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PERIODIC SOLUTIONS FOR A CLASS OF FUNCTIONAL DIFFERENTIAL SYSTEM

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Abstract. In this paper, we study the existence of periodic solutions to a class of functional differential system. By using Schauder’s fixed point theorem, we show that the system has a periodic solution under given conditions. Finally, four examples are given to demonstrate the validity of our main results.

1. Introduction

In this article, we study the existence of \( \omega \)-periodic solutions to the following functional differential system

\[
\begin{align*}
x_i'(t) &= a_i(t)g_i(x_i(t)) - f_i(t,x_1(t-\tau_1(t)),\ldots,x_n(t-\tau_n(t))), \quad i = 1, 2, \ldots, n,
\end{align*}
\]

where \( a_i, \tau_i : \mathbb{R} \to \mathbb{R} \) are \( \omega \)-periodic continuous functions and \( a_i(t) > 0 \) for any \( t \in [0, \omega] \), \( f_i(t,u_1,\ldots,u_n) : \mathbb{R}^{n+1} \to \mathbb{R} \) is \( \omega \)-periodic in \( t \) and \( g_i : \mathbb{R} \to \mathbb{R} \).

When \( n = 1 \), the problem (1.1) reduces to the functional differential equation

\[
\begin{align*}
x'(t) &= a(t)g(x(t)) - h(t,x(t-\tau(t))).
\end{align*}
\]

The existence of periodic solutions for the special cases of (1.2) have been considered extensively by many authors, because (1.2) includes many important models in mathematical biology, such as, Hematopoiesis models; Nicholson’s blowflies models; models for blood cell production, see [2, 3, 4, 8, 9, 7] and the references therein. Recently, Wang [5] investigated existence, multiplicity and nonexistence of positive periodic solutions for the periodic differential equation

\[
\begin{align*}
x'(t) &= a(t)p(x(t))x(t) - \lambda h(t)h(x(t-\tau(t))).
\end{align*}
\]

His approach depended on fixed point theorem in a cone. An essential condition on the function \( p \) in [5] is that \( p \) is bounded above and below by positive constants on \([0, +\infty)\). Hence, the method in [5] is not necessarily suitable for functional differential equation with general nonlinear term \( p \). For example, to our best knowledge, results about periodic solutions for the following functional differential equation

\[
\begin{align*}
x'(t) &= a(t)x^\alpha(t) - \lambda h(t)f(x(t-\tau(t))).
\end{align*}
\]
are few, here $\alpha \neq 0$ is a constant and $\lambda > 0$ is a positive real parameter.

In the paper, we obtain sufficient conditions for the existence of periodic solutions for the system (1.1) by using Schauder’s fixed point theorem. Our results improve and generalize the corresponding results of [1, 6, 10].

2. Main results

The following well-known Schauder’s fixed point theorem is crucial in our arguments.

Lemma 2.1. Let $X$ be a Banach space with $D \subset X$ closed and convex. Assume that $T: D \to D$ is a completely continuous map, then $T$ has a fixed point in $D$.

Put $C_\omega = \{ u \in C(R, R) : u(t + \omega) = u(t), t \in R \}$ with the norm defined by $\|u\|_{C_\omega} = \max_{0 \leq t \leq \omega} |u(t)|$ and

$$E = \{ x = (x_1(t), \ldots, x_n(t)) : x_i \in C_\omega \}, \quad \|x\|_E = \sum_{i=1}^n \|x_i\|_{C_\omega}.$$ 

Then $C_\omega$ and $E$ are Banach spaces.

Let $p, q \in C_\omega$ and consider the following two differential equations

$$x'(t) = -p(t)x(t) + q(t), \quad (2.1)$$

$$x'(t) = p(t)x(t) - q(t). \quad (2.2)$$

Lemma 2.2. Assume that $\int_0^\omega p(t)dt \neq 0$, then (2.1) has a unique $\omega$-periodic solution

$$x(t) = \int_t^{t+\omega} \frac{\exp \int_t^s p(r)dr}{\exp \int_0^\omega p(r)dr - 1} q(s) \, ds$$

and (2.2) has a unique $\omega$-periodic solution

$$x(t) = \int_t^{t+\omega} \frac{\exp \int_t^t p(r)dr}{\exp \int_0^\omega p(r)dr - 1} q(s) \, ds.$$ 

Let $M \in R, m \in R: M > m$ and define

$$\prec_{[m,M]} = \{ i : g_i(m) \leq g_i(M), 1 \leq i \leq n \},$$

$$\succ_{[m,M]} = \{ i : g_i(m) > g_i(M), 1 \leq i \leq n \}.$$ 

By using Schauder’s fixed point theorem, we obtain the following existence result on the periodic solution for (1.1).

Theorem 2.1. Assume that there exist constants $M_i > m_i, i = 1, 2, \ldots, n$ such that $g_i \in C^1([m_i, M_i], R), f_i \in C(R \times \Lambda, R)$, here $\Lambda = [m_1, M_1] \times \cdots \times [m_n, M_n]$, and for any $u_i \in [m_i, M_i]$ and $t \in [0, \omega]$,

$$g_i(M_i) \leq \frac{f_i(t, u_1, \ldots, u_n)}{a_i(t)} \leq g_i(m_i) \quad \text{if} \quad i \in \succ_{[m_i, M_i]}.$$
Then \((1.1)\) has at least one periodic solution \((x_1^*(t), \ldots, x_n^*(t)) \in E\) with \(m_i \leq x_i^* \leq M_i (1 \leq i \leq n)\).

**Proof.** Without loss of the generality, we assume that there exists a \(k\): \(0 \leq k \leq n\) such that

\[
i \in \succ_{[m_i, M_i]} \quad \text{for} \quad 1 \leq i \leq k, \quad i \in \prec_{[m_i, M_i]} \quad \text{for} \quad k + 1 \leq i \leq n,
\]

here if \(i \leq 0\), \(\succ_{[m_i, M_i]} = \phi\), if \(i \geq n + 1\), \(\prec_{[m_i, M_i]} = \phi\).

Since \(g_i \in C^1([m_i, M_i], R)\), there exist \(l_i > 0\) such that

\[
1 + \frac{1}{l_i} g'_i(u) > 0, \quad u \in [m_i, M_i], \quad i = 1, 2, \ldots, k,
\]

\[
1 - \frac{1}{l_i} g'_i(u) > 0, \quad u \in [m_i, M_i], \quad i = 1 + k, \ldots, n.
\]

Assume that \((x_1(t), \ldots, x_n(t)) \in E\) is a solution of \((1.1)\), then

\[
x'_i(t) = -l_i a_i(t) x_i(t) + a_i(t) \left[ g_i(x_i(t)) + l_i x_i(t) - \frac{f_i(t, X(t - \tau(t)))}{a_i(t)} \right], \quad i = 1, 2, \ldots, k,
\]

\[
x'_i(t) = l_i a_i(t) x_i(t) - a_i(t) \left[ l_i x_i(t) - g_i(x_i(t)) + \frac{f_i(t, X(t - \tau(t)))}{a_i(t)} \right], \quad i = k + 1, \ldots, n
\]

and

\[
x_i(t) = \int_t^{t+\omega} a_i(s) \exp \int_t^s l_i a_i(r) dr - 1 \left[ g_i(x_i(s)) + l_i x_i(s) - \frac{f_i(s, X(s - \tau(s)))}{a_i(s)} \right] ds,
\]

\[
i = 1, 2, \ldots, k,
\]

\[
x_i(t) = \int_t^{t+\omega} a_i(s) \exp \int_t^s l_i a_i(r) dr - 1 \left[ l_i x_i(s) - g_i(x_i(s)) + \frac{f_i(s, X(s - \tau(s)))}{a_i(s)} \right] ds,
\]

\[
i = k + 1, \ldots, n,
\]

where \(f_i(t, X(t - \tau(t)) = f_i(t, x_1(t - \tau_1(t)), \ldots, x_n(t - \tau_n(t)))\).

Define a set \(\Omega\) in \(E\) and an operator \(T: E \to E\) by

\[
\Omega = \{ x \in E : m_i \leq x_i \leq M_i, i = 1, 2, \ldots, n \},
\]

\[
(Tx)(t) = (T x_1(t), T x_2(t), \ldots, T x_n(t)), \quad x = (x_1(t), \ldots, x_n(t)) \in E.
\]
where
\[
(Tx_i)(t) = \int_t^{t+\omega} \frac{a_i(s) \exp \int_s^t l_i a_i(r) \, dr}{\exp \int_0^\omega l_i a_i(r) \, dr - 1} \left[ g_i(x_i(s)) + l_i x_i(s) - \frac{f_i(s, X(s - \tau(s)))}{a_i(s)} \right] \, ds, \quad 1 \leq i \leq k,
\]
\[
(Tx_i)(t) = \int_t^{t+\omega} \frac{a_i(s) \exp \int_s^t l_i a_i(r) \, dr}{\exp \int_0^\omega l_i a_i(r) \, dr - 1} \left[ l_i x_i(s) - g_i(x_i(s)) + \frac{f_i(s, X(s - \tau(s)))}{a_i(s)} \right] \, ds, \quad k + 1 \leq i \leq n.
\]

First, we show that \(T(\Omega) \subseteq \Omega\). Using (2.3) and (2.4), we obtain that for \(x \in \Omega\),
\[
m_i + \frac{1}{t} g_i(m_i) \leq x_i(t) + \frac{1}{t} g_i(x_i(t)) \leq M_i + \frac{1}{t} g_i(M_i), \quad i = 1, 2, \ldots, k,
\]
\[
m_i - \frac{1}{t} g_i(m_i) \leq x_i(t) - \frac{1}{t} g_i(x_i(t)) \leq M_i - \frac{1}{t} g_i(M_i), \quad i = k + 1, \ldots, n.
\]

Using (2.3) and (2.4), we have
\[
(Tx_i)(t) = \int_t^{t+\omega} \frac{a_i(s) \exp \int_s^t l_i a_i(r) \, dr}{\exp \int_0^\omega l_i a_i(r) \, dr - 1} \left[ g_i(x_i(s)) + x_i(s) - \frac{f_i(s, X(s - \tau(s)))}{l_i a_i(s)} \right] \, ds
\]
\[
\leq \left[ m_i \int_t^{t+\omega} \frac{a_i(s) \exp \int_s^t l_i a_i(r) \, dr}{\exp \int_0^\omega l_i a_i(r) \, dr - 1} \, ds, M_i \int_t^{t+\omega} \frac{a_i(s) \exp \int_s^t l_i a_i(r) \, dr}{\exp \int_0^\omega l_i a_i(r) \, dr - 1} \, ds \right]
\]
\[
= [m_i, M_i], \quad i = 1, 2, \ldots, k,
\]
\[
(Tx_i)(t) = \int_t^{t+\omega} \frac{a_i(s) \exp \int_s^{t+\omega} l_i a_i(r) \, dr}{\exp \int_0^\omega l_i a_i(r) \, dr - 1} \left[ l_i x_i(s) - g_i(x_i(s)) + \frac{f_i(s, X(s - \tau(s)))}{a_i(s)} \right] \, ds
\]
\[
\leq \left[ m_i \int_t^{t+\omega} \frac{a_i(s) \exp \int_s^{t+\omega} l_i a_i(r) \, dr}{\exp \int_0^\omega l_i a_i(r) \, dr - 1} \, ds, M_i \int_t^{t+\omega} \frac{a_i(s) \exp \int_s^{t+\omega} l_i a_i(r) \, dr}{\exp \int_0^\omega l_i a_i(r) \, dr - 1} \, ds \right]
\]
\[
= [m_i, M_i], \quad i = k + 1, k + 2, \ldots, n.
\]

Next, we show that \(T : \Omega \rightarrow \Omega\) is completely continuous. Obviously, \(T(\Omega)\) is a uniformly bounded set and \(T\) is continuous on \(\Omega\), so it suffices to show \(T(\Omega)\) is equi-continuous by Ascoli-Arzela theorem. For any \(x \in \Omega\), we have
\[
(Tx_i)'(t) = -l_i a_i(t)(Tx_i)(t) + a_i(t) \left[ g_i(x_i(t)) + l_i x_i(t) - \frac{f_i(t, X(t - \tau(t)))}{a_i(t)} \right],
\]
\[
i = 1, 2, \ldots, k,
\]
\[
(Tx_i)'(t) = l_i a_i(t)(Tx_i)(t) - a_i(t) \left[ l_i x_i(t) - g_i(x_i(t)) + \frac{f_i(t, X(t - \tau(t)))}{a_i(t)} \right],
\]
\[
i = k + 1, \ldots, n.
\]
Since $T(\Omega)$ is bounded and $f_i, g_i, a_i$ are continuous, there exists $\rho > 0$ such that
\[
|(Tx_i)'(t)| \leq \rho, \quad x \in \Omega, \quad i = 1, 2, \ldots, n,
\]
which implies that $T(\Omega)$ is equi-continuous. So $T$ is a completely continuous operator on $\Omega$. Clearly, $\Omega$ is a close and convex set in $E$. Therefore, $T$ has a fixed point $x^* \in \Omega$ by Lemma 2.1. Furthermore, $m_i \leq x^*_i(t) \leq M_i$, which means $(x^*_1(t), \ldots, x^*_n(t)) \in E$ is an $\omega$-periodic solution of (1.1). The proof is complete. \hfill $\square$

**Remark 2.1.** Assume that all conditions of Theorem 2.1 are satisfied. Further suppose that there exist $1 \leq i_0 \leq n$ and $t_0 \in [0, \omega]$ such that any $u_i \in [m_i, M_i]$,\n
\[
(2.7) \quad \frac{f_{i_0}(t_0, u_1, \ldots, u_n)}{a_{i_0}(t_0)} < g_{i_0}(m_{i_0}) \quad \text{if} \quad i_0 \in ]m_{i_0}, M_{i_0}|
\]

and

\[
(2.8) \quad \frac{f_{i_0}(t_0, u_1, \ldots, u_n)}{a_{i_0}(t_0)} > g_{i_0}(m_{i_0}) \quad \text{if} \quad i_0 \in [m_{i_0}, M_{i_0}].
\]

Then $x^*_i > m_{i_0}$ for any $t \in [0, \omega]$.

**Proof.** Assume that there is a $t^* \in [0, \omega]$ such that $x^*_{i_0}(t^*) = m_{i_0}$. Then

\[
m_{i_0} = \int_{t^*}^{t^* + \omega} a_{i_0}(s) \exp \int_{t^*}^{s} l_{i_0} a_{i_0}(r) \, dr \exp \int_{0}^{s} l_{i_0} a_{i_0}(r) \, dr - 1 \times \left[ g_{i_0}(x^*_{i_0}(s)) + l_{i_0} x^*_{i_0}(s) - \frac{f_{i_0}(s, X^*(s - \tau(s)))}{a_{i_0}(s)} \right] ds, \quad i_0 \leq k,
\]
or

\[
m_{i_0} = \int_{t^*}^{t^* + \omega} a_{i_0}(s) \exp \int_{s}^{t^* + \omega} l_{i_0} a_{i_0}(r) \, dr \exp \int_{0}^{s} l_{i_0} a_{i_0}(r) \, dr - 1 \times \left[ l_{i_0} x^*_{i_0}(s) - g_{i_0}(x^*_{i_0}(s)) + \frac{f_{i_0}(s, X^*(s - \tau(s)))}{a_{i_0}(s)} \right] ds, \quad i_0 > k,
\]

where $f_i(t, X^*(t - \tau(t)) = f_i(t, x^*_1(t - \tau_1(t)), \ldots, x^*_n(t - \tau_n(t)))$.

On the other hand, since for $s \in [0, \omega]$,

\[
\frac{g_{i_0}(x^*_{i_0}(s))}{l_{i_0}(s)} + x^*_{i_0}(s) - \frac{f_{i_0}(s, X^*(s - \tau(s)))}{l_{i_0} a_{i_0}(s)} - m_{i_0} \geq 0 \quad \text{for} \quad i_0 \leq k,
\]

\[
x^*_{i_0}(s) - \frac{g_{i_0}(x^*_{i_0}(s))}{l_{i_0}(s)} + \frac{f_{i_0}(s, X^*(s - \tau(s)))}{l_{i_0} a_{i_0}(s)} - m_{i_0} \geq 0 \quad \text{for} \quad i_0 > k,
\]

one can obtain that for any $s \in [0, \omega]$,

\[
\frac{g_{i_0}(x^*_{i_0}(s))}{l_{i_0}(s)} + x^*_{i_0}(s) - \frac{f_{i_0}(s, X^*(s - \tau(s)))}{l_{i_0} a_{i_0}(s)} - m_{i_0} \equiv 0 \quad \text{for} \quad i_0 \leq k
\]

\[
x^*_{i_0}(s) - \frac{g_{i_0}(x^*_{i_0}(s))}{l_{i_0}(s)} + \frac{f_{i_0}(s, X^*(s - \tau(s)))}{l_{i_0} a_{i_0}(s)} - m_{i_0} \equiv 0 \quad \text{for} \quad i_0 > k,
\]
which is a contradiction since
\[
0 \geq \frac{g_{i_0}(m_{i_0})}{l_{i_0}} - \frac{f_{i_0}(t_0, X^*(t_0 - \tau(t_0)))}{l_{i_0} a_{i_0}(t_0)} > 0 \text{ for } i_0 \leq k,
\]
\[
0 \geq \frac{g_{i_0}(m_{i_0})}{l_{i_0}} + \frac{f_{i_0}(t_0, X^*(t_0 - \tau(t_0)))}{l_{i_0} a_{i_0}(t_0)} > 0 \text{ for } i_0 > k.
\]

\[\square\]

**Remark 2.2.** Assume that all conditions of Theorem 2.1 are satisfied. Further suppose that there exist constants \( m, M \) such that
\[
(2.9) \quad \frac{f_{r_0}(t_1, u_1, \ldots, u_n)}{a_{r_0}(t_1)} > g_{r_0}(M_{r_0}) \quad \text{if} \quad r_0 \in (m_{r_0}, M_{r_0})
\]
and
\[
(2.10) \quad \frac{f_{r_0}(t_1, u_1, \ldots, u_n)}{a_{r_0}(t_1)} < g_{r_0}(M_{r_0}) \quad \text{if} \quad r_0 \in (m_{r_0}, M_{r_0}).
\]
Then \( x^*_{r_0} < M_{r_0} \) for any \( t \in [0, \omega] \).

Consider the equations
\[
(2.11) \quad x'(t) = -a(t)x(t) + f(t, x(t - \tau(t))) \quad \text{if} \quad f \text{ is } \omega\text{-periodic in } t, \quad a, \tau \text{ are } \omega\text{-periodic continuous functions and } a(t) > 0 
\]
for all \( t \in R \).

**Corollary 2.1.** Assume that there exist constants \( M > m \) such that \( f \in C(R \times [m, M], R) \) and for any \( u \in [m, M] \) and \( t \in [0, \omega] \)
\[
ma(t) \leq f(t, u) \leq Ma(t).
\]
Then (2.11) (or 2.12) has at least one periodic solution \( m \leq x \leq M \).

Next, we consider the existence of a positive \( \omega\)-periodic solution for problem (1.4). We give explicit intervals of \( \lambda \) such that (1.4) has at least one positive \( \omega\)-periodic solution.

In the following, we assume that \( a, h, \tau: R \to R \) are \( \omega\)-periodic continuous functions and \( a(t) > 0, h(t) > 0 \) for any \( t \in [0, \omega] \). \( f: (0, +\infty) \to (0, +\infty) \) is continuous.

Put
\[
\bar{f}_0 = \limsup_{t \to 0^+} \frac{f(t)}{t^\alpha}, \quad \underline{f}_0 = \liminf_{t \to 0^+} \frac{f(t)}{t^\alpha}, \quad \bar{f}_\infty = \limsup_{t \to +\infty} \frac{f(t)}{t^\alpha}, \quad \underline{f}_\infty = \liminf_{t \to +\infty} \frac{f(t)}{t^\alpha},
\]
\[
\delta^* = \max_{t \in [0, \omega]} \frac{h(t)}{a(t)}, \quad \delta = \min_{t \in [0, \omega]} \frac{h(t)}{a(t)}.
\]

**Theorem 2.2.** The problem (1.4) has at least one positive periodic solution if one of the following conditions holds:
\((H_1)\quad \alpha < 0, \liminf_{t \to 0^+} f(t) > 0, \limsup_{t \to +\infty} f(t) < +\infty \text{ and } \lambda < (f_0(\delta))^{-1};\)

\((f_\infty)^{-1} < \lambda < (f_0(\delta^*))^{-1};\)

\((H_2)\quad \alpha > 0, \limsup_{t \to 0^+} f(t) < +\infty, \liminf_{t \to +\infty} f(t) > 0 \text{ and } \lambda > (f_\infty(\delta^*)^{-1}.\)

**Proof.** Assume that \((H_1)\) holds. From the definition of \(f_0, f_\infty\) and \((H_1)\), there exist \(r_1 > 0\) and \(\bar{r}_1 > r_1\) such that

\[
\frac{\lambda h(t) f(u)}{a(t)} \leq u^\alpha, \quad 0 < u \leq r_1, \quad \inf_{u \in (0, r_1]} f(u) > 0,
\]

\[
\frac{\lambda h(t) f(u)}{a(t)} \geq u^\alpha, \quad u \geq \bar{r}_1, \quad \sup_{u \in [\bar{r}_1, +\infty)} f(u) < +\infty.
\]

Let \(\lambda \in \left(\frac{1}{f_\infty(\delta)}, \frac{1}{f_0(\delta^*)}\right).\) It is easy to check that

\[
\inf \left\{ \frac{\lambda h(t) f(u)}{a(t)} : t \in [0, \omega], u \in (0, \bar{r}_1] \right\} := \mu_1 > 0,
\]

\[
\sup \left\{ \frac{\lambda h(t) f(u)}{a(t)} : t \in [0, \omega], u \in [\bar{r}_1, +\infty) \right\} := \bar{\mu}_1 < +\infty.
\]

Put

\[m = \min \left\{ \frac{r_1}{2}, \bar{\mu}_1^\frac{1}{\alpha} \right\}, \quad M = \max \left\{ 2\bar{r}_1, \mu_1^\frac{1}{\alpha} \right\}, \]

then

\[
M^{\alpha} \leq \mu_1 \leq \frac{\lambda h(t) f(u)}{a(t)} \leq x^{\alpha} \leq m^{\alpha}, \quad m \leq u \leq r_1,
\]

\[
M^{\alpha} \leq x^{\alpha} \leq \frac{\lambda h(t) f(u)}{a(t)} \leq \bar{\mu}_1 \leq m^{\alpha}, \quad \bar{r}_1 \leq u \leq M.
\]

On the other hand,

\[
M^{\alpha} \leq \mu_1 \leq \frac{\lambda h(t) f(u)}{a(t)} \leq \bar{\mu}_1 \leq m^{\alpha}, \quad r_1 \leq u \leq \bar{r}_1.
\]

Hence,

\[
M^{\alpha} \leq \frac{\lambda h(t) f(u)}{a(t)} \leq m^{\alpha}, \quad m \leq u \leq M.
\]

By Theorem 2.1 (1.4) has at least one periodic solution \(x \in C_\omega : 0 < m \leq x \leq M.\)

Assume that \((H_2)\) holds. There exist \(0 < r_3 < 1\) and \(\bar{r}_3 > 1\) such that

\[
\frac{\lambda h(t) f(u)}{a(t)} \geq u^\alpha, \quad 0 < u \leq r_3, \quad \sup_{u \in (0, r_3]} f(u) < +\infty,
\]

\[
\frac{\lambda h(t) f(u)}{a(t)} \leq u^\alpha, \quad u \geq \bar{r}_3, \quad \inf_{u \in [\bar{r}_3, +\infty)} f(u) > 0.
\]
Let \( \lambda \in \left( \frac{1}{f_0}, \frac{1}{f_\infty} \right) \), then
\[
\inf \left\{ \frac{\lambda h(t)f(u)}{a(t)} : t \in [0, \omega], u \in [r_3, +\infty) \right\} := \mu_3 > 0,
\]
\[
\sup \left\{ \frac{\lambda h(t)f(u)}{a(t)} : t \in [0, \omega], u \in (0, \bar{r}_3) \right\} := \bar{\mu}_3 < +\infty.
\]
Put
\[
m = \min \left\{ \frac{r_3}{2}, \frac{\bar{\mu}_3}{\alpha} \right\}, \quad M = \max \left\{ 2\bar{r}_3, \bar{\mu}_3^\frac{1}{\alpha} \right\},
\]
then
\[
m^\alpha \leq \frac{\lambda h(t)f(u)}{a(t)} \leq M^\alpha, \quad m \leq u \leq M.
\]
By Theorem 2.1, (1.4) has at least one periodic solution \( x \in C_\omega : 0 < m \leq x \leq M \).
The proof is complete. \( \Box \)

**Corollary 2.2.**

1. Assume that \( \alpha < 0 \) and \( 0 < \liminf_{t \to 0^+} f(t) \leq \limsup_{t \to 0^+} f(t) < +\infty \),
then (1.4) has at least one positive periodic solution for sufficiently large \( \lambda > 0 \).
2. Assume that \( \alpha < 0 \) and \( 0 < \liminf_{t \to +\infty} f(t) \leq \limsup_{t \to +\infty} f(t) < +\infty \),
then (1.4) has at least one positive periodic solution for sufficiently small \( \lambda > 0 \).
3. Assume that \( \alpha > 0 \) and \( 0 < \liminf_{t \to 0^+} f(t) \leq \limsup_{t \to 0^+} f(t) < +\infty \),
then (1.4) has at least one positive periodic solution for sufficiently small \( \lambda > 0 \).
4. Assume that \( \alpha > 0 \) and \( 0 < \liminf_{t \to +\infty} f(t) \leq \limsup_{t \to +\infty} f(t) < +\infty \),
then (1.4) has at least one positive periodic solution for sufficiently large \( \lambda > 0 \).

**Proof.** Here we only prove case (1). Since \( 0 < \liminf_{t \to 0^+} f(t) \leq \limsup_{t \to 0^+} f(t) < +\infty \), there exists \( 0 < r < 1 \) such that
\[
\mu := \inf_{t \in (0, r]} f(t) \leq \sup_{t \in (0, r]} f(t) := \nu < +\infty.
\]
Let \( \lambda > 0 \) such that \( (\lambda \delta \mu)^{\frac{1}{\alpha}} < r \) and set
\[
m = (\lambda \delta \nu)^{\frac{1}{\alpha}}, \quad M = (\lambda \delta \mu)^{\frac{1}{\alpha}},
\]
then \( r > M > m > 0 \) and
\[
M^\alpha \leq \frac{\lambda h(t)f(u)}{a(t)} \leq m^\alpha, \quad m \leq u \leq M.
\]
By Theorem 2.1, (1.4) has at least one periodic solution \( x \in C_\omega : 0 < m \leq x \leq M \).
The proof is complete. \( \Box \)
3. Some examples

In this section, we apply the main results obtained in previous section to several examples.

Example 3.1. Consider the differential equation

\[ x'(t) = \frac{1}{\sqrt{\sin x(t)}} + b(t), \]  

where \( b(t) \) is a \( \omega \)-periodic continuous function.

It is easy to verify from Theorem 2.1 that (3.1) has least two periodic solutions

\[ 0 < |x_1| < 0.5\pi < |x_2| < \pi \]  

if \( |b(t)| > 1 \) for all \( t \in \mathbb{R} \). Since \( x_i + 2k\pi (i = 1, 2, k \in \mathbb{Z}) \) is also the periodic solutions of (3.1), (3.1) has infinitely many periodic solutions when \( |b(t)| > 1 \).

Example 3.2. Consider the differential equation

\[ x'(t) = \left(1 + \frac{\sin t}{100}\right)x^3(t) - f(x(t - \cos t)), \]

where

\[ f(u) = \begin{cases} 
0.1, & u < \frac{2}{3}, \\
 u^2 - u + \frac{5}{4}, & u > 1.
\end{cases} \]

In (3.2), \( a(t) = 1 + 0.01\sin t \) and \( g(x) = x^3 \). Put \( m_1 = 0.1, M_1 = 0.6, m_2 = 1.1, M_2 = 2 \), then

\[ g(m_i) \leq \frac{f(u)}{a(t)} \leq g(M_i), \quad \forall u \in [m_i, M_i], \ t \in \mathbb{R}, \ i = 1, 2. \]

By Theorem 2.1, (3.2) has two positive \( 2\pi \)-periodic solutions \( x_1, x_2 \) such that

\[ m_1 \leq x_1 \leq M_1, \ n_2 \leq x_2 \leq M_2. \]

Example 3.3. Consider the differential equation

\[ x'(t) = x^3(t) + \frac{1}{x(t)} - \lambda \left(1 + \frac{\sin t}{2}\right)(2 - \sin x(t - \cos t)) \]

where \( \lambda > 0 \) is a positive real parameter.

In (3.3), \( a(t) = 1, g(x) = x^3 + x^{-1} \) and \( f(t, u) = \lambda (1 + \frac{\sin t}{2})(2 - \sin u) \). Put

\[ m_1 = \frac{2}{9\lambda}, \quad M_1 = 1, \quad m_2 = 1, \quad M_2 = \sqrt{\frac{9\lambda}{2}}, \]

then for sufficiently large \( \lambda > 0 \),

\[ g(M_1) \leq f(t, u) \leq g(m_1), \quad u \in [m_1, M_1], \ t \in \mathbb{R}, \]

\[ g(m_2) \leq f(t, u) \leq g(M_2), \quad u \in [m_2, M_2], \ t \in \mathbb{R}. \]

By Theorem 2.1, (3.3) has two positive \( 2\pi \)-periodic solutions \( x_1 \in [m_1, M_1], x_2 \in [m_2, M_2] \) for sufficiently large \( \lambda > 0 \).
Example 3.4. Consider the differential system

\[
\begin{aligned}
  x'(t) &= (2 - \cos t)x(t) - y^2(t), \\
  y'(t) &= -2\sin y(t) + \exp(0.5x(t) - y(t)).
\end{aligned}
\]  

(3.4)

Put \( D = [0.01, 0.29] \times [0.2, 0.53] \), then for \((u_1, u_2) \in D\) and \( t \in [0, 2\pi] \),

\[
0.01 \leq \frac{u_2^2}{2 - \cos t} \leq 0.29, \quad 2\sin 0.2 \leq e^{0.5u_1-u_2} \leq 2\sin 0.53.
\]

By Theorem 2.1, (3.4) has a \( 2\pi \)-periodic solution \((x(t), y(t))\) such that \( 0.01 \leq x(t) \leq 0.29 \) and \( 0.2 \leq y(t) \leq 0.53 \).

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