P}_{n,m}$ given by

$$\widehat{\xi} = (\xi_{n-m+1}, \dots, \xi_n, \xi_1, \dots, \xi_{n-m}).$$

Then there exists a family $\mathbf{C}^{\widehat{\xi}} = \{C_{\mathbf{t}}^{\widehat{\xi}}\}_{\mathbf{t}\in\mathbb{I}^{m-1}}$ in $\mathcal{C}_{n-m+1}$ such that

$$C(\mathbf{u}) = \int_0^{u_{\xi_{n-m+1}}} \cdots \int_0^{u_{\xi_{n-1}}} C_{\mathbf{t}}^{\widehat{\xi}}(u_{\xi_n}, u_{\xi_1} \dots, u_{\xi_{n-m}})\, \mathrm{d}t_1 \dots \mathrm{d}t_{m-1}. \tag{5}$$

Since $C_{\mathbf{t}}^{\widehat{\xi}}$ satisfies (C2), equality (5) implies that $C^m = \Pi_m$. For the arbitrariness of $\xi_1, \xi_2, \dots, \xi_{n-m}$, it follows that $C \in \mathcal{C}_n(\Pi_m)$. □

Remark 3.2. Since Theorem 3.1, if $C \in \mathcal{C}_n(\Pi_m)$, then there exist $\binom{n}{m}$ families $\mathbf{C}^\sigma = \{C_{\mathbf{t}}^\sigma\}_{\mathbf{t}\in\mathbb{I}^{m-1}}$ in $\mathcal{C}_{n-m+1}$, each family associated with $\sigma \in \mathcal{P}_{n,m}$, such that C can be written in $\binom{n}{m}$ different forms by means of (4). Moreover, for a fixed $\sigma \in \mathcal{P}_{n,m}$, $\{C_{\mathbf{t}}^\sigma\}_{\mathbf{t}\in\mathbb{I}^{m-1}}$ is not uniquely determined: in fact, there exist infinitely many families $\mathbf{D}^\sigma = \{D_{\mathbf{t}}^\sigma\}_{\mathbf{t}\in\mathbb{I}^{m-1}}$ such that $C_{\mathbf{t}}^\sigma \neq D_{\mathbf{t}}^\sigma$ for every $\mathbf{t}$ belonging to a subset of $\mathbb{I}^{m-1}$ with $(m-1)$–dimensional Lebesgue measure 0, and C can be represented in terms of $\mathbf{D}^\sigma$ by means of (4).

In the case $n = 3$ and $m = 2$, Theorem 3.1 can be reformulated in this form.

Corollary 3.3. A 3–copula $C_3 \in \mathcal{C}_3(\Pi_2)$ if, and only if, there exist three families of 2–copulas $\{C_t^{(1)}\}_{t\in\mathbb{I}}$, $\{C_t^{(2)}\}_{t\in\mathbb{I}}$ and $\{C_t^{(3)}\}_{t\in\mathbb{I}}$, such that

$$C_3(u_1, u_2, u_3) = \int_0^{u_1} C_t^{(1)}(u_2, u_3)\, \mathrm{d}t = \int_0^{u_2} C_t^{(2)}(u_1, u_3)\, \mathrm{d}t = \int_0^{u_3} C_t^{(3)}(u_1, u_2)\, \mathrm{d}t.$$

In particular, we have that, for every $i \in \{1, 2, 3\}$,

$$\int_0^1 C_t^{(i)}(u_1, u_2)\, \mathrm{d}t = u_1 u_2. \tag{6}$$

A method for constructing families of 2–copulas that satisfy (6) is provided in [8, Example 3.10]. Specifically, for any 2–copula C, we can construct the family of 2–copula $\{C_t\}_{t\in\mathbb{I}}$ given by

$$C_t(u_1, u_2) = \begin{cases} C(1 - t + u_1, u_2) - C(1 - t, u_2), & u_1 \le t, \\ u_2 - C(1 - t, u_2) + C(u_1 - t, u_2), & u_1 > t, \end{cases}$$

which satisfies condition (6).

Example 3.4. Let C_θ be a member of the Eyraud–Farlie–Gumbel–Morgenstern family of 3–copulas given by

$$C_\theta(u_1, u_2, u_3) = u_1 u_2 u_3 (1 + \theta(1 - u_1)(1 - u_2)(1 - u_3)),$$

where $\theta \in [-1, 1]$ (see [8]). Then C_θ has all the 2–marginals equal to Π_2 and it can be expressed, for example, into the form

$$C_\theta(u_1, u_2, u_3) = \int_0^{u_{\sigma_1}} C_t^\sigma(u_{\sigma_2}, u_{\sigma_3})\, \mathrm{d}t,$$

where $\sigma = (\sigma_1, \sigma_2, \sigma_3) \in \mathcal{P}_{3,2}$, and $\mathbf{C} = \{C_t^\sigma\}_{t\in\mathbb{I}}$ is the family of 2–copulas given by

$$C_t^\sigma(u,v) = uv + \theta uv(1-u)(1-v)(1-2t),$$

for every $t \in \mathbb{I}$ and $\sigma \in \mathcal{P}_{3,2}$.

Theorem 3.1 can be rewritten in a simpler form if we suppose that $C \in \mathcal{C}_n(\Pi_m)$ is exchangeable, viz. it does not change under permutation of its arguments.

Corollary 3.5. Let n, m be in $\mathbb{N}$, $2 \le m < n$. Let C be an exchangeable copula. Then $C \in \mathcal{C}_n(\Pi_m)$ if, and only if, there exists a family $\mathbf{C} = \{C_{\mathbf{t}}\}_{\mathbf{t}\in\mathbb{I}^{m-1}}$ in $\mathcal{C}_{n-m+1}$ such that, for every $\mathbf{u} \in \mathbb{I}^n$,

$$C(\mathbf{u}) = \int_0^{u_1} \dots \int_0^{u_{m-1}} C_{\mathbf{t}}(u_m, \dots, u_n)\, \mathrm{d}t_1 \dots \mathrm{d}t_{m-1}. \tag{7}$$

P r o o f. Let n, m be in $\mathbb{N}$, $2 \le m < n$. Let C be exchangeable. If $C \in \mathcal{C}_n(\Pi_m)$, then Theorem 3.1 ensures that there exists a family $\mathbf{C} = \{C_{\mathbf{t}}\}_{\mathbf{t}\in\mathbb{I}^{m-1}}$ in $\mathcal{C}_{n-m+1}$ such that, C admits the representation (4). Conversely, if C can be represented in the form (7), then

$$C(u_1, \dots, u_m, 1, \dots, 1) = \prod_{i=1}^m u_i,$$

and, because C is exchangeable, all its m–marginal d.f.'s are equal to $\prod_{i=1}^m u_i$, and, thus, $C \in \mathcal{C}_n(\Pi_m)$. □

Pointwise upper and lower bounds for the class $\mathcal{C}_n(\Pi_m)$ have been given in [4] (when $n = 3$ and $m = 2$, see also [5]). Theorem 3.1 provides also a way for obtaining them. In fact, for every $\sigma \in \mathcal{P}_{n,m}$ there exists a family $\mathbf{C}^\sigma = \{C_{\mathbf{t}}^\sigma\}_{\mathbf{t}\in\mathbb{I}^{m-1}}$ in $\mathcal{C}_{n-m+1}$ such that $C \in \mathcal{C}_n$ can be represented in the form (4). Now, because every copula satisfies the inequalities (1), it follows that, for every $\mathbf{u} \in \mathbb{I}^{n-m+1}$ and for every $\mathbf{t} \in \mathbb{I}^{m-1}$,

$$W_{n-m+1}(\mathbf{u}) \le C_{\mathbf{t}}^\sigma(\mathbf{u}) \le M_{n-m+1}(\mathbf{u}).$$

Thus, the following inequalities can be easily derived:

$$C_L(\mathbf{u}) \le C(\mathbf{u}) \le C_U(\mathbf{u}), \tag{8}$$

where we define

$$\begin{aligned} C_L(\mathbf{u}) &= \max_{\sigma\in\mathcal{P}_{n,m}} \left\{ \left(\prod_{i=1}^{m-1} u_{\sigma_i} \right) \cdot W_{n-m+1}(u_{\sigma_m}, \dots, u_{\sigma_n}) \right\}, \\ C_U(\mathbf{u}) &= \min_{\sigma\in\mathcal{P}_{n,m}} \left\{ \left(\prod_{i=1}^{m-1} u_{\sigma_i} \right) \cdot M_{n-m+1}(u_{\sigma_m}, \dots, u_{\sigma_n}) \right\}. \end{aligned}$$

An improvement of these bounds can be achieved by writing the expression for the survival d.f. associated with C and impose that it is non–negative (see [7] for this procedure).

ACKNOWLEDGEMENTS

The support of School of Economics and Management, Free University of Bozen–Bolzano, via the project "Risk and Dependence" is acknowledged.

(Received August 4, 2011)

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