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ON THE COMPLETENESS OF THE SYSTEM $\{t^{\lambda_n} \log^{m_n} t\}$ IN $C_0(E)$

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Abstract. Let $E = \bigcup\limits_{n=1}^{\infty} I_n$ be the union of infinitely many disjoint closed intervals where $I_n = [a_n, b_n]$, $0 < a_1 < b_1 < a_2 < b_2 < \ldots < b_n < \ldots$, $\lim\limits_{n\to\infty} b_n = \infty$. Let $\alpha(t)$ be a nonnegative function and $\{\lambda_n\}_{n=1}^{\infty}$ a sequence of distinct complex numbers. In this paper, a theorem on the completeness of the system $\{t^{\lambda_n} \log^{m_n} t\}$ in $C_0(E)$ is obtained where $C_0(E)$ is the weighted Banach space consists of complex functions continuous on E with $f(t)\mathrm{e}^{-\alpha(t)}$ vanishing at infinity.

Keywords: completeness, Banach space, complex Müntz theorem

MSC 2010: 30E10, 41A10

1. Introduction

Fix a *weight* $\alpha(t)$, that is a nonnegative continuous function defined on $\mathbb{R}$ such that

$$\lim_{|t|\to\infty} \frac{\alpha(t)}{\log |t|} = \infty. \tag{1}$$

The weighted Banach space C_α consists of complex continuous functions f defined on the real axis $\mathbb{R}$ with $f(t)\exp(-\alpha(t))$ vanishing at infinity, and normed by

$$\|f\|_\alpha = \sup\{|f(t)\exp(-\alpha(t))|\colon\ t \in \mathbb{R}\}$$

for $f \in C_\alpha$. Denote by $\mathbf{M}(\Lambda)$ the set of functions which are finite linear combinations of the exponential system $\{t^\lambda\colon\ \lambda \in \Lambda\}$ where $\Lambda = \{\lambda_n\colon\ n = 1, 2, \ldots\}$ is a sequence of

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complex numbers. Condition (1) guarantees that $\mathbf{M}(\Lambda)$ is a subspace of C_α. When $\Lambda = \{\lambda_n\colon\ n = 1, 2, \ldots\}$ are just all of the positive integers, the problem of density of $\mathbf{M}(\Lambda)$ in C_α in the norm $\|.\|_\alpha$ is the classical Bernstein problem on polynomial approximation in [3] and [4]. A well-known result which was obtained by S. Izumi and T. Kawata in 1937 in [9] is described as follows.

Theorem 1.1. *Suppose $\alpha(t)$ is an even function satisfying (1) and $\alpha(\mathrm{e}^t)$ is a convex function on $\mathbb{R}$. Then a necessary and sufficient condition for polynomials to be dense in the space C_α is*

$$\int_{-\infty}^{+\infty} \frac{\alpha(t)}{1+t^2}\,\mathrm{d}t = \infty.$$

Motivated by the Bernstein problem and the Müntz theorem in [3], combining Malliavin's uniqueness theorem in [11], by the approach of Fourier transform, in the papers [5]–[7], a series of intriguing results related to the Berstein polynomial approximation problem were obtained. When $\Lambda = \{\lambda_n\colon\ n = 1, 2, \ldots\}$ are a selected part of the positive integers, one particularly interesting result in [6] is described below.

Theorem 1.2. *Suppose $\alpha(t)$ is an even function satisfying (1) and $\alpha(\mathrm{e}^t)$ is a convex function on $\mathbb{R}$. Let $\Lambda = \{\lambda_n\colon\ n = 1, 2, \ldots\}$ be a sequence of strictly increasing positive integers and let*

$$\Lambda(r) = 2\sum_{\lambda_n \leqslant r} \frac{1}{\lambda_n}, \ \textit{if } r \geqslant \lambda_1; \quad \Lambda(r) = 0, \ \textit{otherwise},$$

$k(r) = \Lambda(r) - \log^+ r$, $\log^+ r = \max\{\log r, 0\}$, $\tilde{k}(r) = \inf\{k(r')\colon\ r' \geqslant r\}$. *If*

$$\int_{0}^{+\infty} \frac{\alpha(\exp\{\tilde{k}(t) - a\})}{1+t^2}\,\mathrm{d}t = \infty, \tag{2}$$

for each $a \in \mathbb{R}$, then $\mathbf{M}(\Lambda)$ is dense in C_α.

Conversely, if the sequence Λ contains all of the odd integers, then for $\mathbf{M}(\Lambda)$ to be dense in C_α, it is necessary that (2) holds for each $a \in \mathbb{R}$.

Recently, there arose an interest in the Riesz basis property in $L^2(E)$ (see [15]), where E is the union of finitely many disjoint intervals:

$$E = \bigcup_{n=1}^{l} I_n, \quad I_n = (a_n,\ b_n), \quad 0 < a_1 < b_1 < a_2 < b_2 < \ldots < b_l,\ l \geqslant 2.$$

There also arose an interest in approximation in weighted Banach spaces consisting of functions continuous on a set E which is an infinite union of closed intervals, that is $E = \bigcup\limits_{n=1}^{\infty} I_n$ and I_n are disjoint closed intervals on $\mathbb{R}$, $\operatorname{dist}(0, I_n) \to \infty$ as $n \to \infty$.

Let $C_0(E)$ denote the weighted Banach space consisting of complex functions f continuous on the union of infinitely many disjoint closed intervals E with $f(t)\exp(-\alpha(t))$ vanishing at infinity, and normed by

$$\|f\|_E = \sup\{|f(t)\exp(-\alpha(t))| \colon\ t \in E\}$$

for $f \in C_0(E)$. Let $|I_n|$ be the length of the interval I_n. In [2], the following result was obtained.

Theorem 1.3. *Suppose that*

$$|I_n| \geqslant c(\operatorname{dist}(0, I_n))^{-M} \tag{3}$$

for some $c > 0$, $M < \infty$. The polynomials are dense in $C_0(E)$ if and only if

$$\int_E \alpha(t)\omega(\mathrm{i}, \mathrm{d}t, \mathbb{C} \setminus E) = +\infty, \tag{4}$$

where $\omega(\mathrm{i}, \mathrm{d}t, \mathbb{C} \setminus E)$ is the harmonic measure for the domain $\mathbb{C} \setminus E$ as seen from i.

Let $\Lambda_1 = \{\lambda_n, m_n\}_{n=1}^{\infty}$ where $\{\lambda_n\}_{n=1}^{\infty}$ is a sequence of complex numbers and $m_n = 0, 1, \ldots, \mu_n - 1$ is the multiplicity of λ_n. By (1), it is obvious that $\{t^{\lambda_n} \log^{m_n} t\}$ is in $C_0(E)$. We say that the system $\{t^{\lambda_n} \log^{m_n} t\}$ is *complete* in $C_0(E)$ if the closure of its linear hull $\mathbf{M}(\Lambda_1)$ coincides with $C_0(E)$ (see [1]–[7], [9]–[10] and [14]–[20]). In view of the Müntz theorem (see, for example, [3] and [4]) and Theorem 1.2, it is natural to ask under what conditions can $\mathbf{M}(\Lambda_1)$ be complete in $C_0(E)$? In this paper, sufficient conditions for $\mathbf{M}(\Lambda_1)$ to be complete in $C_0(E)$ are obtained.

In contrast to the method in [5]–[7] which is a combination of Malliavin's uniqueness theorem in [11] and inverse Fourier transformation that cannot be applied in our situation, we will employ the method in [1] and [16]–[18] from which with a combination of Theorem 1.3 our completeness theorem follows.

Let E be a union of infinitely many disjoint closed intervals

$$E = \bigcup_{n=1}^{\infty} I_n, \quad I_n = [a_n, b_n], \tag{5}$$
$$0 < a_1 < b_1 < a_2 < b_2 < \ldots < b_n \to \infty.$$

Let $\alpha(t)$ be a nonnegative function satisfying

$$\alpha(t) = \alpha(a_1) + \int_{a_1}^{t} \frac{\varphi(\zeta)}{\zeta}\,\mathrm{d}\zeta \tag{6}$$

with $\varphi(t) \geqslant 0$ and $\varphi(t) \uparrow \infty$ as $t \to \infty$.

In order to present the completeness theorem, we need some definitions from [21]. We denote by $\mathbf{L}(\mathbf{c}, \mathbf{D})$ the class of all complex sequences $\mathbf{A} = \{a_n\}$, $|a_n| \leqslant |a_{n+1}|$ satisfying the following properties: (1) $n/|a_n| \to D \geqslant 0$, (2) for $n \neq k$ one has that $|a_n - a_k| \geqslant c|n-k|$ for some constant c, and (3) $\sup|\arg(a_n)| < \pi/2$. The following definition is from [21].

Definition 1.1. Let the sequence $\mathbf{A} \in \mathbf{L}(\mathbf{c}, \mathbf{D})$ and a, b be real positive numbers such that $a + b < 1$. We say that a sequence $\mathbf{B} = \{b_n\}_{n=1}^{\infty}$ belongs to the class $\mathbf{A}_{a,b}$ if for all $n \in \mathbb{N}$ we have

$$b_n \in \{z\colon\ |z - a_n| \leqslant a_n^a\},$$

and for all $k \neq n$ one of the following holds

(i) $b_k = b_n$.

(ii) $|b_k - b_n| \geqslant \max\{\mathrm{e}^{-|a_k|^b}, \mathrm{e}^{-|a_n|^b}\}$.

We may write $\mathbf{B}$ in the form of a multiplicity sequence $\Lambda_1 = \{\lambda_n, m_n\}_{n=1}^{\infty}$, by grouping together all those terms that have the same modulus, and ordering them so that $|\lambda_n| < |\lambda_{n+1}|$; this form of $\mathbf{B}$ is called $\{\lambda, m\}$ *reordering* (see [21]).

Recall $\mathbf{M}(\Lambda_1)$ is the linear hull of the system $\{t^{\lambda_n} \log^{m_n} t\}$. The main result of this paper is as follows.

Theorem 1.4. *Suppose $\alpha(t)$ is a nonnegative function satisfying* (1), (4) *and* (6) *where E is defined in* (5) *and satisfies* (3). *Moreover, suppose $\Lambda_1 = \{\lambda_n, m_n\}_{n=1}^{\infty}$ is a sequence of complex numbers which is a $\{\lambda, m\}$ reordering of $\mathbf{B} = \{b_n\} \in \mathbf{A}_{a,b}$ for a sequence $\mathbf{A} = \{a_n\} \in \mathbf{L}(\mathbf{c}, \mathbf{D})$ such that $\arg(a_n) \to 0$ as $n \to \infty$, satisfying*

$$|\arg(\lambda_n)| < \beta < \frac{\pi}{2}. \tag{7}$$

For some positive number λ, let

$$1/\eta = \max_{0<\delta<D\cos\beta} \frac{2\delta}{\sqrt{D^2\sin^2\beta + \delta^2}}(D\cos\beta - \delta)(1-\lambda). \tag{8}$$

If

$$\int_E \frac{\alpha(t)}{t^{1+\eta}}\,\mathrm{d}t = +\infty, \tag{9}$$

then $\mathbf{M}(\Lambda)$ is complete in $C_0(E)$.

The paper is organized as follows. In Section 2, a useful function which is a generalization of multiply a function in [16]–[18] and [20] will be constructed. Some preliminary lemmas will also be provided. In Section 3, the completeness theorem below will be proved.

2. Some Lemmas

In this section we prove some auxiliary results that are the basic ingredients to prove our completeness theorem. We will use the arguments similar to [16]–[18] and [20].

We consider the function

$$G(z)=\prod_{n=1}^{\infty}\left(1-\frac{z^2}{\lambda_n^2}\right)^{\mu_n} \tag{10}$$

where μ_n denotes the multiplicity of the term $1-z^2/\lambda_n^2$ and the integral

$$K_\gamma(s)=\frac{1}{2\pi \mathrm{i}}\int_{\arg(z)=\pm\gamma}\frac{\mathrm{e}^{-zs}}{G(z)}\,\mathrm{d}z,\quad s=u+\mathrm{i}v, \tag{11}$$

where γ satisfies $\beta<\gamma<\pi-\beta$ while β is defined in (7) satisfying $0<\beta<\pi/2$, the integral being taking first on $\arg(z)=\gamma$ from ∞ to 0 and then on $\arg(z)=-\gamma$ from 0 to ∞ (see Figure 2.1).

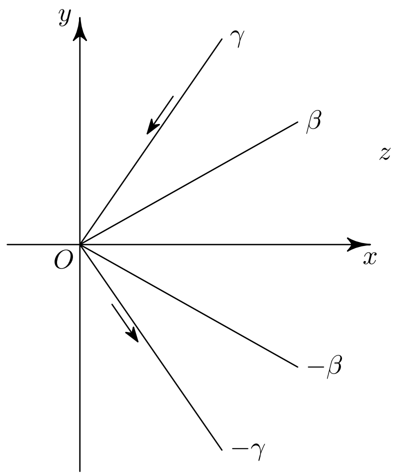

Figure 2.1

We fix some notations. Let A denote a positive constant, which may be different at each occurrence. Let $s=u+\mathrm{i}v$, $\varepsilon>0$ be a small positive number, and $D>0$ be defined in Definition 1.1. Let

$$D_\gamma^\varepsilon = \begin{cases} \{s\colon\ -u\cos\gamma + |v|\sin\gamma - \pi D\sin(\gamma-\beta) \leqslant -2\varepsilon\}, & \text{for } \beta < \gamma \leqslant \pi/2; \\ \{s\colon\ -u\cos\gamma + |v|\sin\gamma - \pi D\sin(\gamma+\beta) \leqslant -2\varepsilon\}, & \text{for } \pi/2 \leqslant \gamma < \pi - \beta, \end{cases}$$

and let

$$D_\gamma = \begin{cases} \{s\colon\ -u\cos\gamma + |v|\sin\gamma - \pi D\sin(\gamma-\beta) < 0\}, & \text{for } \beta < \gamma \leqslant \pi/2; \\ \{s\colon\ -u\cos\gamma + |v|\sin\gamma - \pi D\sin(\gamma+\beta) < 0\}, & \text{for } \pi/2 \leqslant \gamma < \pi - \beta. \end{cases}$$

We shall establish an analytic function which is related to $K_\gamma(s)$ and independent of γ. The first step is to show that $K_\gamma(s)$ is analytic in D_γ. We need the extension of a theorem of N. Levinson from [21].

Lemma 2.1. *Suppose $\Lambda_1 = \{\lambda_n, m_n\}_{n=1}^\infty$ is a sequence of complex numbers which is a $\{\lambda, m\}$ reordering of $\mathbf{B} = \{b_n\} \in \mathbf{A}_{a,b}$ for a sequence $\mathbf{A} = \{a_n\} \in \mathbf{L}(\mathbf{c}, \mathbf{D})$ such that $\arg(a_n) \to 0$ as $n \to \infty$. Then the entire function defined in (10) satisfies the following for every $\varepsilon > 0$ as $r \to \infty$:*

$$|G(r\mathrm{e}^{\mathrm{i}\theta})| = O(\exp(\pi r(D|\sin\theta| + \varepsilon))), \tag{12}$$

and whenever $r\mathrm{e}^{\mathrm{i}\theta} \notin \bigcup\limits_{n=1}^{\infty} B(\pm b_n, \frac{1}{3}\mathrm{e}^{-|a_n|^\beta})$ (where $B(z_0, r) = \{z\colon\ |z - z_0| < r\}$),

$$\frac{1}{|G(r\mathrm{e}^{\mathrm{i}\theta})|} = O(\exp(\pi r(-D|\sin\theta| + \varepsilon))). \tag{13}$$

Furthermore for every $\varepsilon > 0$ as $n \to \infty$:

$$\frac{\mu_n!}{|G^{[\mu_n]}(\lambda_n)|} = O(\exp(\varepsilon|\lambda_n|)).$$

With the aid of Lemma 2.1, we may begin with a line of investigation on $K_\gamma(s)$.

Lemma 2.2. *For $\beta < \gamma < \pi - \beta$, $K_\gamma(s)$ is analytic in D_γ and bounded in D_γ^ε.*

Proof. Since for $\beta < \gamma \leqslant \pi/2$, we have $\sin\gamma \geqslant \sin(\gamma - \beta)$, thus for $s \in D_\gamma^\varepsilon$, by (13),

$$\left|\frac{\mathrm{e}^{-sr\mathrm{e}^{\pm\mathrm{i}\gamma}}}{G(r\mathrm{e}^{\pm\mathrm{i}\gamma})}\right| \leqslant A\mathrm{e}^{(-u\cos\gamma + |v|\sin\gamma - \pi D\sin(\gamma-\beta)+\varepsilon)r} < A\mathrm{e}^{-\varepsilon r}, \tag{14}$$

where A is a constant that only depends on γ and ε. The same estimate holds for $s \in D_\gamma^\varepsilon$, $\pi/2 \leqslant \gamma < \pi - \beta$. Hence the integral on the right hand side of (11) converges absolutely and uniformly for $s \in D_\gamma^\varepsilon$. From the proof of Lemma 3.2.2 in [20] we know that $K_\gamma(s)$ is analytic and bounded in D_γ^ε (see, for example, [13], Vol. I, Theorem 17.20). Since the choice of ε is arbitrary, $K_\gamma(s)$ is analytic in D_γ. □

Taking $\gamma = \pi/2$, we have

Lemma 2.3. *The function*

$$K_{\pi/2}(s) = \frac{1}{2\pi i}\int_{\infty}^{0} \frac{e^{-isr}}{G(ir)} i\,dr + \frac{1}{2\pi i}\int_{0}^{\infty} \frac{e^{isr}}{G(-ir)}(-i)\,dr$$
$$= -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-isy}}{G(iy)}\,dy$$

is analytic in

$$D_{\pi/2} = \{s\colon\ |v| < \pi D\cos\beta\} \tag{15}$$

and bounded in

$$D_{\pi/2}^{\varepsilon} = \{s\colon\ |v| \leqslant \pi D\cos\beta - 2\varepsilon\}. \tag{16}$$

Proof. Let $\gamma = \pi/2$ in Lemma 2.2. □

Lemma 2.4. *For $\beta < \gamma_1 < \gamma_2 < \pi - \beta$, we have*

$$K_{\gamma_1}(s) = K_{\gamma_2}(s)$$

in $D_{\gamma_1}^{\varepsilon} \cap D_{\gamma_2}^{\varepsilon}$.

Proof. The proof is a method of contour integration similar to Lemma 3.2.4 in [20], here we write it down for the reader's convenience. The convergence of $K_{\gamma_1}(s)$ and $K_{\gamma_2}(s)$ follows from Lemma 2.2 immediately. Since (7) is satisfied, the function

$$\frac{e^{-zs}}{G(z)}$$

is analytic with respect to z in the domain $\{z\colon\ \beta < |\arg(z)| < \pi - \beta\}$, so we have by Cauchy's theorem (see Figure 2.2)

$$K_{\gamma_2}(s) - K_{\gamma_1}(s) = \lim_{r\to\infty}\frac{1}{2\pi i}\left(\int_{\Gamma_{\gamma_2}} + \int_{\Gamma_{-\gamma_2}} + \int_{-\Gamma_{\gamma_1}} + \int_{-\Gamma_{-\gamma_1}}\right)\frac{e^{-zs}}{G(z)}\,dz$$
$$= -\lim_{r\to\infty}\frac{1}{2\pi i}\left(\int_{C_{-\gamma_1,-\gamma_2}} + \int_{C_{\gamma_1,\gamma_2}}\right)\frac{e^{-zs}}{G(z)}\,dz$$
$$= -\lim_{r\to\infty}\frac{1}{2\pi i}\left(\int_{-\gamma_2}^{-\gamma_1} + \int_{\gamma_1}^{\gamma_2}\right)\frac{e^{-re^{i\theta}s}re^{i\theta}i}{G(re^{i\theta})}\,d\theta.$$

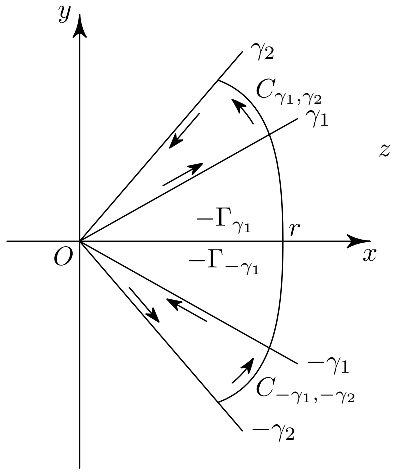

Figure 2.2

By (14), we have for $\gamma_1 \leqslant \gamma \leqslant \gamma_2$ and $-\gamma_2 \leqslant \gamma \leqslant -\gamma_1$,

$$\left|\frac{\mathrm{e}^{-sr\mathrm{e}^{\pm\mathrm{i}\gamma}}}{G(r\mathrm{e}^{\pm\mathrm{i}\gamma})}\right| < A\mathrm{e}^{-\varepsilon r},$$

where A is a constant that only depends on γ and ε. Thus

$$|K_{\gamma_1}(s) - K_{\gamma_2}(s)| \leqslant \lim_{r\to+\infty} \int_{\gamma_2}^{\gamma_1} A\mathrm{e}^{-\varepsilon r} r\,\mathrm{d}\theta = 0.$$

□

It is shown by Lemma 2.4 that $K_{\gamma_1}(s)$ and $K_{\gamma_2}(s)$ are analytic continuations of each other. Letting γ vary continuously in $(\beta, \pi-\beta)$, a function $K(s)$ which is defined by $K_\gamma(s)$ and analytic on the domain

$$F = \bigcup_{\beta<\gamma<\pi-\beta} D_\gamma$$

is obtained. It is obvious that

Lemma 2.5. *In $D_{\pi/2}$, $K(s) = K_{\pi/2}(s)$.*

For sufficiently small $\delta > 0$, denote

(17) $$B_\delta = \{s = u + \mathrm{i}v\colon\ |v| \leqslant \pi D\cos\beta - \delta\pi\}.$$

We shall consider an approximation problem in the strip B_δ which is crucial in the proof of Theorem 1.4. For fixed δ and B_δ, let $\varepsilon < (\delta\pi)/2$. By (16) and (17), it is not hard to see that

$$B_\delta \subset D^{\varepsilon}_{\pi/2}. \tag{18}$$

Let μ be a small positive number, $\gamma_1 = \pi/2 - \mu\pi$ and $\gamma_2 = \pi/2 + \mu\pi$. From the definition of $D^{\varepsilon}_{\gamma_k}$ $(k = 1, 2)$, it is not hard to verify that for sufficiently small μ and ε such that

$$\tan(\mu\pi) < \frac{\delta}{D\sin\beta}, \tag{19}$$

while D_{γ_1} and $D^{\varepsilon}_{\gamma_1}$ must contain the right half strip

$$B^+_\delta = \{s = u + \mathrm{i}v\colon\ u \geqslant 0,\ |v| \leqslant \pi D\cos\beta - \delta\pi\}, \tag{20}$$

D_{γ_2} and $D^{\varepsilon}_{\gamma_2}$ must contain the left half strip

$$B^-_\delta = \{s = u + \mathrm{i}v\colon\ u \leqslant 0,\ |v| \leqslant \pi D\cos\beta - \delta\pi\}. \tag{21}$$

Lemma 2.6. *In B_δ, the integral*

$$K(s) = K_{\pi/2}(s) = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{-\mathrm{i}sy}}{G(\mathrm{i}y)}\,\mathrm{d}y$$

is convergent uniformly and absolutely, and the function $K(s) = K_{\pi/2}(s)$ is analytic and bounded.

Proof. It is a combination of (18), Lemma 2.3 and Lemma 2.5. $\square$

Let us recall an interesting result in complex analysis (see [10], p. 79) for future use.

Lemma 2.7. *Let $f(z)$ be a function analytic in the disk $|z| \leqslant 2\mathrm{e}R$ with $|f(0)| = 1$ and let τ be an arbitrary small positive number. Then the estimate*

$$\log|f(z)| > -A(\tau)\log M_f(2\mathrm{e}R), \quad A(\tau) = \log\frac{15\mathrm{e}^3}{\tau}$$

is valid everywhere in the disk $|z| \leqslant R$ except on a set of discs with sum of diameters less than $8\tau R$.

With the aid of Lemma 2.7, we can prove the following:

Lemma 2.8. *If $G(z)$ is an entire function of exponential type with $G(0) = 1$, then there exists a sequence $\{t_k\}$ with $k \geqslant t_k \geqslant (1-\lambda)k$, $k = 1, 2, \ldots$, where λ is a sufficiently small positive number, such that*

$$\log |G(t_k \mathrm{e}^{\mathrm{i}\theta})| > -At_k,$$

where A is a constant not related to t_k.

Proof. Choosing $8\tau = \lambda < 1$ and $R = k$ in the annulus $R \geqslant |z| \geqslant (1-8\tau)R$, and applying the estimate in Lemma 2.7, the conclusion follows (see, for example, [20], p. 50). $\square$

By the same method as in [16]–[18] and [20], we get ready to verify the following estimate which will play an important role in the proof of Theorem 1.4.

Lemma 2.9. *There exists a sequence $\{t_k\}$ with $k \geqslant t_k \geqslant (1-\lambda)k$ (λ is some sufficiently small positive number) such that for $s = u + \mathrm{i}v \in B_\delta$, $\operatorname{Re} s = u \geqslant 0$,*

$$\left| K(s) - \sum_{|\lambda_n| < t_k} \sum_{m=0}^{\mu_n - 1} a_{n,m} s^m \mathrm{e}^{-\lambda_n s} \right| \leqslant A^{t_k} \mathrm{e}^{-ut_k \sin(\mu\pi)}, \tag{22}$$

and for $s = u + \mathrm{i}v \in B_\delta$, $\operatorname{Re} s = u \leqslant 0$,

$$\left| K(s) - \sum_{|\lambda_n| < t_k} \sum_{m=0}^{\mu_n - 1} a_{n,m} s^m \mathrm{e}^{-\lambda_n s} \right| \leqslant A^{t_k} \mathrm{e}^{-ut_k}, \tag{23}$$

where A is a constant independent of s and t_k, while μ is a small positive number satisfying (19).

Proof. We use the method of contour integration which is similar to [18] and [20]. From Lemma 2.1, we know that the function $G(z)$ defined in (10) is an entire function of exponential type with $G(0) = 1$. By Lemma 2.8, there exists a sequence $\{t_k\}$ with $n \geqslant t_k \geqslant (1-\lambda)k$ ($k = 1, 2, \ldots$) such that

$$\frac{1}{|G(t_k \mathrm{e}^{\mathrm{i}\theta})|} < \mathrm{e}^{-At_k}, \tag{24}$$

where A is a constant not related to t_k and λ is a sufficiently small positive number.

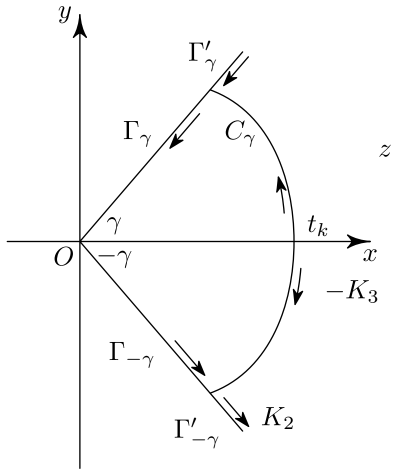

Figure 2.3

Recall $\beta < \gamma < \pi - \beta$. Choose some μ satisfying (19) and let $\gamma = \gamma_1 = \pi/2 - \mu\pi$ or $\gamma = \gamma_2 = \pi/2 + \mu\pi$. Considering (7), by the residue theorem, we have (see Figure 2.3)

$$\begin{aligned}\sum_{|\lambda_n|<t_k} \sum_{m=0}^{\mu_n-1} a_{n,m} s^m \mathrm{e}^{-\lambda_n s} &= \sum_{|\lambda_n|<t_k} \mathrm{Res}\Big[\frac{\mathrm{e}^{-zs}}{G(z)}, t_k\Big] \\ &= \frac{1}{2\pi \mathrm{i}}\bigg(\int_{\Gamma_\gamma} + \int_{\Gamma_{-\gamma}} + \int_{C_\gamma}\bigg) \frac{\mathrm{e}^{-zs}}{G(z)}\,\mathrm{d}z.\end{aligned}$$

From Lemma 2.2, we know that $K_\gamma(s)$ converges whenever $s \in D_\gamma$, thus

$$\begin{aligned}K_\gamma(s) - \sum_{|\lambda_n|<t_k} \sum_{m=0}^{\mu_n-1} a_{n,m} s^m \mathrm{e}^{-\lambda_n s} &= \frac{1}{2\pi \mathrm{i}}\bigg(\int_{\Gamma'_\gamma} + \int_{\Gamma'_{-\gamma}} - \int_{C_\gamma}\bigg) \frac{\mathrm{e}^{-zs}}{G(z)}\,\mathrm{d}z \\ &=: K_1(s) + K_2(s) - K_3(s),\end{aligned}$$

hence

$$\bigg|K(s) - \sum_{|\lambda_n|<t_k} \sum_{m=0}^{\mu_n-1} a_{n,m} s^m \mathrm{e}^{-\lambda_n s}\bigg| \leqslant |K_1(s)| + |K_2(s)| + |K_3(s)|. \tag{25}$$

In the case of $s \in B_\delta^+$ where B_δ^+ is defined in (20), we know that $s \in D_{\gamma_1}^\varepsilon$. Taking $\gamma = \gamma_1 = \pi/2 - \mu\pi$, by (14), we have

$$
\begin{aligned}
(26) \qquad |K_1(s)| &< \int_{t_k}^{\infty} A\mathrm{e}^{(-u\cos\gamma + |v|\sin\gamma - \pi D\sin(\gamma-\beta)+\varepsilon)r}\,\mathrm{d}r \\
&= \int_{t_k}^{\infty} A\mathrm{e}^{(-u\sin(\mu\pi) + |v|\cos\mu\pi - \pi D\cos(\beta+\mu\pi)+\varepsilon)r}\,\mathrm{d}r \\
&\leqslant \int_{t_k}^{\infty} A\mathrm{e}^{(-u\sin(\mu\pi) - (\delta\pi\cos\mu\pi - \pi D\sin\beta\sin(\mu\pi)-\varepsilon)r}\,\mathrm{d}r \\
&< A^{t_k}\mathrm{e}^{-ut_k\sin(\mu\pi)},
\end{aligned}
$$

where last inequality follows from choosing ε such that

$$
\delta\pi\cos\mu\pi - \pi D\sin\beta\sin(\mu\pi) - \varepsilon > 0.
$$

Applying the same reasoning to $|K_2(s)|$, a similar estimate can be obtained. From (24), we have the estimate

$$
(27) \qquad |K_3(s)| < \frac{t_k}{2\pi}\int_{-(\pi/2-\mu\pi)}^{\pi/2-\mu\pi} \frac{\mathrm{e}^{t_k(-u\cos\theta+v\sin\theta)}}{|G(t_k\mathrm{e}^{\mathrm{i}\theta})|}\mathrm{d}\theta < A^{t_k}\mathrm{e}^{-ut_k\sin(\mu\pi)},
$$

where A is a constant independent of t_k. Thus (22) follows from (25), (26) and (27). For $s \in B_\delta^-$ in (21), taking $\gamma = \gamma_2 = \pi/2 + \mu\pi$, choosing ε sufficiently such that

$$
\delta\pi\cos\mu\pi + \pi D\sin\beta\sin(\mu\pi) - \varepsilon > 0,
$$

applying the same reasoning, we can get (23). □

To prove Theorem 1.4, we also need other lemmas. The following lemma is so-called Carleman's Theorem (see [10], p. 103).

Lemma 2.10. *Let* $\log^- r = \max\{-\log r, 0\}$. *If* $g(w)$ *is analytic and bounded in the half-plane* $\mathrm{Im}(w) \geqslant 0$ *and*

$$
\int_{-\infty}^{+\infty} \frac{\log^-|g(t)|}{1+t^2}\,\mathrm{d}t = \infty,
$$

then $g(w) \equiv 0$.

We also need a result of M. M. Dzhrbasian (see [12], Sect. 10, Lemma 1]).

Lemma 2.11. *Suppose* $\alpha(t)$ *is given as in* (6), *let*

$$
M_n = \sup_{t\geqslant 0} \mathrm{e}^{-\alpha(t)}t^n
$$

and

$$\Phi(t) = \sup_{n\geqslant 1} \frac{t^n}{M_n}.$$

Then there exists some constant $A > 0$ *such that for* t *sufficiently large*

$$\log \Phi(t) \geqslant A\alpha(t).$$

Let E' denote the image of E under the transformation $\xi = \ln t$, and let ν denote a measure supported on E' satisfying

$$\int_{E'} \mathrm{e}^{\alpha(\mathrm{e}^{\xi})}\, \mathrm{d}|\nu|(\mathrm{e}^{\xi}) < \infty.$$

We define a function for $s \in B_\delta$ by

$$F(s) = \int_{E'} K(s-\xi)\, \mathrm{d}\nu(\mathrm{e}^{\xi}). \tag{28}$$

Remark 2.12. By Lemma 2.6, when $\xi \in E'$ is fixed $K(s-\xi)$ is analytic for $s \in B_\delta$; when $s \in B_\delta$ is fixed, $K(s-\xi)$ is both measurable and bounded for $\xi \in E'$. Thus, it is not hard to prove that $F(s)$ is analytic and bounded in B_δ (see [14], Chap. 10, Exercise 16; 1, Sect. 3 and [1], p. 8; also [20], Sec. 2.4).

The following lemma will be crucial in our proof of Theorem 1.4.

Lemma 2.13. *Let* ν *denote a measure supported on* E' *satisfying*

$$\int_{E'} \mathrm{e}^{\alpha(\mathrm{e}^{\xi})}\, \mathrm{d}|\nu|(\mathrm{e}^{\xi}) < \infty,$$

where E' *is the image of* E *under the transformation* $\xi = \ln t$ *and* $\alpha(t)$ *is a nonnegative function satisfying* (1), (4) *and* (6). E *is defined in* (5) *and satisfies* (3). *If for* $s \in B_\delta$, $F(s) \equiv 0$ *where* $F(s)$ *is defined by* (28), *then*

$$\int_E t^n\, \mathrm{d}\nu(t) = 0, \quad n = 0, 1, 2, \ldots. \tag{29}$$

P r o o f. It is obvious that $s - \xi \in B_\delta$ for $\xi \in E'$ and $s \in B_\delta$. From Lemma 2.6, we know that the integral

$$K(s-\xi) = -\frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\mathrm{e}^{-\mathrm{i}(s-\xi)y}}{G(\mathrm{i}y)}\, \mathrm{d}y$$

converges uniformly and absolutely with respect to $\xi \in E'$. Interchanging the order of the integrations in (28), we have

$$F(s) = -\frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{\mathrm{e}^{-\mathrm{i}sy}}{G(\mathrm{i}y)}\left[\int_{E'} \mathrm{e}^{\mathrm{i}y\xi}\,\mathrm{d}\nu(\mathrm{e}^{\xi})\right]\mathrm{d}y \equiv 0. \tag{30}$$

Define

$$k(y) = \frac{1}{G(\mathrm{i}y)}\int_{E'} \mathrm{e}^{\mathrm{i}y\xi}\,\mathrm{d}\nu(\mathrm{e}^{\xi}).$$

By properly choosing the constant $\varepsilon > 0$ such that $\varepsilon_1 = \pi D - \varepsilon > 0$, it follows from Lemma 2.1 and the definition of the measure ν that for some positive constant A, we have

$$\begin{aligned} |k(y)| &\leqslant \frac{1}{|G(\mathrm{i}y)|}\int_{E'} \mathrm{d}|\nu|(\mathrm{e}^{\xi}) \sup_{\xi\in E'} |\mathrm{e}^{\mathrm{i}y\xi}| \\ &\leqslant A\mathrm{e}^{(-\pi D+\varepsilon)|y|} \\ &< A\mathrm{e}^{-\varepsilon_1|y|}. \end{aligned}$$

By the Plancherel Theorem (see [14] Theorem 9.13) and (30), we have

$$\int_{-\infty}^{+\infty} |k(y)|^2\,\mathrm{d}y = 0$$

and for $y \in \mathbb{R}$, $k(y) = 0$, i.e.,

$$\int_{E'} \mathrm{e}^{\mathrm{i}y\xi}\,\mathrm{d}\nu(\mathrm{e}^{\xi}) = 0 \tag{31}$$

follows from the continuity of $k(y)$ in $\mathbb{R}$. Define the function

$$L(z) = \int_{E'} \mathrm{e}^{z\xi}\,\mathrm{d}\nu(\mathrm{e}^{\xi}),$$

take the transformation $t = \mathrm{e}^{\xi}$, from the theory of transformations (see [8], p. 163), we have

$$L(z) = \int_{E} t^z\,\mathrm{d}\nu(t).$$

We claim that $L(z)$ is analytic in the closed right half plane $\operatorname{Re} z \geqslant 0$. Actually, by the definition of the measure ν, the analyticity of $L(z)$ follows from Fubini's theorem and Morera's theorem. By (31), $L(\mathrm{i}y) = 0$ for any $y \in \mathbb{R}$. Thus, $L(z) = 0$ for $\operatorname{Re} z \geqslant 0$. In particularly, $L(n) = 0$, $n = 0, 1, 2, \ldots$, i.e.,

$$\int_{E} t^n\,\mathrm{d}\nu(t) = 0, \quad n = 0, 1, 2, \ldots.$$

$\square$

3. Proof of Theorem 1.4

In this section, we will prove Theorem 1.4.

Proof. If $\mathbf{M}(\Lambda_1)$ is not complete in $C_0(E)$, there exists a non-trivial bounded linear functional L such that $L(t^{\lambda_n}\log^{m_n} t) = 0$ for $\Lambda_1 = \{\lambda_n, m_n\}_{n=1}^{\infty}$ where $m_n = 0, 1, \ldots, \mu_n - 1$. (For a discussion of the bounded linear functionals in $C_0(E)$, we refer to [18] for more details.) That is, by the Riesz's representation theorem (see [14], p. 40), there exists a complex measure ν satisfying

$$\|\nu\| = \int_E \mathrm{e}^{\alpha(t)}\,\mathrm{d}|\nu|(t) = \|L\|$$

and

$$L(h) = \int_E h(t)\,\mathrm{d}\nu(t), \quad h \in C_0(E).$$

Define

$$L(z) = \int_E t^z\,\mathrm{d}\nu(t),$$

by Fubini's theorem and Morera's theorem we know that $L(z)$ is analytic in the closed right half plane $\operatorname{Re} z \geqslant 0$. Taking the transformations $t = \mathrm{e}^{\xi}$, from the theory of transformations (see [8], p. 163), we have

$$L^{(m_n)}(\lambda_n) = \int_E t^{\lambda_n}\log^{m_n} t\,\mathrm{d}\nu(t) = \int_{E'} \xi^{m_n}\mathrm{e}^{\lambda_n\xi}\,\mathrm{d}\nu(\mathrm{e}^{\xi}) = 0, \tag{32}$$

where $m_n = 0, 1, \ldots, \mu_n - 1$ and E' is the image of E.

Recall the definition of $F(s)$ in (28). To prove Theorem 1.4, it suffices to prove that if (32) holds, then $F(z) \equiv 0$ for $s \in B_\delta$. Indeed by Lemma 2.12, it will then follows that $L(n) \equiv 0$, and then from Theorem 1.3 that $L \equiv 0$, proving that $M(\Lambda)$ is complete.

For $s \in B_\delta$, let $\{t_k\}$ be the sequence defined in Lemma 2.9, with $k \geqslant t_k \geqslant (1-\lambda)k$ (λ is a sufficiently small positive number), by (11), Lemma 2.3 and Lemma 2.5, we have

$$\begin{aligned}
F(s) &= \int_{E'} K(s-\xi)\,\mathrm{d}\nu(\mathrm{e}^{\xi})\\
&= \int_{E'}\Bigg[K(s-\xi) - \sum_{|\lambda_n|<t_k}\sum_{m=0}^{\mu_n-1} a_{n,m}(s-\xi)^m\mathrm{e}^{-\lambda_n(s-\xi)}\Bigg]\,\mathrm{d}\nu(\mathrm{e}^{\xi})\\
&\quad + \int_{E'}\sum_{|\lambda_n|<t_k}\sum_{m=0}^{\mu_n-1} a_{n,m}(s-\xi)^m\mathrm{e}^{-\lambda_n(s-\xi)}\,\mathrm{d}\nu(\mathrm{e}^{\xi})\\
&=: F_{1,k}(s) + F_{2,k}(s).
\end{aligned}$$

By (32), we have

$$F_{2,k}(s) = 0.$$

Hence, for $s = u + \mathrm{i}v \in B_\delta$, $F(s) = F_{1,k}(s)$. By (22) and (23) in Lemma 2.9, we have

$$\begin{aligned}|F(s)| = |F_{1,k}(s)| &\leqslant A^{t_k} \Bigg(\mathrm{e}^{-ut_k \sin(\mu\pi)} \int_{E' \cap \{\mathrm{Re}(s-\xi) \geqslant 0\}} |\mathrm{e}^\xi|^{t_k \sin(\mu\pi)} \, \mathrm{d}\nu(\mathrm{e}^\xi) \\ &\quad + \mathrm{e}^{-ut_k} \int_{E' \cap \{\mathrm{Re}(s-\xi) \leqslant 0\}} |\mathrm{e}^\xi|^{t_k} \, \mathrm{d}\nu(\mathrm{e}^\xi) \Bigg),\end{aligned}$$

where A is a constant independent of k and s. Hence for $\mathrm{Re}\, s = u \geqslant 0$,

$$\begin{aligned}|F(s)| &\leqslant A^{t_k} \left(\frac{\int_E |t|^{t_k \sin(\mu\pi)} \mathrm{d}\nu(t)}{|\mathrm{e}^s|^{t_k \sin(\mu\pi)}} + \frac{\int_E |t|^{t_k} \, \mathrm{d}\nu(t)}{|\mathrm{e}^s|^{t_k}} \right) \\ &\leqslant A_1^{t_k} \frac{\int_E |t|^{t_k} \, \mathrm{d}\nu(t)}{|\mathrm{e}^s|^{t_k \sin(\mu\pi)}}.\end{aligned}$$

Thus, by $k \geqslant t_k \geqslant (1-\lambda)k$,

$$|F(s)| \leqslant \inf_{k \geqslant 1} A_1^k \frac{\int_E |t|^k \, \mathrm{d}\nu(t)}{|\mathrm{e}^s|^{(1-\lambda)k \sin(\mu\pi)}} \leqslant \inf_{k \geqslant 1} A_1^k \|\nu\| \frac{\sup_{t \geqslant 0} |t|^k \mathrm{e}^{-\alpha(t)}}{|\mathrm{e}^s|^{(1-\lambda)k \sin(\mu\pi)}}.$$

Let

$$M_n = \sup_{t \geqslant 0} \mathrm{e}^{-\alpha(t)} t^n$$

and

$$\Phi(t) = \sup_{n \geqslant 1} \frac{t^n}{M_n}.$$

By Lemma 2.11, for $\mathrm{Re}\, s \geqslant 0$, there exists some constant $A_2 > 0$ such that

$$|F(s)| \leqslant \mathrm{e}^{-A_2 \alpha(\overline{t})}, \tag{33}$$

where $\overline{t} = A_1 |\mathrm{e}^s|^{(1-\lambda)\sin(\mu\pi)}$. In order to use Lemma 2.10, we transform the domain B_δ into the upper half-plane $\mathrm{Im}\, z \geqslant 0$.

Let

$$m = D \cos\beta - \delta, \tag{34}$$

then B_δ is transformed into an angle $|\arg z_1| \leqslant m\pi$ by $z_1 = \mathrm{e}^s$, and the angle is transformed in the right half-plane $\mathrm{Re}\, z_2 \geqslant 0$ by $z_2 = z_1^{1/2m}$. Finally, let $z = \mathrm{i} z_2$, the domain B_δ is transformed into the upper half-plane $\mathrm{Im}\, z \geqslant 0$. More accurately, we have

$$|\mathrm{e}^s| = |z_1| = |z_2^{2m}| = |(-\mathrm{i}z)^{2m}| = |z^{2m}|$$

and

$$F(s) = F(\log z_1) = F(\log z_2^{2m}) = F(\log(-\mathrm{i}z)^{2m}).$$

Define $g(z) = F(\log(-\mathrm{i}z)^{2m})$; it is obvious that $g(z)$ is analytic and bounded in the upper half-plane $\operatorname{Im} z \geqslant 0$ (see Remark 2.1). By (33), for $\operatorname{Im} z \geqslant 0$ and $|z|$ sufficiently large, we have

$$|g(z)| \leqslant \mathrm{e}^{-A_2\alpha(A_3|z|^{2m(1-\lambda)\sin(\mu\pi)})} = \mathrm{e}^{-A_2\alpha(A_3|z|^{m'})}, \tag{35}$$

where A_3 is some positive constant independent of z, m is given by (34), and

$$m' = 2m(1-\lambda)\sin(\mu\pi) = 2(D\cos\beta - \delta)(1-\lambda)\sin(\mu\pi). \tag{36}$$

Let $\tan(\mu\pi) \to \delta/D\sin\beta$ in (19), then

$$\sin(\mu\pi) \to \frac{\delta}{\sqrt{D^2\sin^2\beta + \delta^2}}.$$

Denote

$$m'' = \frac{2\delta}{\sqrt{D^2\sin^2\beta + \delta^2}}(D\cos\beta - \delta)(1-\lambda). \tag{37}$$

By (35), for $\operatorname{Im} z \geqslant 0$ and $|z|$ sufficiently large, we have

$$|g(z)| \leqslant \mathrm{e}^{-A_2\alpha(A_3|z|^{m''})}. \tag{38}$$

It is obvious that δ can be chosen such that $0 < \delta < D\cos\beta$.

Recall the definition of η,

$$1/\eta = \max_{0<\delta<D\cos\beta} m''.$$

By (38), for $\operatorname{Im} z \geqslant 0$ and $|z|$ sufficiently large, we have

$$|g(z)| \leqslant \mathrm{e}^{-A_2\alpha(A_3|z|^{1/\eta})}. \tag{39}$$

Thus, by (39)

$$\begin{aligned}\int^{\infty} \frac{\log|g(t)|}{t^2}\,\mathrm{d}t &\leqslant \int^{\infty} \frac{-A_2\alpha(A_3t^{1/\eta})}{t^2}\,\mathrm{d}t \\ &= -A_2\frac{\eta}{A_3}\int^{\infty} \frac{\alpha(w)}{(w/A_3)^{1+\eta}}\,\mathrm{d}w \\ &= -A_4\int^{\infty} \frac{\alpha(w)}{w^{1+\eta}}\,\mathrm{d}w \\ &\leqslant -A_4\int_E \frac{\alpha(w)}{w^{1+\eta}}\,\mathrm{d}w,\end{aligned}$$

where A_4 is some positive constant independent of w. Thus, by (9), we have

$$\int^{\infty} \frac{\log |g(t)|}{t^2}\,\mathrm{d}t = -\infty.$$

Hence

$$\int^{\infty} \frac{\log |g(t)|}{1+t^2}\,\mathrm{d}t = -\infty.$$

Let $\int_{-\infty}$ mean that the upper limit of the integral is a negative number with sufficiently large magnitude. Similarly, we have

$$\int_{-\infty} \frac{\log |g(t)|}{t^2}\,\mathrm{d}t \leqslant \int_{-\infty} \frac{-A_2\alpha(A_3|t|^{1/\eta})}{t^2}\,\mathrm{d}t = \int^{\infty} \frac{-A_2\alpha(A_3 t^{1/\eta})}{t^2}\,\mathrm{d}t = -\infty.$$

Hence

$$\int_{-\infty} \frac{\log |g(t)|}{1+t^2}\,\mathrm{d}t = -\infty.$$

By Remark 2.1, we know that

$$\int \frac{\log |g(t)|}{1+t^2}\,\mathrm{d}t$$

is bounded near zero, thus

$$\int_{-\infty}^{\infty} \frac{\log |g(t)|}{1+t^2}\,\mathrm{d}t = -\infty,$$

and by Lemma 2.10, $g(z) \equiv 0$. $\square$

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