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ON THE CAUCHY PROBLEM FOR LINEAR HYPERBOLIC FUNCTIONAL-DIFFERENTIAL EQUATIONS

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Abstract. We study the question of the existence, uniqueness, and continuous dependence on parameters of the Carathéodory solutions to the Cauchy problem for linear partial functional-differential equations of hyperbolic type. A theorem on the Fredholm alternative is also proved. The results obtained are new even in the case of equations without argument deviations, because we do not suppose absolute continuity of the function the Cauchy problem is prescribed on, which is rather usual assumption in the existing literature.

Keywords: functional-differential equation of hyperbolic type, Cauchy problem, Fredholm alternative, well-posedness, existence of solutions

MSC 2010: 35L15, 35L10

1. Introduction

On the rectangle $\mathcal{D} = [a,b] \times [c,d]$, we consider the linear hyperbolic functional-differential equation

$$\frac{\partial^2 u(t,x)}{\partial t\,\partial x} = \ell(u)(t,x) + q(t,x), \tag{1.1}$$

where $\ell\colon C(\mathcal{D};\mathbb{R}) \to L(\mathcal{D};\mathbb{R})$ is a linear bounded operator and $q \in L(\mathcal{D};\mathbb{R})$. Under a solution to the equation (1.1) were understand a function $u\colon \mathcal{D} \to \mathbb{R}$ absolutely continuous on $\mathcal{D}$ in the sense of Carathéodory (see Proposition 2.1) which satisfies the equality (1.1) almost everywhere on the set $\mathcal{D}$.

For the hyperbolic equation

$$u_{tx} = p(t,x)u + q(t,x), \tag{1.2}$$

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which is a particular case of (1.1), a number of results is known especially in the case where the coefficients p and q are continuous and the solution u to the equation (1.2) is supposed to have continuous all derivatives up to the second order (see, e.g., [6], [7], [8], [12], [13], [18], [19], [20], [22] and references therein). In this case one can pass from the canonical form (1.2) to the wave equation

$$u_{tt} - u_{xx} = \tilde{p}(t,x)u + \tilde{q}(t,x)$$

and vice versa.

If the coefficients p and q in the equation (1.2) are discontinuous, the situation is much more complicated. Nevertheless, the concept of Carathéodory solutions was used and the results generalizing those known in the continuous case were obtained (see, e.g., [1], [3], [4], [9], [10], [11], [21], [22]). We follow these results and consider solutions to the equation (1.1) in the class of functions absolutely continuous on $\mathcal{D}$ in the sense of Carathéodory (see Proposition 2.1). Various initial and boundary value problems have been studied in the literature for hyperbolic equations and their systems (see, e.g., [1], [3], [4], [6], [7], [8], [9], [10], [11], [12], [13], [18], [19], [20], [21], [22], and references therein). In this paper, we investigate the Cauchy problem for the equation (1.1) formulated in the following way: Let $\mathcal{H}$ be a strictly monotone curve connecting the vertices (a,d) and (b,c) of the rectangle $\mathcal{D}$, which is defined as the graph of a decreasing continuous (not absolutely continuous in general) function $h\colon [a,b] \to [c,d]$ such that $h(a) = d$ and $h(b) = c$. The values u and $u'_{[2]}$ are prescribed on $\mathcal{H}$ as follows:[1]

$$u\,(t,h(t)) = g(t) \quad \text{for } t \in [a,b], \tag{1.3}$$

$$u'_{[2]}\left(h^{-1}(x),x\right) = \psi(x) \quad \text{for a.e. } x \in [c,d], \tag{1.4}$$

where $g \in C([a,b];\mathbb{R})$ and $\psi \in L([c,d];\mathbb{R})$. The functions g and ψ cannot be chosen arbitrarily, they must satisfy the so-called *consistency condition* (see Section 3). We should mention here that every solution u to the problem (1.1), (1.3), (1.4) verifies also the initial condition

$$u'_{[1]}\,(t,h(t)) = \varphi(t) \quad \text{for a.e. } t \in [a,b], \tag{1.5}$$

where[2]

$$\varphi(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(g(t) + \int_{h(t)}^{d} \psi(\eta)\,\mathrm{d}\eta\right) \quad \text{for a.e. } t \in [a,b]$$

[1] The symbol $u'_{[2]}$ stands for the partial derivative of u with respect to the second argument.

[2] The existence of the derivative on the right-hand side of this equality is ensured by the consistency conddition (see Proposition 3.1 below).

(see Lemmas 3.3 and 3.4). Observe that the condition (1.4) is equivalent to

$$u'_{[2]}(t,h(t)) = \psi(h(t)) \quad \text{for a.e. } t \in [a,b] \tag{1.4'}$$

provided that $h \in AC([a,b];\mathbb{R})$ and $h^{-1} \in AC([c,d];\mathbb{R})$ (see Lemma 3.1 below).

In [3], K. Deimling formulates the Cauchy problem for the hyperbolic equation with Carathéodory right-hand side as follows:

$$\begin{cases} u\,(t,h(t)) = g(t), \\ u'_{[1]}\,(t,h(t)) = \varphi(t), \\ u'_{[2]}\,(t,h(t))\,h'(t) = g'(t) - \varphi(t), \end{cases} \tag{1.6}$$

where $h \in CD([a,b];[c,d])$ is an absolutely continuous function, $g \in AC([a,b];\mathbb{R})$, and $\varphi \in L([a,b];\mathbb{R})$. He proves, among other, that the problem (1.2), (1.6) has a unique solution under the assumption $h^{-1} \in AC([c,d];\mathbb{R})$. Formulation of the Cauchy problem in the form of the initial conditions (1.3), (1.4) is more general, because we do not need to suppose that the function h is absolutely continuous. However, if we assume that $h \in AC([a,b];\mathbb{R})$ and $h^{-1} \in AC([c,d];\mathbb{R})$, then both the formulations coincide (see Remark 3.2 below).

The aim of this paper is to prove the Fredholm alternative and theorems on the continuous dependence of solutions to the problem (1.1), (1.3), (1.4) on the initial conditions and parameters (see Sections 5 and 8). Moreover, some solvability conditions for the problem considered are given in Section 7, and equations with the so-called Volterra operators are studied as well.

The results obtained are applied to the equation with argument deviations

$$\frac{\partial^2 u(t,x)}{\partial t\,\partial x} = p(t,x)u(\tau(t,x),\mu(t,x)) + q(t,x), \tag{1.1'}$$

where $p, q \in L(\mathcal{D};\mathbb{R})$ and $\tau\colon \mathcal{D} \to [a,b]$, $\mu\colon \mathcal{D} \to [c,d]$ are measurable functions.

2. Notation and preliminary results

The following notation is used throughout the paper.

(i) $\mathbb{N}$ is the set of all natural numbers. $\mathbb{R}$ is the set of all real numbers, $\mathbb{R}_+ = [0,+\infty[$. $\operatorname{Ent}(x)$ denotes the entire part of the number $x \in \mathbb{R}$.

(ii) $\mathcal{D} = [a,b] \times [c,d]$, where $-\infty < a < b < +\infty$ and $-\infty < c < d < +\infty$.

(iii) The first and the second order partial derivatives of a function $v\colon \mathcal{D} \to \mathbb{R}$ at a point $(t,x) \in \mathcal{D}$ are denoted by $v'_{[1]}(t,x)$ (or $v_t(t,x)$, $\partial v(t,x)/\partial t$), $v'_{[2]}(t,x)$ (or

$v_x(t,x)$, $\partial v(t,x)/\partial x$), $v''_{[1,2]}(t,x)$ (or $v_{tx}(t,x)$, $\partial^2 v(t,x)/\partial t\,\partial x$), and $v''_{[2,1]}(t,x)$ (or $v_{xt}(t,x)$, $\partial^2 v(t,x)/\partial x\,\partial t$).

(iv) $C(\mathcal{D};\mathbb{R})$ is the Banach space of continuous functions $v\colon \mathcal{D}\to\mathbb{R}$ equipped with the norm $\|v\|_C = \max\{|v(t,x)|\colon (t,x)\in\mathcal{D}\}$.

(v) $CD([a,b];[c,d])$ is the set of continuous decreasing functions $v\colon [a,b]\to[c,d]$ such that $v(a)=d$ and $v(b)=c$.

(vi) $AC([\alpha,\beta];\mathbb{R})$, where $-\infty<\alpha<\beta<+\infty$, is the set of absolutely continuous functions $u\colon [\alpha,\beta]\to\mathbb{R}$.

(vii) $C^*(\mathcal{D};\mathbb{R})$ is the set of functions $v\colon \mathcal{D}\to\mathbb{R}$ admitting the representation

$$v(t,x) = e + \int_a^t k(s)\,\mathrm{d}s + \int_c^x l(\eta)\,\mathrm{d}\eta + \int_a^t\int_c^x f(s,\eta)\,\mathrm{d}\eta\,\mathrm{d}s \quad \text{for } (t,x)\in\mathcal{D},$$

where $e\in\mathbb{R}$, $k\in L([a,b];\mathbb{R})$, $l\in L([c,d];\mathbb{R})$, and $f\in L(\mathcal{D};\mathbb{R})$. Equivalent definitions of the class $C^*(\mathcal{D};\mathbb{R})$ are given in Proposition 2.1 below.

(viii) $L(\mathcal{D};\mathbb{R})$ is the Banach space of Lebesgue integrable functions $p\colon \mathcal{D}\to\mathbb{R}$ equipped with the norm $\|p\|_L = \iint_{\mathcal{D}} |p(t,x)|\,\mathrm{d}t\,\mathrm{d}x$.

(ix) $\mathcal{L}(\mathcal{D})$ is the set of linear bounded operators $\ell\colon C(\mathcal{D};\mathbb{R})\to L(\mathcal{D};\mathbb{R})$.

(x) $\operatorname{mes} A$ denotes the Lebesgue measure of the set $A\subset R^m$, $m=1,2$.

(xi) If X, Y are Banach spaces and $T\colon X\to Y$ is a linear bounded operator then $\|T\|$ denotes the norm of the operator T, i.e.,

$$\|T\| = \sup\{\|T(z)\|_Y\colon z\in X, \|z\|_X\leqslant 1\}.$$

(xii) $A\div B$ stands for the symmetric difference of the sets A and B, i.e., $A\div B = (A\setminus B)\cup(B\setminus A)$.

The following proposition dealing with equivalent characterizations of functions absolutely continuous in the sense of Carathéodory plays a very important role in our investigation.

Proposition 2.1 ([16, Theorem 3.1]). *The following three statements are equivalent:*

(1) *the function $v\colon \mathcal{D}\to\mathbb{R}$ is absolutely continuous on $\mathcal{D}$ in the sense of Carathéodory*[3]*;*

(2) $v\in C^*(\mathcal{D};\mathbb{R})$*;*

(3) *the function $v\colon \mathcal{D}\to\mathbb{R}$ satisfies the conditions*

  (a) $v(\cdot,x)\in AC([a,b];\mathbb{R})$ *for every* $x\in[c,d]$, $v(a,\cdot)\in AC([c,d];\mathbb{R})$*;*

  (b) $v'_{[1]}(t,\cdot)\in AC([c,d];\mathbb{R})$ *for almost every* $t\in[a,b]$*;*

  (c) $v''_{[1,2]}\in L(\mathcal{D};\mathbb{R})$.

[3] This notion is introduced in [2] (see also [16]).

Remark 2.1. It is clear that the conditions (3a)–(3c) stated in the previous proposition can be replaced by the symmetric ones, i.e.,

(3) the function $v\colon \mathcal{D} \to \mathbb{R}$ satisfies the conditions
- (A) $v(\cdot, c) \in AC([a,b];\mathbb{R})$, $v(t,\cdot) \in AC([c,d];\mathbb{R})$ for every $t \in [a,b]$;
- (B) $v'_{[2]}(\cdot, x) \in AC([a,b];\mathbb{R})$ for almost every $x \in [c,d]$;
- (C) $v''_{[2,1]} \in L(\mathcal{D};\mathbb{R})$.

Moreover, for an arbitrary function $v \in C^*(\mathcal{D};\mathbb{R})$, the equality

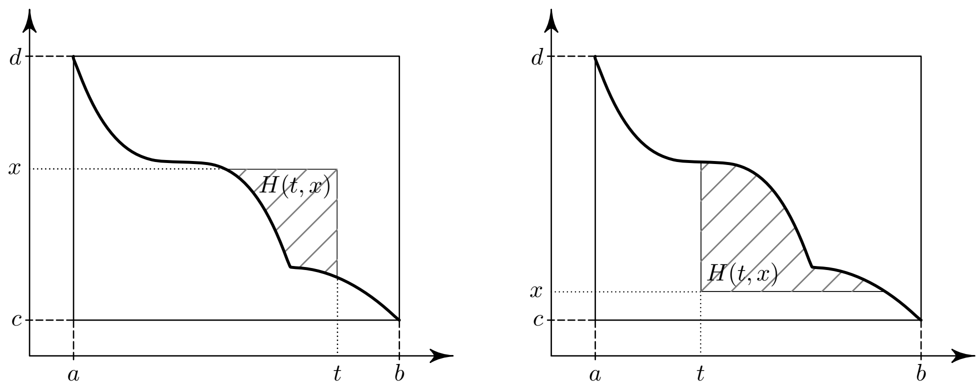
$$v''_{[1,2]}(t,x) = v''_{[2,1]}(t,x) \quad \text{for a.e. } (t,x) \in \mathcal{D}$$

holds.

Notation 2.1. Having a function $h \in CD([a,b];[c,d])$, we put

$$H(t,x) = \{(s,\eta) \in \mathbb{R}^2\colon \ \min\{h^{-1}(x), t\} \leqslant s \leqslant \max\{h^{-1}(x), t\}, \\ \min\{h(s), x\} \leqslant \eta \leqslant \max\{h(s), x\}\} \quad \text{for } (t,x) \in \mathcal{D} \tag{2.1}$$

(see the pictures below as an illustration). It is clear that, for any $(t,x) \in \mathcal{D}$, the set $H(t,x)$ is a measurable subset of $\mathcal{D}$.



3. Consistency condition

We first mention that the formulation of the Cauchy problem for the equation (1.1) in the form of the conditions (1.3) and (1.4) is rather natural. Indeed, if $u\colon \mathcal{D} \to \mathbb{R}$ is a function absolutely continuous on $\mathcal{D}$ in the sense of Carathédory (i.e., if $u \in C^*(\mathcal{D};\mathbb{R})$) then, using conditions (3a)–(3c) of Proposition 2.1 and (3A)–(3C) of Remark 2.1, we get

$$u(\cdot, h(\cdot)) \in C([a,b];\mathbb{R}), \quad u'_{[1]}(\cdot, h(\cdot)) \in L([a,b];\mathbb{R}), \quad u'_{[2]}\left(h^{-1}(\cdot), \cdot\right) \in L([c,d];\mathbb{R})$$

provided $h \in CD([a,b];[c,d])$. As was said above, the functions g and ψ appearing in the initial conditions (1.3) and (1.4) cannot be chosen arbitrarily. The following definition is motivated by the notion of a consistency condition presented in the monograph [22].

Definition 3.1. Let $h \in CD([a,b];[c,d])$, $g \in C([a,b];\mathbb{R})$, and $\psi \in L([c,d];\mathbb{R})$. We say that the pair (g,ψ) is *h-consistent* (in the space $C^*(\mathcal{D};\mathbb{R})$) if there exists a function $u \in C^*(\mathcal{D};\mathbb{R})$ satisfying the conditions (1.3) and (1.4).

Now we introduce conditions sufficient and necessary for a pair (g,ψ) to be h-consistent; their proofs are postponed till Section 3.1 below.

Proposition 3.1. *Let $h \in CD([a,b];[c,d])$, $g \in C([a,b];\mathbb{R})$, and $\psi \in L([c,d];\mathbb{R})$. Then the pair (g,ψ) is h-consistent if and only if the function*

$$t \mapsto g(t) + \int_{h(t)}^{d} \psi(\eta)\,\mathrm{d}\eta \tag{3.1}$$

is absolutely continuous on the interval $[a,b]$.

Proposition 3.2. *Let $h \in CD([a,b];[c,d])$ be an absolutely continuous function, $g \in C([a,b];\mathbb{R})$, and $\psi \in L([c,d];\mathbb{R})$. Then the pair (g,ψ) is h-consistent if and only if the function g is absolutely continuous.*

Remark 3.1. The assumption $h \in AC([a,b];\mathbb{R})$ is not necessary for the existence of an h-consistent pair. Indeed, let $g \in AC([a,b];\mathbb{R})$. Then the pair $(g,0)$ is h-consistent for an arbitrary $h \in CD([a,b];[c,d])$.

Remark 3.2. Let $h \in AC([a,b];\mathbb{R})$ and $h^{-1} \in AC([c,d];\mathbb{R})$. It follows from Proposition 3.2 that the pair (g,ψ) is h-consistent if and only if the function g is absolutely continuous. Moreover, by using Lemma 3.1 below, we easily show that the condition (1.4) is equivalent to (1.4′). Since $h^{-1} \in AC([a,b];\mathbb{R})$, the inequality $h'(t) < 0$ holds for almost every $t \in [a,b]$ (see, e.g., [14, Chapter IX, Example 13]). Therefore, if $u \in C^*(\mathcal{D};\mathbb{R})$ satisfies the initial conditions (1.3) and (1.4) then it also satisfies the conditions (1.6) with

$$\varphi(t) = g'(t) - \psi(h(t))h'(t) \quad \text{for a.e. } t \in [a,b].$$

Conversely, if $u \in C^*(\mathcal{D};\mathbb{R})$ satisfies the initial conditions (1.6) then it satisfies the conditions (1.3) and (1.4), where

$$\psi(x) = \frac{g'(h^{-1}(x)) - \varphi(h^{-1}(x))}{h'(h^{-1}(x))} \quad \text{for a.e. } x \in [c,d].$$

Consequently, the Cauchy problems formulated in the form of conditions (1.3), (1.4) and in the form of conditions (1.6) coincide in this case.

A consistent pair can be also characterized in terms of the unique solvability of the problem (1.1), (1.3), (1.4) with the zero operator ℓ. More precisely, the following statement is true.

Proposition 3.3. *Let $h \in CD([a,b];[c,d])$, $g \in C([a,b];\mathbb{R})$, and $\psi \in L([c,d];\mathbb{R})$. Then the pair (g,ψ) is h-consistent if and only if the problem* (1.1), (1.3), (1.4) *with $\ell = 0$ has a unique solution for every $q \in L(\mathcal{D};\mathbb{R})$.*[4]

3.1. Proofs. In order to prove propositions stated above we need the following lemmas.

Lemma 3.1 ([14, Chapter IX, §3, Theorem 3])**.** *Let $f \in AC([\alpha,\beta];\mathbb{R})$ be a decreasing function. Then the relation $\operatorname{mes} f(E) = f(\alpha) - f(\beta)$ holds for an arbitrary measurable set $E \subseteq [\alpha,\beta]$ such that $\operatorname{mes} E = \beta - \alpha$.*

Lemma 3.2 ([16, Proposition 3.5])**.** *Let $f \in L(\mathcal{D};\mathbb{R})$ and*

$$u(t,x) = \int_a^t \int_c^x f(s,\eta)\,\mathrm{d}\eta\,\mathrm{d}s \quad \text{for } (t,x) \in \mathcal{D}.$$

Then

(i) *there exists a set $E \subseteq [a,b]$ such that $\operatorname{mes} E = b - a$ and*

$$u'_{[1]}(t,x) = \int_c^x f(t,\eta)\,\mathrm{d}\eta \quad \text{for } t \in E \text{ and } x \in [c,d];$$

(ii) *there exists a set $F \subseteq [c,d]$ such that $\operatorname{mes} F = d - c$ and*

$$u'_{[2]}(t,x) = \int_a^t f(s,x)\,\mathrm{d}s \quad \text{for } t \in [a,b] \text{ and } x \in F;$$

(iii) *there exists a set $G \subseteq \mathcal{D}$ such that $\operatorname{mes} G = (b-a)(d-c)$ and*

$$u''_{[1,2]}(t,x) = f(t,x) \quad \text{for } (t,x) \in G.$$

[4] The symbol 0 stands here for the zero operator.

Lemma 3.3. *Let $h \in CD([a,b];[c,d])$, $g \in C([a,b];\mathbb{R})$, and $\psi \in L([c,d];\mathbb{R})$. Then an arbitrary function $u \in C^*(\mathcal{D};\mathbb{R})$ fulfilling the conditions* (1.3) *and* (1.4) *satisfies*

$$u(t,x) = g(t) + \int_{h(t)}^{x} \psi(\eta)\,\mathrm{d}\eta + \iint_{H(t,x)} u''_{[1,2]}(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta \quad \text{for } (t,x) \in \mathcal{D} \tag{3.2}$$

and

$$\begin{aligned} u(t,x) = g\left(h^{-1}(x)\right) &+ \int_{h^{-1}(x)}^{t} u'_{[1]}(s,h(s))\,\mathrm{d}s \\ &+ \iint_{H(t,x)} u''_{[1,2]}(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta \quad \text{for } (t,x) \in \mathcal{D}, \end{aligned} \tag{3.3}$$

where the mapping H is defined by the formula (2.1).

P r o o f. Let a function $u \in C^*(\mathcal{D};\mathbb{R})$ satisfy the conditions (1.3) and (1.4). Then, using properties (3A)–(3C) of Remark 2.1, we obtain

$$\begin{aligned} \iint_{H(t,x)} u''_{[1,2]}(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta &= \int_{h(t)}^{x}\int_{h^{-1}(\eta)}^{t} u''_{[2,1]}(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta \\ &= \int_{h(t)}^{x} [u'_{[2]}(t,\eta) - u'_{[2]}(h^{-1}(\eta),\eta)]\,\mathrm{d}\eta \\ &= u(t,x) - u(t,h(t)) - \int_{h(t)}^{x} u'_{[2]}(h^{-1}(\eta),\eta)\,\mathrm{d}\eta \end{aligned}$$

for $(t,x) \in \mathcal{D}$. Consequently, by virtue of the initial conditions (1.3) and (1.4), the relation (3.2) holds.

On the other hand, using properties (3a)–(3c) of Proposition 2.1, we get

$$\begin{aligned} \iint_{H(t,x)} u''_{[1,2]}(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta &= \int_{h^{-1}(x)}^{t}\int_{h(s)}^{x} u''_{[1,2]}(s,\eta)\,\mathrm{d}\eta\,\mathrm{d}s \\ &= \int_{h^{-1}(x)}^{t} [u'_{[1]}(s,x) - u'_{[1]}(s,h(s))]\,\mathrm{d}s \\ &= u(t,x) - u\left(h^{-1}(x),x\right) - \int_{h^{-1}(x)}^{t} u'_{[1]}(s,h(s))\,\mathrm{d}s \end{aligned}$$

for $(t,x) \in \mathcal{D}$ and thus, in view of the initial condition (1.3), the relation (3.3) is satisfied. □

Lemma 3.4. *Let $h \in CD([a,b];[c,d])$, $g \in C([a,b];\mathbb{R})$, and $\psi \in L([c,d];\mathbb{R})$ be such that the function* (3.1) *is absolutely continuous on the interval $[a,b]$. Moreover, let*

$$u(t,x) = g(t) + \int_{h(t)}^{x} \psi(\eta)\,\mathrm{d}\eta + \iint_{H(t,x)} f(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta \quad \textit{for } (t,x) \in \mathcal{D}, \tag{3.4}$$

where $f \in L(\mathcal{D};\mathbb{R})$ and the mapping H is defined by the formula (2.1). *Then $u \in C^*(\mathcal{D};\mathbb{R})$ and u satisfies the conditions* (1.3), (1.4),

$$u'_{[1]}(t,x) = \varphi(t) + \int_{h(t)}^{x} f(t,\eta)\,\mathrm{d}\eta \quad \textit{for a.e. } t \in [a,b] \textit{ and all } x \in [c,d], \tag{3.5}$$

$$u'_{[2]}(t,x) = \psi(x) + \int_{h^{-1}(x)}^{t} f(s,x)\,\mathrm{d}s \quad \textit{for } t \in [a,b] \textit{ and a.e. } x \in [c,d], \tag{3.6}$$

and

$$u''_{[1,2]}(t,x) = f(t,x) \quad \textit{for a.e. } (t,x) \in \mathcal{D}, \tag{3.7}$$

where the function φ is given by the relation

$$\varphi(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(g(t) + \int_{h(t)}^{d} \psi(\eta)\,\mathrm{d}\eta\right) \quad \textit{for a.e. } t \in [a,b]. \tag{3.8}$$

Proof. In view of the formula (2.1), it follows immediately from the relation (3.4) that the function u satisfies the condition (1.3). It is clear that the equality (3.4) can be rewritten to the form

$$\begin{aligned} u(t,x) = g(t) &- \int_{c}^{h(t)} \psi(\eta)\,\mathrm{d}\eta - \int_{c}^{h(t)}\int_{h^{-1}(\eta)}^{t} f(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta \\ &+ \int_{c}^{x} \psi(\eta)\,\mathrm{d}\eta - \int_{c}^{x}\int_{a}^{h^{-1}(\eta)} f(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta + \int_{c}^{x}\int_{a}^{t} f(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta \end{aligned}$$

for $(t,x) \in \mathcal{D}$. Therefore, $u(t,\cdot) \in AC([c,d];\mathbb{R})$ for every $t \in [a,b]$. Moreover, in view of Lemma 3.2 (ii) there exists a set $E_1 \subseteq [c,d]$, $\operatorname{mes} E_1 = d - c$, such that

$$u'_{[2]}(t,x) = \psi(x) - \int_{a}^{h^{-1}(x)} f(s,x)\,\mathrm{d}s + \int_{a}^{t} f(s,x)\,\mathrm{d}s \quad \text{for } t \in [a,b],\ x \in E_1,$$

whence we get $u'_{[2]}\left(h^{-1}(x),x\right) = \psi(x)$ for $x \in E_1$, and thus the function u satisfies the conditions (1.4) and (3.6).

On the other hand, the equality (3.4) can be rewritten to the form

$$
\begin{aligned}
u(t,x) = & -\int_x^d \psi(\eta)\,\mathrm{d}\eta - \int_a^{h^{-1}(x)}\int_{h(s)}^x f(s,\eta)\,\mathrm{d}\eta\,\mathrm{d}s \\
& + g(t) + \int_{h(t)}^d \psi(\eta)\,\mathrm{d}\eta - \int_a^t\int_c^{h(s)} f(s,\eta)\,\mathrm{d}\eta\,\mathrm{d}s + \int_a^t\int_c^x f(s,\eta)\,\mathrm{d}\eta\,\mathrm{d}s
\end{aligned}
$$

for $(t,x)\in\mathcal{D}$. Consequently, by using the assumption on the function (3.1), we obtain that $u(\cdot,x)\in AC([a,b];\mathbb{R})$ for every $x\in[c,d]$. Moreover, in view of Lemma 3.2 (i) there exists $E_2\subseteq[a,b]$, $\operatorname{mes}E_2=b-a$, such that

$$
u'_{[1]}(t,x)=\varphi(t)-\int_c^{h(t)} f(t,\eta)\,\mathrm{d}\eta+\int_c^x f(t,\eta)\,\mathrm{d}\eta \quad \text{for } t\in E_2,\ x\in[c,d],
$$

where the function φ is given by the relation (3.8). Therefore, $u'_{[1]}(t,\cdot)\in AC([c,d];\mathbb{R})$ for every $t\in E_2$ and, by virtue of Lemma 3.2 (iii), there exists $E_3\subseteq\mathcal{D}$ such that $\operatorname{mes}E_3=(b-a)(d-c)$ and $u''_{[1,2]}(t,x)=f(t,x)$ for $(t,x)\in E_3$. It means that the conditions (3.5) and (3.7) are fulfilled and $u''_{[1,2]}\in L(\mathcal{D};\mathbb{R})$.

We have shown that the function u satisfies the relations (1.3), (1.4), (3.5)–(3.7) and the conditions (3a)–(3c) of Proposition 2.1, and thus $u\in C^*(\mathcal{D};\mathbb{R})$. □

Lemma 3.5. *Let $f\in CD([a,b];[c,d])$ be an absolutely continuous function and $w\in C^*(\mathcal{D};\mathbb{R})$. Then the function z defined by the formula*

$$
z(t)=w(t,f(t)) \quad \text{for } t\in[a,b] \tag{3.9}
$$

is absolutely continuous.

Proof. Let $\varepsilon>0$ be arbitrary. Then there exists $\delta_1>0$ such that

$$
\iint_P |w''_{[1,2]}(s,\eta)|\,\mathrm{d}s\,\mathrm{d}\eta<\frac{\varepsilon}{6} \quad \text{for } P\subseteq\mathcal{D},\ \operatorname{mes}P<\delta_1^2. \tag{3.10}
$$

Moreover, there exists $\delta_2>0$, $\delta_2\leqslant\delta_1$, such that

$$
\int_I |w'_{[1]}(s,f(s))|\,\mathrm{d}s<\frac{\varepsilon}{3} \quad \text{for } I\subseteq[a,b],\ \operatorname{mes}I<\delta_2, \tag{3.11}
$$

$$
\int_J |w'_{[2]}\left(f^{-1}(\eta),\eta\right)|\,\mathrm{d}\eta<\frac{\varepsilon}{3} \quad \text{for } J\subseteq[c,d],\ \operatorname{mes}J<\delta_2. \tag{3.12}
$$

Since the function f is absolutely continuous, there exists $\delta>0$, $\delta\leqslant\delta_2$, such that the relation

$$
\sum_{k=1}^n |f(b_k)-f(a_k)|<\delta_2 \tag{3.13}
$$

holds for an arbitrary system $\{]a_k, b_k[\}_{k=1}^n$ of disjoint intervals in $[a,b]$ with the property

$$\sum_{k=1}^{n}(b_k - a_k) < \delta. \tag{3.14}$$

Now let $\{]a_k, b_k[\}_{k=1}^n$ be a system of disjoint intervals in $[a,b]$ satisfying (3.14). Since the function f is decreasing, $\{]f(b_k), f(a_k)[\}_{k=1}^n$ forms a system of disjoint intervals in $[c,d]$ such that (3.13) holds, and $\{[a_k, b_k] \times [f(b_k), f(a_k)]\}_{k=1}^n$ is a system of non-overlapping rectangles contained in $\mathcal{D}$ fulfilling

$$\sum_{k=1}^{n}(b_k - a_k)(f(a_k) - f(b_k)) \leqslant \delta \sum_{k=1}^{n}(f(a_k) - f(b_k)) < \delta\delta_2 \leqslant \delta_1^2. \tag{3.15}$$

It is not difficult to verify that, for any $k = 1, 2, \ldots, n$, we have

$$\begin{aligned} z(b_k) - z(a_k) &= w(b_k, f(b_k)) - w(a_k, f(a_k)) \\ &= \int_{a_k}^{b_k} w'_{[1]}(s, f(b_k))\,\mathrm{d}s - \int_{f(b_k)}^{f(a_k)} w'_{[2]}(a_k, \eta)\,\mathrm{d}\eta \\ &= \int_{a_k}^{b_k} w'_{[1]}(s, f(s))\,\mathrm{d}s - \int_{a_k}^{b_k}\int_{f(b_k)}^{f(s)} w''_{[1,2]}(s,\eta)\,\mathrm{d}\eta\,\mathrm{d}s \\ &\quad - \int_{f(b_k)}^{f(a_k)} w'_{[2]}\left(f^{-1}(\eta), \eta\right)\,\mathrm{d}\eta + \int_{f(b_k)}^{f(a_k)}\int_{a_k}^{f^{-1}(\eta)} w''_{[2,1]}(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta, \end{aligned}$$

whence we get

$$\begin{aligned} |z(b_k) - z(a_k)| \leqslant \int_{a_k}^{b_k} \left|w'_{[1]}(s, f(s))\right|\,\mathrm{d}s + \int_{f(b_k)}^{f(a_k)} \left|w'_{[2]}\left(f^{-1}(\eta), \eta\right)\right|\,\mathrm{d}\eta \\ + 2\int_{a_k}^{b_k}\int_{f(b_k)}^{f(a_k)} |w''_{[1,2]}(s,\eta)|\,\mathrm{d}\eta\,\mathrm{d}s \quad \text{for } k = 1, 2, \ldots, n. \end{aligned}$$

Consequently,

$$\begin{aligned} \sum_{k=1}^{n} |z(b_k) - z(a_k)| \leqslant \int_I \left|w'_{[1]}(s, f(s))\right|\,\mathrm{d}s \\ + \int_J \left|w'_{[2]}\left(f^{-1}(\eta), \eta\right)\right|\,\mathrm{d}\eta + 2\iint_E |w''_{[1,2]}(s,\eta)|\,\mathrm{d}s\,\mathrm{d}\eta, \end{aligned}$$

where $I = \bigcup_{k=1}^{n} [a_k, b_k]$, $J = \bigcup_{k=1}^{n} [f(b_k), f(a_k)]$, and $E = \bigcup_{k=1}^{n} [a_k, b_k] \times [f(b_k), f(a_k)]$. The last relation, together with (3.10)–(3.15), guarantees that

$$\sum_{k=1}^{n} |z(b_k) - z(a_k)| \leqslant \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + 2\,\frac{\varepsilon}{6} = \varepsilon,$$

and thus the function z is absolutely continuous. □

Now we are in a position to prove Propositions 3.1–3.3.

P r o o f of Proposition 3.1. First suppose that the pair (g, ψ) is h-consistent. Then there exists a function $u \in C^*(\mathcal{D}; \mathbb{R})$ satisfying the conditions (1.3) and (1.4). According to Lemma 3.3, the function u admits the representations (3.2) and (3.3), whose comparing and setting $x = d$ yields

$$g(t) + \int_{h(t)}^{d} \psi(\eta)\,\mathrm{d}\eta = g(a) + \int_a^t u'_{[1]}(s, h(s))\,\mathrm{d}s \quad \text{for } t \in [a, b],$$

whereas $u'_{[1]}(\cdot, h(\cdot)) \in L([a, b]; \mathbb{R})$. It means, however, that the function (3.1) is absolutely continuous on the interval $[a, b]$.

Now suppose that h, g, and ψ are such that the function (3.1) is absolutely continuous on the interval $[a, b]$. Then, by virtue of Lemma 3.4, the function u defined by the formula

$$u(t, x) = g(t) + \int_{h(t)}^{x} \psi(\eta)\,\mathrm{d}\eta \quad \text{for } (t, x) \in \mathcal{D} \tag{3.16}$$

belongs to the set $C^*(\mathcal{D}; \mathbb{R})$ and satisfies the initial conditions (1.3) and (1.4). Consequently, the pair (g, ψ) is h-consistent. □

P r o o f of Proposition 3.2. If the pair (g, ψ) is h-consistent then, using Lemma 3.5, we obtain that the function g is absolutely continuous on the interval $[a, b]$.

Conversely, let $g \in AC([a, b]; \mathbb{R})$. Then the function

$$(t, x) \mapsto g(a) + \int_c^d \psi(\eta)\,\mathrm{d}\eta + \int_a^t g'(s)\,\mathrm{d}s - \int_c^x \psi(\eta)\,\mathrm{d}\eta$$

is of the class $C^*(\mathcal{D}; \mathbb{R})$. Therefore, by virtue of Lemma 3.5, it is clear that the function (3.1) is absolutely continuous on the interval $[a, b]$ and thus the pair (g, ψ) is h-consistent (see Proposition 3.1). □

P r o o f of Proposition 3.3. If the problem (1.1), (1.3), (1.4) with $\ell = 0$ has a unique solution for every $q \in L(\mathcal{D};\mathbb{R})$ then it is clear that the pair (g,ψ) is h-consistent.

Conversely, let the pair (g,ψ) be h-consistent and let $q \in L(\mathcal{D};\mathbb{R})$. Then, according to Proposition 3.1, the function (3.1) is absolutely continuous on the interval $[a,b]$ and thus, by virtue of Lemma 3.4, the problem (1.1), (1.3), (1.4) with $\ell = 0$ has at least one solution. The uniqueness follows from Lemma 3.3. □

4. Auxiliary statements

The following proposition plays a crucial role in the proofs of statements given in Sections 5, 7, and 8.

Proposition 4.1. *Suppose that $h \in CD([a,b];[c,d])$ and $\ell \in \mathcal{L}(\mathcal{D})$. Then the operator $T\colon C(\mathcal{D};\mathbb{R}) \to C(\mathcal{D};\mathbb{R})$ defined by the formula*

$$T(v)(t,x) = \iint_{H(t,x)} \ell(v)(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta \quad \text{for } (t,x) \in \mathcal{D},\ v \in C(\mathcal{D};\mathbb{R}), \tag{4.1}$$

where the mapping H is given by (2.1)*, is completely continuous.*

The above statement can be easily proved in the case where the operator ℓ is strongly bounded, i.e., if there exists a function $\eta \in L(\mathcal{D};\mathbb{R}_+)$ such that

$$|\ell(v)(t,x)| \leqslant \eta(t,x)\|v\|_C \quad \text{for a.e. } (t,x) \in \mathcal{D} \text{ and all } v \in C(\mathcal{D};\mathbb{R}). \tag{4.2}$$

However, Schaefer proved that there exists an operator $\ell \in \mathcal{L}(\mathcal{D})$ which is not strongly bounded (see [15]). In order to prove Proposition 4.1 without the additional requirement (4.2) we need several notions and statements from functional analysis.

Definition 4.1 ([5, Definition II.3.25])**.** Let X be a Banach space, X^* its dual space.

We say that a sequence $\{x_n\}_{n=1}^{+\infty} \subseteq X$ is weakly convergent if there exists $x \in X$ such that $f(x) = \lim\limits_{n\to+\infty} f(x_n)$ for every $f \in X^*$. The element x is said to be the weak limit of this sequence.

A set $M \subseteq X$ is said to be weakly sequentially compact if every sequence of elements from M contains a subsequence which is weakly convergent in X.

A sequence $\{x_n\}_{n=1}^{+\infty}$ of elements from X is called a weak Cauchy sequence if $\{f(x_n)\}_{n=1}^{+\infty}$ is a Cauchy sequence in $\mathbb{R}$ for every $f \in X^*$.

We say that the space X is weakly complete if every weak Cauchy sequence of elements from X possesses a weak limit in X.

Definition 4.2 ([5, Definition VI.4.1]). Let X and Y be Banach spaces, $T\colon X \to Y$ a linear bounded operator. The operator T is said to be weakly compact if it maps bounded sets in X into weakly sequentially compact subset of Y.

Definition 4.3. We say that a set $M \subseteq L(\mathcal{D};\mathbb{R})$ has a property of absolutely continuous integral if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that the relation

$$\left|\iint_E p(t,x)\,\mathrm{d}t\,\mathrm{d}x\right| < \varepsilon \quad \text{for every } p \in M$$

holds whenever the measurable set $E \subseteq \mathcal{D}$ is such that $\operatorname{mes} E < \delta$.

The following three lemmas can be found in [5].

Lemma 4.1 (Theorem IV.8.6). *The space $L(\mathcal{D};\mathbb{R})$ is weakly complete.*

Lemma 4.2 (Theorem VI.7.6). *A linear bounded operator mapping the space $C(\mathcal{D};\mathbb{R})$ into a weakly complete Banach space is weakly compact.*

Lemma 4.3 (Corollary IV.8.11). *If a set $M \subseteq L(\mathcal{D};\mathbb{R})$ is weakly sequentially compact then it has a property of absolutely continuous integral.*

Proof of Proposition 4.1. Let $M \subseteq C(\mathcal{D};\mathbb{R})$ be a bounded set. We shall show that the set $T(M) = \{T(v)\colon v \in M\}$ is relatively compact in the space $C(\mathcal{D};\mathbb{R})$. According to the Arzelà-Ascoli lemma, it is sufficient to show that the set $T(M)$ is bounded and equicontinuous.

Boundedness. It is clear that

$$|T(v)(t,x)| \leqslant \iint_{H(t,x)} |\ell(v)(s,\eta)|\,\mathrm{d}s\,\mathrm{d}\eta \leqslant \|\ell(v)\|_L \leqslant \|\ell\|\,\|v\|_C$$

for $(t,x) \in \mathcal{D}$ and every $v \in M$. Therefore, the set $T(M)$ is bounded in $C(\mathcal{D};\mathbb{R})$.

Equicontinuity. Let $\varepsilon > 0$ be arbitrary. Lemmas 4.1 and 4.2 yield that the operator ℓ is weakly completely continuous, that is, the set $\ell(M) = \{\ell(v)\colon v \in M\}$ is a weakly relatively compact subset of $L(\mathcal{D};\mathbb{R})$. Therefore, Lemma 4.3 guarantees that there exists $\delta > 0$ such that the relation

$$\left|\iint_E \ell(v)(t,x)\,\mathrm{d}t\,\mathrm{d}x\right| < \frac{\varepsilon}{4} \quad \text{for } v \in M \tag{4.3}$$

holds for every measurable set $E \subseteq \mathcal{D}$ satisfying $\operatorname{mes} E < \max\{b-a, d-c\}\delta$.

On the other hand, we have

$$
\begin{aligned}
&|T(v)(t_2,x_2)-T(v)(t_1,x_1)| \\
&\qquad = \left|\iint_{H(t_2,x_2)} \ell(v)(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta - \iint_{H(t_1,x_1)} \ell(v)(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta\right| \\
&\qquad \leqslant \sum_{k=1}^{4}\left|\iint_{E_k} \ell(v)(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta\right| \quad \text{for } (t_1,x_1),(t_2,x_2)\in\mathcal{D},\ v\in M,
\end{aligned}
$$

where measurable sets $E_k\subseteq\mathcal{D}$ $(k=1,\dots,4)$ are such that $\operatorname{mes}E_k\leqslant(d-c)|t_2-t_1|$ for $k=1,2$ and $\operatorname{mes}E_k\leqslant(b-a)|x_2-x_1|$ for $k=3,4$. Hence, by virtue of the relation (4.3), we get

$$
\begin{aligned}
&|T(v)(t_2,x_2)-T(v)(t_1,x_1)|<\varepsilon \\
&\qquad\qquad \text{for } (t_1,x_1),(t_2,x_2)\in\mathcal{D},\ |t_2-t_1|+|x_2-x_1|<\delta, \text{ and } v\in M,
\end{aligned}
$$

i.e., the set $T(M)$ is equicontinuous in $C(\mathcal{D};\mathbb{R})$. □

5. Fredholm alternative

Throughout this section, we fix a function $h\in CD([a,b];[c,d])$. Along with the problem (1.1), (1.3), (1.4) we consider the corresponding homogeneous problem

$$\frac{\partial^2 u(t,x)}{\partial t\,\partial x}=\ell(u)(t,x), \tag{1.1_0}$$

$$u\,(t,h(t))=0 \quad \text{for } t\in[a,b], \tag{1.3_0}$$

$$u'_{[2]}\left(h^{-1}(x),x\right)=0 \quad \text{for a.e. } x\in[c,d]. \tag{1.4_0}$$

Observe that the homogeneous problem (1.1_0), (1.3_0), (1.4_0) is well-defined because the pair $(0,0)$ is obviously h-consistent.

Now we establish the main result of this section, namely, the statement on the Fredholmity of the problem (1.1), (1.3), (1.4).

Theorem 5.1. *The problem* (1.1), (1.3), (1.4) *has a unique solution for an arbitrary h-consistent pair (g,ψ) and every $q\in L(\mathcal{D};\mathbb{R})$ if and only if the corresponding homogeneous problem* (1.1_0), (1.3_0), (1.4_0) *has only the trivial solution.*

P r o o f. Let u be a solution to the problem (1.1), (1.3), (1.4). According to Lemma 3.3, u is a solution to the equation

$$v=T(v)+f \tag{5.1}$$

in the space $C(\mathcal{D};\mathbb{R})$, where the operator T is defined by the formula (4.1),

$$f(t,x) = g(t) + \int_{h(t)}^{x} \psi(\eta)\,\mathrm{d}\eta + \iint_{H(t,x)} q(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta \quad \text{for } (t,x) \in \mathcal{D}, \tag{5.2}$$

and the mapping H is given by the formula (2.1).

Conversely, if the pair (g,ψ) is h-consistent, $q \in L(\mathcal{D};\mathbb{R})$, and $v \in C(\mathcal{D};\mathbb{R})$ is a solution to the equation (5.1) with f given by (5.2) then, by virtue of Lemma 3.4, $v \in C^*(\mathcal{D};\mathbb{R})$ and v is a solution to the problem (1.1), (1.3), (1.4). Hence, the problem (1.1), (1.3), (1.4) and the equation (5.1) are equivalent in this sense.

Note also that u is a solution to the homogeneous problem (1.1_0), (1.3_0), (1.4_0) if and only if u is a solution to the homogeneous equation

$$v = T(v) \tag{5.3}$$

in the space $C(\mathcal{D};\mathbb{R})$.

According to Proposition 4.1, the operator T is completely continuous. It follows from the Riesz-Schauder theory that the equation (5.1) is uniquely solvable for every $f \in C(\mathcal{D};\mathbb{R})$ if and only if the homogeneous equation (5.3) has only the trivial solution. Therefore, the assertion of the theorem holds. □

Definition 5.1. Let the problem (1.1_0), (1.3_0), (1.4_0) have only the trivial solution. An operator $\Omega\colon L(\mathcal{D};\mathbb{R}) \to C(\mathcal{D};\mathbb{R})$ which assigns to every $q \in L(\mathcal{D};\mathbb{R})$ the solution u to the problem (1.1), (1.3_0), (1.4_0) is called the Cauchy operator of the problem (1.1_0), (1.3_0), (1.4_0).

Remark 5.1. It is clear that the Cauchy operator is linear.

If the homogeneous problem (1.1_0), (1.3_0), (1.4_0) has a nontrivial solution then, by virtue of Theorem 5.1, there exist a function q and an h-consistent pair (g,ψ) such that the problem (1.1), (1.3), (1.4) has either no solution or infinitely many solutions. However, as follows from the proof of Theorem 5.1, a stronger assertion can be shown in this case.

Proposition 5.1. *Let the problem* (1.1_0), (1.3_0), (1.4_0) *have a nontrivial solution. Then, for an arbitrary h-consistent pair (g,ψ), there exists a function $q \in L(\mathcal{D};\mathbb{R})$ such that the problem* (1.1), (1.3), (1.4) *has no solution.*

Proof. Let u_0 be a nontrivial solution to the problem (1.1_0), (1.3_0), (1.4_0) and let (g,ψ) be an h-consistent pair.

It follows from the proof of Theorem 5.1 that u_0 is also a nontrivial solution to the homogeneous equation (5.3) in the space $C(\mathcal{D};\mathbb{R})$. Therefore, by the Riesz-Schauder theory, there exists $f \in C(\mathcal{D};\mathbb{R})$ such that the equation (5.1) has no solution.

Then the problem (1.1), (1.3), (1.4) has no solution for $q \equiv \ell(z)$, where

$$z(t,x) = f(t,x) - g(t) - \int_{h(t)}^{x} \psi(\eta)\,\mathrm{d}\eta \quad \text{for } (t,x) \in \mathcal{D}.$$

Indeed, if the problem indicated had a solution u then the function $u + z$ would be a solution to the equation (5.1), which would lead to a contradiction. □

6. Volterra operators

The following definition introduces the notion of a $[t_0, h]$-Volterra operator which is useful in the investigation of the Cauchy problem for the equation (1.1) (see, e.g., Theorem 7.2 below).

Definition 6.1. Let $t_0 \in [a,b]$ and $h \in CD([a,b];[c,d])$. We say that $\ell \in \mathcal{L}(\mathcal{D})$ is a $[t_0, h]$-Volterra operator if the relation

$$\ell(v)(t,x) = 0 \quad \text{for a.e. } (t,x) \in [a_0, b_0] \times [h(b_0), h(a_0)]$$

holds for an arbitrary interval $[a_0, b_0] \subseteq [a,b]$ and every function $v \in C(\mathcal{D};\mathbb{R})$ such that $t_0 \in [a_0, b_0]$ and

$$v(t,x) = 0 \quad \text{for } (t,x) \in [a_0, b_0] \times [h(b_0), h(a_0)].$$

Remark 6.1. If the operator ℓ in the equation (1.1) is a $[t_0, h]$-Volterra one then the Cauchy problem (1.1), (1.3), (1.4) can be restricted to an arbitrary rectangle $[a_0, b_0] \times [h(b_0), h(a_0)] \subseteq \mathcal{D}$ containing the point $(t_0, h(t_0))$.

Let the operator ℓ be defined by the formula

$$(6.1) \quad \ell(v)(t,x) = p(t,x)v(\tau(t,x), \mu(t,x)) \quad \text{for a.e. } (t,x) \in \mathcal{D} \text{ and all } v \in C(\mathcal{D};\mathbb{R}),$$

where $p \in L(\mathcal{D};\mathbb{R})$ and $\tau\colon \mathcal{D} \to [a,b]$, $\mu\colon \mathcal{D} \to [c,d]$ are measurable functions.

Then clearly $\ell \in \mathcal{L}(\mathcal{D})$. A sufficient and necessary condition for the operator ℓ to be a $[t_0, h]$-Volterra one is given in the next proposition.

Proposition 6.1. *Let $t_0 \in [a,b]$ and $h \in CD([a,b];[c,d])$. Then the operator ℓ defined by the formula (6.1) is a $[t_0,h]$-Volterra one if and only if the conditions*

$$|p(t,x)|\min\{t_0,t,h^{-1}(x)\} \leqslant |p(t,x)|\tau(t,x) \leqslant |p(t,x)|\max\{t_0,t,h^{-1}(x)\} \quad \text{for a.e. } (t,x)\in\mathcal{D} \tag{6.2}$$

and

$$|p(t,x)|\min\{h(t_0),h(t),x\} \leqslant |p(t,x)|\mu(t,x) \leqslant |p(t,x)|\max\{h(t_0),h(t),x\} \quad \text{for a.e. } (t,x)\in\mathcal{D}. \tag{6.3}$$

are satisfied.

To prove this proposition we need the following lemma.

Lemma 6.1. *Let $\alpha,\beta\colon \mathcal{D}\to[\gamma_1,\gamma_2]$ be measurable functions and $E\subseteq\mathcal{D}$ a set of positive measure such that*

$$\alpha(t,x) < \beta(t,x) \quad \text{for } (t,x)\in E. \tag{6.4}$$

Then there exist $E_0\subseteq E$ and $z_0\in\,]\gamma_1,\gamma_2[$ such that $\operatorname{mes} E_0>0$ and

$$\alpha(t,x) < z_0 < \beta(t,x) \quad \text{for } (t,x)\in E_0. \tag{6.5}$$

P r o o f. Let

$$E_n = \Big\{(t,x)\in E\colon\ \alpha(t,x)-\beta(t,x) \leqslant -\frac{1}{n}\Big\} \quad \text{for } n\in\mathbb{N}. \tag{6.6}$$

Clearly, $E_1\subseteq E_2\subseteq\ldots\subseteq E$ and $\bigcup_{n=1}^{+\infty} E_n = E$. Therefore, in view of the assumption $\operatorname{mes} E>0$, there exists $n_0\in\mathbb{N}$ such that

$$\operatorname{mes} E_{n_0} > 0. \tag{6.7}$$

Moreover, (6.6) yields that

$$\alpha(t,x)-\beta(t,x) \leqslant -\frac{1}{n_0} \quad \text{for } t\in E_{n_0}. \tag{6.8}$$

Now we put

$$I(z) = \{(t,x)\in E_{n_0}\colon\ \alpha(t,x)<z\} \quad \text{for } z\in[\gamma_1,\gamma_2].$$

Clearly, $I(\gamma_1) = \emptyset$ and $(z_1) \subseteq I(z_2) \subseteq E_{n_0}$ for $\gamma_1 \leqslant z_1 \leqslant z_2 \leqslant \gamma_2$, and thus we can set

$$z^* = \sup\{z \in [\gamma_1, \gamma_2] \colon \operatorname{mes} I(z) = 0\}. \tag{6.9}$$

It can be easily verified that

$$I(z^*) = \bigcup_{k=1}^{+\infty} I\Big(z^* - \frac{z^* - \gamma_1}{k}\Big) \quad \text{and} \quad \operatorname{mes} I\Big(z^* - \frac{z^* - \gamma_1}{k}\Big) = 0 \quad \text{for } k \in \mathbb{N},$$

which guarantees that

$$\operatorname{mes} I(z^*) = 0. \tag{6.10}$$

Moreover, in view of (6.7) and (6.10) we have $E_{n_0} \setminus I(z^*) \neq \emptyset$, whence, on account of (6.8), we get

$$z^* \leqslant \alpha(t,x) \leqslant \beta(t,x) - \frac{1}{n_0} \leqslant \gamma_2 - \frac{1}{n_0} \quad \text{for } (t,x) \in E_{n_0} \setminus I(z^*)$$

and, in particular,

$$\gamma_1 < z^* + \frac{1}{2n_0} < \gamma_2. \tag{6.11}$$

Put

$$E_0 = \Big\{(t,x) \in E_{n_0} \colon z^* \leqslant \alpha(t,x) < z^* + \frac{1}{2n_0}\Big\}. \tag{6.12}$$

Then $I(z^* + 1/(2n_0)) = I(z^*) \cup E_0$ whereas the relation (6.9) guarantees that $\operatorname{mes} I(z^* + 1/(2n_0)) > 0$. Consequently, by using (6.10), we get

$$\operatorname{mes} E_0 > 0. \tag{6.13}$$

On the other hand, by virtue of (6.8) and (6.12), we have

$$z^* + \frac{1}{2n_0} \leqslant \alpha(t,x) + \frac{1}{2n_0} \leqslant \beta(t,x) - \frac{1}{2n_0} < \beta(t,x) \quad \text{for } (t,x) \in E_0$$

which yields

$$\alpha(t,x) < z^* + \frac{1}{2n_0} < \beta(t,x) \quad \text{for } (t,x) \in E_0,$$

and thus the relation (6.5) holds with $z_0 = z^* + 1/(2n_0)$. □

P r o o f of Proposition 6.1. Let the operator ℓ be defined by the formula (6.1).

If the inequalities (6.2) and (6.3) are satisfied then it is easy to verify that the operator ℓ is a $[t_0, h]$-Volterra one.

Conversely, let the operator ℓ be a $[t_0, h]$-Volterra one. Assume that, on the contrary, the first inequality in (6.2) does not hold, i.e., that

$$|p(t,x)|\tau(t,x) < |p(t,x)| \min\{t_0, t, h^{-1}(x)\}$$

on a set of positive measure. Then, according to Lemma 6.1, there exist a set $E_0 \subseteq \mathcal{D}$ of positive measure and $z_0 \in \,]a,b[$ such that

$$p(t,x) \neq 0, \quad \tau(t,x) < z_0 < \min\{t_0, t, h^{-1}(x)\} \quad \text{for } (t,x) \in E_0. \tag{6.14}$$

Therefore, for every $(t,x) \in E_0$ we have $z_0 < t$, $z_0 < t_0$, and $x < h(z_0)$, which guarantees that $E_0 \subseteq [z_0, b] \times [c, h(z_0)]$. Put

$$v(t,x) = \begin{cases} t - z_0 & \text{for } a \leqslant t \leqslant z_0, \ x \in [c,d], \\ 0 & \text{for } z_0 < t \leqslant b, \ x \in [c,d]. \end{cases}$$

Then, clearly, $v \in C(\mathcal{D}; \mathbb{R})$ and

$$v(t,x) = 0 \quad \text{for } (t,x) \in [z_0, b] \times [c, h(z_0)].$$

However, the relations (6.14) yield that

$$p(t,x)v(\tau(t,x), \mu(t,x)) = p(t,x)(\tau(t,x) - z_0) < 0 \quad \text{for } (t,x) \in E_0,$$

which is a contradiction because the operator ℓ is supposed to be a $[t_0, h]$-Volterra one. The contradiction obtained proves that the first inequality in (6.2) holds. The validity of the second inequality in (6.2) and the inequalities (6.3) can be proved analogously. □

Proposition 6.1 yields

Corollary 6.1. *Let $t_0 \in [a,b]$ and $h \in CD([a,b];[c,d])$. Assume that*

$$(\tau(t,x) - t_0)(\tau(t,x) - t) \leqslant 0 \quad \textit{for a.e. } (t,x) \in \mathcal{D}$$

and

$$(\mu(t,x) - h(t_0))(\mu(t,x) - x) \leqslant 0 \quad \textit{for a.e. } (t,x) \in \mathcal{D}.$$

Then the operator ℓ defined by the formula (6.1) *is a $[t_0, h]$-Volterra one.*

7. Existence and uniqueness theorems

In this section, we fix a function $h \in CD([a,b];[c,d])$ and give some efficient conditions guaranteeing the unique solvability of the problem (1.1), (1.3), (1.4) as well as (1.1′), (1.3), (1.4). We first formulate all the results, their proofs being postponed till Section 7.1 below.

Introduce the following notation.

Notation 7.1. Let $\ell \in \mathcal{L}(\mathcal{D})$. Define the operators $\vartheta_k \colon C(\mathcal{D};\mathbb{R}) \to C(\mathcal{D};\mathbb{R})$, $k = 0,1,2,\dots$, by setting

$$\vartheta_0(v) = v, \quad \vartheta_k(v) = T(\vartheta_{k-1}(v)) \quad \text{for } v \in C(\mathcal{D};\mathbb{R}),\ k \in \mathbb{N}, \tag{7.1}$$

where the operator T is given by the formula (4.1).

Theorem 7.1. *Let there exist $m \in \mathbb{N}$ and $\alpha \in [0,1[$ such that the inequality*

$$\|\vartheta_m(u)\|_C \leqslant \alpha \|u\|_C \tag{7.2}$$

is satisfied for every solution u to the homogeneous problem (1.1_0), (1.3_0), (1.4_0). *Then the problem* (1.1), (1.3), (1.4) *has a unique solution for an arbitrary h-consistent pair (g,ψ) and every $q \in L(\mathcal{D};\mathbb{R})$.*

Remark 7.1. The assumption $\alpha \in [0,1[$ in the previous theorem cannot be replaced by the assumption $\alpha \in [0,1]$ (see Example 9.1).

Corollary 7.1. *Let there exist $j \in \mathbb{N}$ such that*

$$\max\left\{ \int_a^b \int_c^{h(t)} p_j(t,x)\,\mathrm{d}x\,\mathrm{d}t, \int_a^b \int_{h(t)}^d p_j(t,x)\,\mathrm{d}x\,\mathrm{d}t \right\} < 1, \tag{7.3}$$

where $p_1 \equiv |p|$,

$$p_{k+1}(t,x) = |p(t,x)| \iint_{H(\tau(t,x),\mu(t,x))} p_k(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta \quad \text{for a.e. } (t,x) \in \mathcal{D},\ k \in \mathbb{N}, \tag{7.4}$$

and the mapping H is defined by the formula (2.1). *Then the problem* (1.1′), (1.3), (1.4) *has a unique solution for an arbitrary h-consistent pair (g,ψ) and every $q \in L(\mathcal{D};\mathbb{R})$.*

Remark 7.2. Example 9.1 shows that the strict inequality (7.3) in Corollary 7.1 cannot be replaced by the nonstrict one.

Theorem 7.2. *Let ℓ be a $[t_0, h]$-Volterra operator for some $t_0 \in [a, b]$. Then the problem* (1.1), (1.3), (1.4) *has a unique solution for an arbitrary h-consistent pair (g, ψ) and every $q \in L(\mathcal{D}; \mathbb{R})$.*

Corollary 7.2. *Let there exist $t_0 \in [a, b]$ such that the conditions* (6.2) *and* (6.3) *are satisfied. Then the problem* (1.1′), (1.3), (1.4) *has a unique solution for an arbitrary h-consistent pair (g, ψ) and every $q \in L(\mathcal{D}; \mathbb{R})$.*

Corollary 7.3. *Let either*

$$\tau(t, x) \leqslant t, \quad \mu(t, x) \geqslant x \quad \text{for a.e. } (t, x) \in \mathcal{D},$$

or

$$\tau(t, x) \geqslant t, \quad \mu(t, x) \leqslant x \quad \text{for a.e. } (t, x) \in \mathcal{D}.$$

Then, for an arbitrary $h \in CD([a, b]; [c, d])$, the problem (1.1′), (1.3), (1.4) *has a unique solution for every h-consistent pair (g, ψ) and all $q \in L(\mathcal{D}; \mathbb{R})$.*

Remark 7.3. Let $h \in AC([a, b]; \mathbb{R})$ and $h^{-1} \in AC([c, d]; \mathbb{R})$. The previous corollary guarantees that the problem (1.2), (1.3), (1.4) is uniquely solvable without any additional assumption imposed on the coefficient p. Since the problems (1.2), (1.3), (1.4) and (1.2), (1.6) are equivalent in this case (see Remark 3.2), the corollary obtained coincides with the result of K. Deimling established in the paper [3].

7.1. Proofs. Now we prove the statements formulated above.

Proof of Theorem 7.1. According to Theorem 5.1, it is sufficient to show that the homogeneous problem (1.1_0), (1.3_0), (1.4_0) has only the trivial solution.

Let u be a solution to the problem (1.1_0), (1.3_0), (1.4_0). Then, by virtue of Lemma 3.3, u satisfies

$$u(t, x) = \iint_{H(t,x)} \ell(u)(s, \eta)\, \mathrm{d}s\, \mathrm{d}\eta = T(u)(t, x) = \vartheta_1(u)(t, x) \quad \text{for } (t, x) \in \mathcal{D}.$$

Therefore, we get

$$u(t, x) = T(\vartheta_1(u))(t, x) = \vartheta_2(u)(t, x) \quad \text{for } (t, x) \in \mathcal{D},$$

and thus $u = \vartheta_k(u)$ for every $k \in \mathbb{N}$. Consequently, the relation (7.2) implies

$$\|u\|_C = \|\vartheta_m(u)\|_C \leqslant \alpha \|u\|_C,$$

which guarantees that $u \equiv 0$, because $\alpha \in [0, 1[$. ☐

P r o o f of Corollary 7.1. It is clear that the equation (1.1′) is a particular case of the equation (1.1), in which the operator ℓ is given by the formula (6.1). It is not difficult to verify that

$$\begin{aligned}|\vartheta_k(v)(t,x)| &\leqslant \iint_{H(t,x)} |p(s,\eta)\vartheta_{k-1}(v)(\tau(s,\eta),\mu(s,\eta))|\,\mathrm{d}s\,\mathrm{d}\eta \\ &\leqslant \|v\|_C \iint_{H(t,x)} p_k(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta \quad \text{for } (t,x)\in\mathcal{D},\ k\in\mathbb{N},\ v\in C(\mathcal{D};\mathbb{R}).\end{aligned}$$

Since the functions p_k are nonnegative, we get for any $k\in\mathbb{N}$ the relation

$$\begin{aligned}&\max_{(t,x)\in\mathcal{D}}\left\{\iint_{H(t,x)} p_k(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta\right\} \\ &\qquad = \max\left\{\iint_{H(a,c)} p_k(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta, \iint_{H(b,d)} p_k(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta\right\}.\end{aligned}$$

Consequently, the assumptions of Theorem 7.1 are satisfied with $m=j$ and

$$\alpha = \max\left\{\int_a^b\int_c^{h(t)} p_j(t,x)\,\mathrm{d}x\,\mathrm{d}t, \int_a^b\int_{h(t)}^d p_j(t,x)\,\mathrm{d}x\,\mathrm{d}t\right\}.$$

□

To prove Theorem 7.2 we need the following lemma.

Lemma 7.1. *Let $t_0\in[a,b]$ and let ℓ be a $[t_0,h]$-Volterra operator. Then*

$$\lim_{k\to+\infty}\|\vartheta_k\| = 0, \tag{7.5}$$

where the operators ϑ_k are defined by the formula (7.1).

P r o o f. Let $\varepsilon\in\,]0,1[$. According to Proposition 4.1, the operator ϑ_1 is completely continuous. Therefore, by virtue of the Arzelà-Ascoli lemma, there exists $\delta>0$ such that

$$\left|\iint_{H(t_2,x_2)} \ell(w)(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta - \iint_{H(t_1,x_1)} \ell(w)(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta\right| \leqslant \varepsilon\,\|w\|_C \tag{7.6}$$
$$\text{for } (t_1,x_1),(t_2,x_2)\in\mathcal{D},\ |t_2-t_1|+|x_2-x_1|<\delta,\ w\in C(\mathcal{D};\mathbb{R}).$$

Since $h\in CD([a,b];[c,d])$, there exists $\delta_0>0$ such that $\delta_0<\delta/2$, $\delta_0<\max\{t_0-a,b-t_0\}$, and

$$|h(t_2)-h(t_1)| < \frac{\delta}{2} \quad \text{for } t_1,t_2\in[a,b],\ |t_2-t_1|\leqslant\delta_0. \tag{7.7}$$

Let

$$n = \max\Big\{\operatorname{Ent}\Big(\frac{t_0-a}{\delta_0}\Big),\ \operatorname{Ent}\Big(\frac{b-t_0}{\delta_0}\Big)\Big\} + 1.$$

Choose $y_{n+1} \in [a, t_0]$ and $y_{n+2} \in [t_0, b]$ such that $y_{n+2} - y_{n+1} = \delta_0$, and put

$$y_k = y_{n+1} - (n+1-k)\,\frac{y_{n+1}-a}{n} \quad \text{for } k = 1, 2, \dots, n,$$
$$y_k = y_{n+2} + (k-n-2)\,\frac{b-y_{n+2}}{n} \quad \text{for } k = n+3, n+4, \dots, 2n+2,$$

and

$$\mathcal{D}_k = [y_{n+2-k}, y_{n+1+k}] \times [h(y_{n+1+k}), h(y_{n+2-k})] \quad \text{for } k = 1, 2, \dots, n+1.$$

Using the relation (7.7) and the definition of the numbers y_k, for any $j, r = 1, 2, \dots, 2n+1$, we get

$$(7.8)\qquad |t_2 - t_1| + |x_2 - x_1| < \delta \quad \text{for } (t_1, x_1), (t_2, x_2) \in [y_j, y_{j+1}] \times [h(y_{r+1}), h(y_r)].$$

Having $w \in C(\mathcal{D}; \mathbb{R})$, we denote

$$\|w\|_i = \|w\|_{C(\mathcal{D}_i;\mathbb{R})} \quad \text{for } i = 1, 2, \dots, n+1.$$

Let $v \in C(\mathcal{D}; \mathbb{R})$ be arbitrary. We shall show that the relation

$$(7.9)\qquad \|\vartheta_k(v)\|_i \leqslant \alpha_i(k)\varepsilon^k \|v\|_C \quad \text{for } k \in \mathbb{N}$$

holds for every $i = 1, 2, \dots, n+1$, where

$$(7.10)\qquad \alpha_i(k) = \alpha_i k^{i-1} \quad \text{for } k \in \mathbb{N},\ i = 1, 2, \dots, n+1,$$
$$(7.11)\qquad \alpha_1 = 1, \quad \alpha_{i+1} = i + 1 + i\alpha_i \quad \text{for } i = 1, 2, \dots, n.$$

By virtue of (7.6) and (7.8), it is easy to verify that, for any $w \in C(\mathcal{D}; \mathbb{R})$ and $i = 1, 2, \dots, n+1$, we have

$$(7.12)\qquad \Big|\iint_{H(t,x)} \ell(w)(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta\Big| \leqslant i\varepsilon\|w\|_C \quad \text{for } (t,x) \in \mathcal{D}_i.$$

Observe that the previous relation immediately implies

$$(7.13)\qquad \|\vartheta_1(v)\|_i \leqslant i\varepsilon\|v\|_C \quad \text{for } i = 1, 2, \dots, n+1.$$

Furthermore, on account of (7.6), (7.8), and the fact that ℓ is a $[t_0, h]$-Volterra operator, we obtain

$$|\vartheta_{k+1}(v)(t,x)| = \left| \iint_{H(t,x)} \ell(\vartheta_k(v))(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta \right| \leqslant \varepsilon \|\vartheta_k(v)\|_1 \quad \text{for } (t,x) \in \mathcal{D}_1,\ k \in \mathbb{N}.$$

Hence, by virtue of (7.13), we get

$$\|\vartheta_k(v)\|_1 \leqslant \varepsilon^k \, \|v\|_C \quad \text{for } k \in \mathbb{N}$$

and thus the relation (7.9) holds for $i = 1$.

Now suppose that the relation (7.9) holds for some $i \in \{1, 2, \dots, n\}$. We shall show that the relation indicated is also true for $i + 1$. With respect to (7.8), we obtain

$$\begin{aligned}
\|\vartheta_{k+1}(v)\|_{i+1} &= \max\left\{ \left| \iint_{H(t,x)} \ell(\vartheta_k(v))(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta \right| :\ (t,x) \in \mathcal{D}_{i+1} \right\} \\
&= \left| \iint_{H(t_k^*, x_k^*)} \ell(\vartheta_k(v))(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta \right| \\
&\leqslant \left| \iint_{H(\hat{t}_k, \hat{x}_k)} \ell(\vartheta_k(v))(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta \right| \\
&\quad + \left| \iint_{H(t_k^*, x_k^*)} \ell(\vartheta_k(v))(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta - \iint_{H(\hat{t}_k, \hat{x}_k)} \ell(\vartheta_k(v))(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta \right|
\end{aligned}$$

for $k \in \mathbb{N}$, where $(t_k^*, x_k^*) \in \mathcal{D}_{i+1}$, $(\hat{t}_k, \hat{x}_k) \in \mathcal{D}_i$, and $|t_k^* - \hat{t}_k| + |x_k^* - \hat{x}_k| < \delta$ for $k \in \mathbb{N}$. Therefore, on account of (7.6), (7.12), and the fact that ℓ is a $[t_0, h]$-Volterra operator, we get

$$\|\vartheta_{k+1}(v)\|_{i+1} \leqslant \varepsilon \, \|\vartheta_k(v)\|_{i+1} + i\varepsilon \|\vartheta_k(v)\|_i \leqslant \varepsilon \, \|\vartheta_k(v)\|_{i+1} + i\alpha_i(k)\varepsilon^{k+1} \|v\|_C$$

for $k \in \mathbb{N}$. Consequently, for any $k \in \mathbb{N}$ we have

$$\|\vartheta_{k+1}(v)\|_{i+1} \leqslant \varepsilon(\varepsilon \|\vartheta_{k-1}(v)\|_{i+1} + i\alpha_i(k-1)\varepsilon^k \|v\|_C) + i\alpha_i(k)\varepsilon^{k+1} \|v\|_C.$$

Continuing this procedure, on account of (7.13) we obtain

$$\|\vartheta_{k+1}(v)\|_{i+1} \leqslant (i + 1 + i(\alpha_i(1) + \ldots + \alpha_i(k)))\varepsilon^{k+1} \|v\|_C \quad \text{for } k \in \mathbb{N}. \tag{7.14}$$

By using (7.10) and (7.11), it is easy to verify that

$$\begin{aligned}
i + 1 + i(\alpha_i(1) + \ldots + \alpha_i(k)) &= i + 1 + i\alpha_i(1^{i-1} + \ldots + k^{i-1}) \\
&\leqslant i + 1 + i\alpha_i k k^{i-1} = i + 1 + i\alpha_i k^i \\
&\leqslant (i + 1 + i\alpha_i)k^i = \alpha_{i+1} k^i \leqslant \alpha_{i+1}(k+1).
\end{aligned}$$

Therefore, (7.13) and (7.14) imply

$$\|\vartheta_k(v)\|_{i+1} \leqslant \alpha_{i+1}(k)\,\varepsilon^k\,\|v\|_C \quad \text{for } k \in \mathbb{N}.$$

Hence, by induction, we have proved that the relation (7.9) holds for every $i = 1, 2, \dots, n+1$.

Now it is already clear that, for any $k \in \mathbb{N}$, the estimate

$$\|\vartheta_k(v)\|_C = \|\vartheta_k(v)\|_{n+1} \leqslant \alpha_{n+1} k^n \varepsilon^k\,\|v\|_C \quad \text{for } v \in C(\mathcal{D}; \mathbb{R})$$

is fulfilled and thus

$$\|\vartheta_k\| \leqslant \alpha_{n+1}\,k^n\,\varepsilon^k \quad \text{for } k \in \mathbb{N}.$$

Since we suppose that $\varepsilon \in\,]0, 1[$, the last relation yields the validity of the condition (7.5). □

P r o o f of Theorem 7.2. According to Lemma 7.1, there exists a number $m_0 \in \mathbb{N}$ such that $\|\vartheta_{m_0}\| < 1$. Moreover, it is clear that

$$\|\vartheta_{m_0}(v)\|_C \leqslant \|\vartheta_{m_0}\|\,\|v\|_C \quad \text{for } v \in C(\mathcal{D}; \mathbb{R}),$$

because the operator ϑ_{m_0} is bounded. Consequently, the assumptions of Theorem 7.1 are satisfied with $m = m_0$ and $\alpha = \|\vartheta_{m_0}\|$. □

P r o o f of Corollary 7.2. It is clear that the equation (1.1′) is a particular case of the equation (1.1) in which the operator ℓ is given by the formula (6.1). By virtue of the assumptions (6.2) and (6.3), Proposition 6.1 guarantees that the operator ℓ is a $[t_0, h]$-Volterra one. Consequently, the assertion of the corollary follows from Theorem 7.2. □

P r o o f of Corollary 7.3. It follows immediately from Corollary 7.2 with $t_0 = a$ and $t_0 = b$, respectively. □

8. Well-posedness

In this section, the well-posedness of the problems (1.1), (1.3), (1.4) and (1.1′), (1.3), (1.4) is studied. We first formulate all the results, their proofs being given in Section 8.1 below.

Throughout this section, we fix a function $h \in CD([a, b]; [c, d])$ for which the mapping H is given by the formula (2.1). On the graph of the function h we consider the Cauchy problem (1.3), (1.4) for the equation (1.1). Call that the pair (g, ψ) is supposed to be h-consistent.

For any $k \in \mathbb{N}$, along with the problem (1.1), (1.3), (1.4) we consider the perturbed problem

$$\frac{\partial^2 u(t,x)}{\partial t \partial x} = \ell_k(u)(t,x) + q_k(t,x), \tag{1.1_k}$$

$$u\,(t, h_k(t)) = g_k(t) \quad \text{for } t \in [a,b], \tag{1.3_k}$$

$$u'_{[2]}\left(h_k^{-1}(x), x\right) = \psi_k(x) \quad \text{for a.e. } x \in [c,d], \tag{1.4_k}$$

where $\ell_k \in \mathcal{L}(\mathcal{D})$, $q_k \in L(\mathcal{D};\mathbb{R})$, $h_k \in CD([a,b];[c,d])$, and $g_k \in C([a,b];\mathbb{R})$, $\psi_k \in L([c,d];\mathbb{R})$ are such that the pair (g_k, ψ_k) is h_k-consistent.

Analogously to Notation 2.1, for given functions h_k we put

$$\begin{aligned} H_k(t,x) = \{(s,\eta) \in \mathbb{R}^2 \colon\ & \min\{h_k^{-1}(x), t\} \leqslant s \leqslant \max\{h_k^{-1}(x), t\}, \\ & \min\{h_k(s), x\} \leqslant \eta \leqslant \max\{h_k(s), x\}\} \quad \text{for } (t,x) \in \mathcal{D},\ k \in \mathbb{N}. \end{aligned} \tag{8.1}$$

It is clear that for any $(t,x) \in \mathcal{D}$ and $k \in \mathbb{N}$, the set $H_k(t,x)$ is a measurable subset of $\mathcal{D}$.

Notation 8.1. Let $\Lambda \in \mathcal{L}(\mathcal{D})$ and $\gamma \in CD([a,b];[c,d])$. Denote by $M(\Lambda, \gamma)$ the set of functions $y \in C^*(\mathcal{D};\mathbb{R})$ admitting the representation

$$y(t,x) = \int_{\gamma(t)}^{x} \int_{\gamma^{-1}(\eta)}^{t} \Lambda(z)(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta \quad \text{for } (t,x) \in \mathcal{D},$$

where $z \in C(\mathcal{D};\mathbb{R})$ and $\|z\|_C = 1$.

Theorem 8.1. *Let the problem* (1.1), (1.3), (1.4) *have a unique solution* u *and*

$$\lim_{k \to +\infty} \lambda_k = 0, \tag{8.2}$$

where

$$\lambda_k = \sup_{\substack{(t,x) \in \mathcal{D} \\ y \in M(\ell_k, h_k)}} \left\{ \left| \iint_{H_k(t,x)} \ell_k(y)(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta - \iint_{H(t,x)} \ell(y)(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta \right| \right\} \tag{8.3}$$

for $k \in \mathbb{N}$. Let, moreover,

$$\lim_{k\to+\infty} \varrho_k \left[\iint_{H_k(t,x)} \ell_k(y)(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta - \iint_{H(t,x)} \ell(y)(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta \right] = 0 \tag{8.4}$$

uniformly on $\mathcal{D}$ for every $y \in C^*(\mathcal{D};\mathbb{R})$,

$$\lim_{k\to+\infty} \varrho_k \left[\iint_{H_k(t,x)} q_k(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta - \iint_{H(t,x)} q(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta \right] = 0 \tag{8.5}$$

uniformly on $\mathcal{D}$,

$$\lim_{k\to+\infty} \varrho_k \int_c^x [\psi_k(\eta) - \psi(\eta)]\,\mathrm{d}\eta = 0 \quad \textit{uniformly on } [c,d], \tag{8.6}$$

$$\lim_{k\to+\infty} \varrho_k \int_{h_k(t)}^{h(t)} \psi(x)\,\mathrm{d}\eta = 0 \quad \textit{uniformly on } [a,b], \tag{8.7}$$

and

$$\lim_{k\to+\infty} \varrho_k \|g_k - g\|_C = 0, \tag{8.8}$$

where

$$\varrho_k = 1 + \|\ell_k\| \quad \textit{for } k \in \mathbb{N}. \tag{8.9}$$

Then there exists $k_0 \in \mathbb{N}$ such that for every $k > k_0$, the problem (1.1_k), (1.3_k), (1.4_k) has a unique solution u_k and

$$\lim_{k\to+\infty} \|u_k - u\|_C = 0. \tag{8.10}$$

Remark 8.1. By using Lemma 3.3, it can be easily verified that the functions u_k and u in Theorem 8.1 also satisfy the condition

$$\lim_{k\to+\infty} \varrho_k \int_a^t [u'_{k[1]}(s,h(s)) - u'_{[1]}(s,h(s))]\,\mathrm{d}s = 0 \quad \text{uniformly on } [a,b].$$

Note also that the sequence $\{h_k\}$ in the previous theorem does not necessarily converge to the function h. Indeed, let $\ell_k = \ell = 0$,[5] $q_k \equiv q \equiv 0$, $\psi_k \equiv \psi \equiv 0$, and let $g_k, g \in AC([a,b];\mathbb{R})$ fulfil the condition (8.8). Then the assumptions of Theorem 8.1 are satisfied for arbitrary functions $h_k, h \in CD([a,b];[c,d])$.

If we suppose that the operators ℓ_k are "uniformly bounded" in the sense of the relation (8.11) then we obtain the following statement.

[5] The symbol 0 stands here for the zero operator.

Corollary 8.1. *Let the problem* (1.1), (1.3), (1.4) *have a unique solution u, let there exist a function $\omega \in L(\mathcal{D}; \mathbb{R}_+)$ such that*

$$(8.11) \quad |\ell_k(y)(t,x)| \leqslant \omega(t,x)\|y\|_C \quad \text{for a. e. } (t,x) \in \mathcal{D} \text{ and all } y \in C(\mathcal{D}; \mathbb{R}),\ k \in \mathbb{N},$$

and

$$(8.12) \quad \lim_{k\to+\infty} \iint_{H_k(t,x)} \ell_k(y)(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta = \iint_{H(t,x)} \ell(y)(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta$$
$$\text{uniformly on } \mathcal{D} \text{ for every } y \in C^*(\mathcal{D}; \mathbb{R}).$$

Moreover, let

$$(8.13) \quad \lim_{k\to+\infty} \iint_{H_k(t,x)} q_k(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta = \iint_{H(t,x)} q(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta \quad \text{uniformly on } \mathcal{D},$$

$$(8.14) \quad \lim_{k\to+\infty} \int_c^x [\psi_k(\eta) - \psi(\eta)]\,\mathrm{d}\eta = 0 \quad \text{uniformly on } [c,d],$$

$$(8.15) \quad \lim_{k\to+\infty} \int_{h_k(t)}^{h(t)} \psi(x)\,\mathrm{d}\eta = 0 \quad \text{uniformly on } [a,b],$$

and

$$(8.16) \quad \lim_{k\to+\infty} \|g_k - g\|_C = 0.$$

Then the conclusion of Theorem 8.1 *holds.*

Remark 8.2. The assumption (8.11) in the previous corollary is important and cannot be omitted (see Example 9.2).

Corollary 8.2. *Let the problem* (1.1), (1.3), (1.4) *have a unique solution u and let there exist a function $\omega \in L(\mathcal{D}; \mathbb{R}_+)$ such that the relation* (8.11) *holds. Moreover, let the conditions* (8.13), (8.14), *and* (8.16) *be satisfied,*

$$(8.17) \quad \lim_{k\to+\infty} \iint_{H(t,x)} [\ell_k(y)(s,\eta) - \ell(y)(s,\eta)]\,\mathrm{d}s\,\mathrm{d}\eta = 0$$
$$\text{uniformly on } \mathcal{D} \text{ for every } y \in C^*(\mathcal{D}; \mathbb{R}),$$

and

$$(8.18) \quad \lim_{k\to+\infty} \|h_k - h\|_C = 0.$$

Then the conclusion of Theorem 8.1 *holds.*

Remark 8.3. If the functions q_k are such that

$$|q_k(t,x)| \leqslant \sigma(t,x) \quad \text{for a.e. } (t,x) \in \mathcal{D},\ k \in \mathbb{N}$$

with $\sigma \in L(\mathcal{D};\mathbb{R}_+)$, then the condition (8.13) in Corollary 8.2 and other statements containing the assumption (8.18) can be replaced by the more convenient condition

$$\lim_{k\to+\infty} \iint_{H(t,x)} [q_k(s,\eta) - q(s,\eta)]\,\mathrm{d}s\,\mathrm{d}\eta = 0 \quad \text{uniformly on } \mathcal{D}$$

(see Lemma 8.2 below).

Corollary 8.2 immediately yields

Corollary 8.3. *Let the homogeneous problem* (1.1_0), (1.3_0), (1.4_0) *have only the trivial solution. Then the Cauchy operator*[6] *of the problem* (1.1_0), (1.3_0), (1.4_0) *is continuous.*

Now we give a statement on the well-posedness of the problem $(1.1')$, (1.3), (1.4). For any $k \in \mathbb{N}$, along with the equation $(1.1')$ we consider the perturbed equation

$$(1.1'_k) \qquad \frac{\partial^2 u(t,x)}{\partial t\,\partial x} = p_k(t,x)u(\tau_k(t,x),\mu_k(t,x)) + q_k(t,x),$$

where $p_k, q_k \in L(\mathcal{D};\mathbb{R})$ and $\tau_k\colon \mathcal{D} \to [a,b]$, $\mu_k\colon \mathcal{D} \to [c,d]$ are measurable functions.

Corollary 8.4. *Let the problem* $(1.1')$, (1.3), (1.4) *have a unique solution* u, *let there exist a function* $\omega \in L(\mathcal{D};\mathbb{R}_+)$ *such that*

$$(8.19) \qquad |p_k(t,x)| \leqslant \omega(t,x) \quad \text{for a.e. } (t,x) \in \mathcal{D},\ k \in \mathbb{N},$$

and

$$(8.20) \qquad \lim_{k\to+\infty} \iint_{H(t,x)} [p_k(s,\eta) - p(s,\eta)]\,\mathrm{d}s\,\mathrm{d}\eta = 0 \quad \text{uniformly on } \mathcal{D}.$$

Moreover, let the conditions (8.13), (8.14), (8.16), *and* (8.18) *be satisfied, and*

$$(8.21) \qquad \lim_{k\to+\infty} \operatorname{ess\,sup}\{|\tau_k(t,x) - \tau(t,x)|\colon (t,x) \in \mathcal{D}\} = 0,$$

$$(8.22) \qquad \lim_{k\to+\infty} \operatorname{ess\,sup}\{|\mu_k(t,x) - \mu(t,x)|\colon (t,x) \in \mathcal{D}\} = 0.$$

Then there exists $k_0 \in \mathbb{N}$ *such that for every* $k > k_0$, *the problem* $(1.1'_k)$, (1.3_k), (1.4_k) *has a unique solution* u_k *and the relation* (8.10) *holds.*

Remark 8.4. The assumption (8.19) in the previous theorem is important and cannot be omitted (see Example 9.2).

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[6] The notion of the Cauchy operator is introduced in Definition 5.1.

Finally, we consider the hyperbolic equation without argument deviations (1.2) in which $p, q \in L(\mathcal{D}; \mathbb{R})$. For any $k \in \mathbb{N}$, along with the equation (1.2) we consider the perturbed equation

$$u_{tx} = p_k(t,x)u + q_k(t,x) \tag{1.2_k}$$

where $p_k, q_k \in L(\mathcal{D}; \mathbb{R})$.

The following statement can be derived from Theorem 8.1.

Corollary 8.5. *Let the conditions* (8.5)–(8.8) *be satisfied,*

$$\lim_{k\to+\infty} \varrho_k \left[\iint_{H_k(t,x)} p_k(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta - \iint_{H(t,x)} p(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta \right] = 0 \quad \textit{uniformly on } \mathcal{D}, \tag{8.23}$$

and

$$\lim_{k\to+\infty} \varrho_k \iint_{H(t,x)\div H_k(t,x)} |p(s,\eta)|\,\mathrm{d}s\,\mathrm{d}\eta = 0 \quad \textit{uniformly on } \mathcal{D}, \tag{8.24}$$

where

$$\varrho_k = 1 + \|p_k\|_L \quad \textit{for } k \in \mathbb{N}. \tag{8.25}$$

Then the relation (8.10) *holds, where* u *and* u_k *are solutions to the problems* (1.2)–(1.4) *and* (1.2_k)–(1.4_k), *respectively.*

Remark 8.5. If the relation $\sup\{\|p_k\|_L \colon k \in \mathbb{N}\} < +\infty$ holds then the assumption (8.24) of the previous corollary is guaranteed, e.g., by the condition (8.18) (see Lemma 8.2 below).

Corollary 8.5 yields

Corollary 8.6. *Let the conditions* (8.14), (8.16), *and* (8.18) *be satisfied,*

$$\lim_{k\to+\infty} \|p_k - p\|_L = 0, \tag{8.26}$$

and

$$\lim_{k\to+\infty} \|q_k - q\|_L = 0. \tag{8.27}$$

Then the conclusion of Corollary 8.5 *holds.*

8.1. Proofs. In order to prove Theorem 8.1, we need the following lemma.

Lemma 8.1. *Let the problem* (1.1_0), (1.3_0), (1.4_0) *have only the trivial solution and let the condition* (8.2) *hold, where the numbers* λ_k *are defined by the formula* (8.3). *Then for any* $z \in C^*(\mathcal{D};\mathbb{R})$ *there exist* $r_0 > 0$ *and* $k_0 \in \mathbb{N}$ *such that*

$$\|y - z\|_C \leqslant r_0(1 + \|\ell_k\|)[\|\Delta(y, h_k) - \Delta(z, h)\|_C + \|\Gamma_k(y, z)\|_C] \quad \text{for } k > k_0,\ y \in C^*(\mathcal{D};\mathbb{R}), \tag{8.28}$$

where

$$\Delta(v, \gamma)(t, x) = v\,(t, \gamma(t)) + \int_{\gamma(t)}^{x} v'_{[2]}\left(\gamma^{-1}(\eta), \eta\right) \mathrm{d}\eta \quad \text{for } (t, x) \in \mathcal{D},\ v \in C^*(\mathcal{D};\mathbb{R}),\ \gamma \in CD([a, b]; [c, d]) \tag{8.29}$$

and

$$\Gamma_k(v, w)(t, x) = \iint_{H_k(t,x)} [v''_{[1,2]}(s, \eta) - \ell_k(v - w)(s, \eta)]\, \mathrm{d}s\, \mathrm{d}\eta - \iint_{H(t,x)} w''_{[1,2]}(s, \eta)\, \mathrm{d}s\, \mathrm{d}\eta \quad \text{for } (t, x) \in \mathcal{D},\ v, w \in C^*(\mathcal{D};\mathbb{R}),\ k \in \mathbb{N}. \tag{8.30}$$

P r o o f. Let the operators $T, T_k\colon C(\mathcal{D};\mathbb{R}) \to C(\mathcal{D};\mathbb{R})$ be defined by the formulas (4.1) and

$$T_k(v)(t, x) = \iint_{H_k(t,x)} \ell_k(v)(s, \eta)\, \mathrm{d}s\, \mathrm{d}\eta \quad \text{for } (t, x) \in \mathcal{D},\ v \in C(\mathcal{D};\mathbb{R}),\ k \in \mathbb{N}.$$

Clearly, we have

$$\|T_k(y)\|_C \leqslant \|\ell_k(y)\|_L \leqslant \|\ell_k\|\, \|y\|_C \quad \text{for } y \in C(\mathcal{D};\mathbb{R}),\ k \in \mathbb{N}.$$

Therefore, the operators T_k $(k \in \mathbb{N})$ are linear bounded ones, and the relation

$$\|T_k\| \leqslant \|\ell_k\| \quad \text{for } k \in \mathbb{N} \tag{8.31}$$

holds. Moreover, the condition (8.2) with λ_k given by (8.3) can be rewritten in the form

$$\sup\{\|T_k(y) - T(y)\|_C\colon\ y \in M(\ell_k, h_k)\} \to 0 \quad \text{as } k \to +\infty. \tag{8.32}$$

Assume that, on the contrary, the assertion of the lemma is not true. Then there exist $z \in C^*(\mathcal{D};\mathbb{R})$, an increasing sequence $\{k_m\}_{m=1}^{+\infty}$ of natural numbers, and a sequence $\{y_m\}_{m=1}^{+\infty}$ of functions from $C^*(\mathcal{D};\mathbb{R})$ such that

$$\|y_m - z\|_C > m(1 + \|\ell_{k_m}\|)[\|\Delta(y_m, h_{k_m}) - \Delta(z, h)\|_C + \|\Gamma_{k_m}(y_m, z)\|_C] \tag{8.33}$$

for $m \in \mathbb{N}$. For any $m \in \mathbb{N}$ and all $(t,x) \in \mathcal{D}$, we put

$$z_m(t,x) = \frac{y_m(t,x) - z(t,x)}{\|y_m - z\|_C}, \tag{8.34}$$

$$v_m(t,x) = \frac{\Delta(y_m, h_{k_m})(t,x) - \Delta(z,h)(t,x) + \Gamma_{k_m}(y_m, z)(t,x)}{\|y_m - z\|_C}, \tag{8.35}$$

$$z_{0,m}(t,x) = z_m(t,x) - v_m(t,x), \tag{8.36}$$

$$w_m(t,x) = T_{k_m}(z_{0,m})(t,x) - T(z_{0,m})(t,x) + T_{k_m}(v_m)(t,x). \tag{8.37}$$

Obviously,

$$\|z_m\|_C = 1 \quad \text{for } m \in \mathbb{N}. \tag{8.38}$$

Using (8.29) and (8.30) in the relation (8.35), by virtue of Lemma 3.3 we get

$$z_{0,m}(t,x) = T_{k_m}(z_m)(t,x) \quad \text{for } (t,x) \in \mathcal{D},\ m \in \mathbb{N}, \tag{8.39}$$

and thus

$$z_{0,m}(t,x) = T(z_{0,m})(t,x) + w_m(t,x) \quad \text{for } (t,x) \in \mathcal{D},\ m \in \mathbb{N}. \tag{8.40}$$

Moreover, it follows from (8.33) and (8.35) that

$$\|v_m\|_C \leqslant \frac{\|\Delta(y_m, h_{k_m}) - \Delta(z,h)\|_C + \|\Gamma_{k_m}(y_m, z)\|_C}{\|y_m - z\|_C} < \frac{1}{m(1 + \|\ell_{k_m}\|)} \tag{8.41}$$

for $m \in \mathbb{N}$. Now the relations (8.31) and (8.41) yield

$$\|T_{k_m}(v_m)\|_C \leqslant \|T_{k_m}\|\,\|v_m\|_C \leqslant \frac{\|\ell_{k_m}\|}{m(1 + \|\ell_{k_m}\|)} < \frac{1}{m} \quad \text{for } m \in \mathbb{N}. \tag{8.42}$$

Observe that the expression (8.39) and the condition (8.38) guarantee the validity of the inclusion $z_{0,m} \in M(\ell_{k_m}, h_{k_m})$ for $m \in \mathbb{N}$ and thus, in view of (8.32), we obtain

$$\lim_{m \to +\infty} \|T_{k_m}(z_{0,m}) - T(z_{0,m})\|_C = 0. \tag{8.43}$$

According to (8.42) and (8.43), it follows from (8.37) that

$$\lim_{m \to +\infty} \|w_m\|_C = 0 \tag{8.44}$$

and, by virtue of (8.38) and (8.41), the equality (8.36) implies $\|z_{0,m}\|_C < 2$ for $m \in \mathbb{N}$. Since the sequence $\{\|z_{0,m}\|_C\}_{m=1}^{+\infty}$ is bounded and the operator T is completely continuous (see Proposition 4.1), there exists a subsequence of $\{T(z_{0,m})\}_{m=1}^{+\infty}$

which is convergent. We can assume without loss of generality that the sequence $\{T(z_{0,m})\}_{m=1}^{+\infty}$ is convergent, i.e., that there exists $z_0 \in C(\mathcal{D};\mathbb{R})$ such that

$$\lim_{m\to+\infty} \|T(z_{0,m}) - z_0\|_C = 0.$$

Then it is clear that

$$\lim_{m\to+\infty} \|z_{0,m} - z_0\|_C = 0, \tag{8.45}$$

because the functions $z_{0,m}$ admit the representation (8.40) and the relation (8.44) holds. However, the estimate (8.41) is true for v_m and thus, the equality (8.36) yields

$$\lim_{m\to+\infty} \|z_m - z_0\|_C = 0$$

which, together with (8.38), guarantees that $\|z_0\|_C = 1$. Since the operator T is continuous and the conditions (8.44) and (8.45) are fulfilled, the relation (8.40) gives $z_0 = T(z_0)$. Consequently, by virtue of Lemma 3.3, $z_0 \in C^*(\mathcal{D};\mathbb{R})$ and z_0 is a non-trivial solution to the homogeneous problem (1.1_0), (1.3_0), (1.4_0), which is a contradiction. □

P r o o f of Theorem 8.1. Since the problem (1.1), (1.3), (1.4) has a unique solution, the homogeneous problem (1.1_0), (1.3_0), (1.4_0) has only the trivial solution. Therefore, the assumptions of Lemma 8.1 are satisfied and thus there exist $r_0 > 0$ and $k_0 \in \mathbb{N}$ such that

$$\|y\|_C \leqslant r_0(1 + \|\ell_k\|)[\|\Delta(y, h_k)\|_C + \|\Gamma_k(y, 0)\|_C] \quad \text{for } k > k_0,\ y \in C^*(\mathcal{D};\mathbb{R}) \tag{8.46}$$

and

$$\|y - u\|_C \leqslant r_0(1 + \|\ell_k\|)[\|\Delta(y, h_k) - \Delta(u, h)\|_C + \|\Gamma_k(y, u)\|_C] \quad \text{for } k > k_0,\ y \in C^*(\mathcal{D};\mathbb{R}), \tag{8.47}$$

where the operators Δ and Γ_k are given by the formulas (8.29) and (8.30), respectively.

It is easy to verify that if for some $k \in \mathbb{N}$, u_0 is a solution to the problem

$$\begin{aligned} \frac{\partial^2 u(t,x)}{\partial t\, \partial x} &= \ell_k(u)(t,x), \\ u\,(t, h_k(t)) &= 0 \quad \text{for } t \in [a,b], \\ u'_{[2]}\left(h_k^{-1}(x), x\right) &= 0 \quad \text{for a.e. } x \in [c,d], \end{aligned} \tag{8.48}$$

then $\Delta(u_0, h_k) \equiv 0$ and $\Gamma_k(u_0, 0) \equiv 0$. Therefore, the relation (8.46) guarantees that for every $k > k_0$, the homogeneous problem (8.48) has only the trivial solution. Hence, for every $k > k_0$, the problem (1.1_k), (1.3_k), (1.4_k) has a unique solution u_k (see Theorem 5.1). Clearly, we have

$$\Delta(u_k, h_k)(t, x) = g_k(t) + \int_{h_k(t)}^{x} \psi_k(\eta)\,\mathrm{d}\eta \quad \text{for } (t, x) \in \mathcal{D},\ k > k_0,$$
$$\Delta(u, h)(t, x) = g(t) + \int_{h(t)}^{x} \psi(\eta)\,\mathrm{d}\eta \quad \text{for } (t, x) \in \mathcal{D},$$

and

$$\Gamma_k(u_k, u)(t, x) = \iint_{H_k(t,x)} \ell_k(u)(s, \eta)\,\mathrm{d}s\,\mathrm{d}\eta - \iint_{H(t,x)} \ell(u)(s, \eta)\,\mathrm{d}s\,\mathrm{d}\eta + \iint_{H_k(t,x)} q_k(s, \eta)\,\mathrm{d}s\,\mathrm{d}\eta - \iint_{H(t,x)} q(s, \eta)\,\mathrm{d}s\,\mathrm{d}\eta$$

for $(t, x) \in \mathcal{D}$, $k > k_0$. Observe that the assumptions (8.6) and (8.7) yield that

(8.49) $$\lim_{k\to+\infty} (1 + \|\ell_k\|)\left[\int_c^{h_k(t)} \psi_k(\eta)\,\mathrm{d}\eta - \int_c^{h(t)} \psi(\eta)\,\mathrm{d}\eta\right] = 0 \quad \text{uniformly on } [a, b].$$

Therefore, by using the relations (8.4), (8.5), (8.6), (8.8), and (8.49), we get

(8.50) $$\lim_{k\to+\infty} (1 + \|\ell_k\|)[\|\Delta(u_k, h_k) - \Delta(u, h)\|_C + \|\Gamma_k(u_k, u)\|_C] = 0.$$

On the other hand, it follows from the inequality (8.47) that

(8.51) $$\|u_k - u\|_C \leqslant r_0(1 + \|\ell_k\|)[\|\Delta(u_k, h_k) - \Delta(u, h)\|_C + \|\Gamma_k(u_k, u)\|_C] \quad \text{for } k > k_0$$

and thus, by virtue of the relation (8.50), the condition (8.10) holds. □

P r o o f of Corollary 8.1. We shall show that the assumptions of Theorem 8.1 are satisfied. Indeed, the relation (8.11) yields $\|\ell_k\| \leqslant \|\omega\|_L$ for $k \in \mathbb{N}$. Therefore, it is clear that, by virtue of the relations (8.12)–(8.16), the assumptions (8.4)–(8.8) of Theorem 8.1 are fulfilled. It remains to show that the condition (8.2) holds, where the numbers λ_k are given by the formula (8.3).

Assume on the contrary, that the condition (8.2) does not hold. Then there exist $\varepsilon_0 > 0$, an increasing sequence $\{k_m\}_{m=1}^{+\infty}$ of natural numbers, and a sequence $\{y_m\}_{m=1}^{+\infty}$ such that

(8.52) $$y_m \in M(\ell_{k_m}, h_{k_m}) \quad \text{for } m \in \mathbb{N}$$

and

$$\max_{(t,x)\in\mathcal{D}}\left\{\left|\iint_{H_{k_m}(t,x)}\ell_{k_m}(y_m)(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta-\iint_{H(t,x)}\ell(y_m)(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta\right|\right\}\geqslant\varepsilon_0 \quad \text{for } m\in\mathbb{N}. \tag{8.53}$$

In view of (8.52) and Notation 8.1, we get

$$y_m(t,x)=\iint_{H_{k_m}(t,x)}\ell_{k_m}(z_m)(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta \quad \text{for } (t,x)\in\mathcal{D},\ m\in\mathbb{N},$$

where $z_m\in C(\mathcal{D};\mathbb{R})$ and $\|z_m\|_C=1$ for $m\in\mathbb{N}$. Since we suppose that the operators ℓ_k are uniformly bounded in the sense of condition (8.11), we obtain $\|y_m\|_C\leqslant\|\omega\|_L$ for $m\in\mathbb{N}$ and thus the sequence $\{y_m\}_{m=1}^{+\infty}$ is bounded in the space $C(\mathcal{D};\mathbb{R})$. We will show that the sequence indicated is also equicontinuous. Indeed, let $\varepsilon>0$ be arbitrary. Since the function ω is integrable on $\mathcal{D}$, there exists $\delta>0$ such that the relation

$$\iint_E\omega(t,x)\,\mathrm{d}t\,\mathrm{d}x<\frac{\varepsilon}{2} \tag{8.54}$$

holds for every measurable set $E\subseteq\mathcal{D}$ satisfying $\operatorname{mes}E<\max\{b-a,d-c\}\delta$. Using the condition (8.11), for any $(t_1,x_1),(t_2,x_2)\in\mathcal{D}$ and $m\in\mathbb{N}$ we get

$$\left|\iint_{H_{k_m}(t_2,x_2)}\ell_{k_m}(z_m)(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta-\iint_{H_{k_m}(t_1,x_1)}\ell_{k_m}(z_m)(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta\right| \leqslant\sum_{k=1}^{2}\iint_{E_k}\omega(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta,$$

where the measurable sets $E_1,E_2\subseteq\mathcal{D}$ are such that $\operatorname{mes}E_1\leqslant(d-c)|t_2-t_1|$ and $\operatorname{mes}E_2\leqslant(b-a)|x_2-x_1|$. Therefore, by virtue of (8.54), we have

$$|y_m(t_2,x_2)-y_m(t_1,x_1)|<\varepsilon \quad \text{for } (t_1,x_1),(t_2,x_2)\in\mathcal{D},\ |t_2-t_1|+|x_2-x_1|<\delta,\ m\in\mathbb{N}.$$

Consequently, the sequence $\{y_m\}_{m=1}^{+\infty}$ is equicontinuous in the space $C(\mathcal{D};\mathbb{R})$. Therefore, according to the Arzelà-Ascoli lemma, we can assume without loss of generality that the sequence indicated is convergent. Hence, there exists $p_0\in\mathbb{N}$ such that

$$\|y_m-y_{p_0}\|_C<\frac{\varepsilon_0}{2(\|\omega\|_L+\|\ell\|+1)} \quad \text{for } m\geqslant p_0. \tag{8.55}$$

Since $y_{p_0} \in C^*(\mathcal{D};\mathbb{R})$ and the relation (8.12) holds, there exists $p_1 \in \mathbb{N}$ such that

$$\max_{(t,x)\in\mathcal{D}} \left\{ \left| \iint_{H_k(t,x)} \ell_k(y_{p_0})(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta - \iint_{H(t,x)} \ell(y_{p_0})(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta \right| \right\} < \frac{\varepsilon_0}{2} \quad \text{for } k \geqslant p_1. \tag{8.56}$$

Now we choose a number $M \in \mathbb{N}$ satisfying $M \geqslant p_0$ and $k_M \geqslant p_1$. It is clear that

$$\begin{aligned}
&\left| \iint_{H_{k_M}(t,x)} \ell_{k_M}(y_M)(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta - \iint_{H(t,x)} \ell(y_M)(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta \right| \\
&\qquad \leqslant \left| \iint_{H_{k_M}(t,x)} \ell_{k_M}(y_M - y_{p_0})(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta \right| \\
&\qquad\quad + \left| \iint_{H_{k_M}(t,x)} \ell_{k_M}(y_{p_0})(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta - \iint_{H(t,x)} \ell(y_{p_0})(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta \right| \\
&\qquad\quad + \left| \iint_{H(t,x)} \ell(y_{p_0} - y_M)(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta \right| \quad \text{for } (t,x) \in \mathcal{D}.
\end{aligned}$$

Therefore, by virtue of the conditions (8.11), (8.55), and (8.56), the last relation yields

$$\begin{aligned}
&\max_{(t,x)\in\mathcal{D}} \left\{ \left| \iint_{H_{k_M}(t,x)} \ell_{k_M}(y_M)(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta - \iint_{H(t,x)} \ell(y_M)(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta \right| \right\} \\
&\qquad \leqslant \|\omega\|_L \|y_M - y_{p_0}\|_C + \frac{\varepsilon_0}{2} + \|\ell\| \|y_{p_0} - y_M\|_C < \varepsilon_0,
\end{aligned}$$

which contradicts the condition (8.53).

The contradiction obtained proves the validity of the condition (8.2) and thus all the assumptions of Theorem 8.1 are satisfied. □

To prove Corollary 8.2 we need the following lemma.

Lemma 8.2. *Let the condition* (8.18) *hold and let* $\{\sigma_k\}_{k=1}^{+\infty}$ *be a sequence of functions from* $L(\mathcal{D};\mathbb{R})$ *such that*

$$|\sigma_k(t,x)| \leqslant \omega(t,x) \quad \text{for a.e. } (t,x) \in \mathcal{D},\ k \in \mathbb{N}, \tag{8.57}$$

where $\omega \in L(\mathcal{D};\mathbb{R}_+)$. *Then*

$$\lim_{k\to+\infty} \iint_{H(t,x)\div H_k(t,x)} |\sigma_k(s,\eta)|\,\mathrm{d}s\,\mathrm{d}\eta = 0 \quad \text{uniformly on } \mathcal{D} \tag{8.58}$$

and

$$\lim_{k\to+\infty}\left[\iint_{H_k(t,x)}\sigma_k(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta-\iint_{H(t,x)}\sigma_k(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta\right]=0 \quad \text{uniformly on } \mathcal{D}. \tag{8.59}$$

P r o o f. Let $\varepsilon>0$ be arbitrary. Then there exists $\delta>0$ such that the relation

$$\iint_E \omega(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta<\varepsilon \tag{8.60}$$

holds for every measurable set $E\subseteq\mathcal{D}$ with the property $\operatorname{mes}E<2(b-a)\delta$. Put $P=\{(t,x)\in\mathcal{D}\colon\ |x-h(t)|\leqslant\delta\}$. It is easy to verify that

$$\operatorname{mes}P<2(b-a)\delta. \tag{8.61}$$

In view of the condition (8.18), there exists $k_0\in\mathbb{N}$ such that

$$|h_k(t)-h(t)|<\delta \quad \text{for } t\in[a,b],\ k\geqslant k_0, \tag{8.62}$$

and thus

$$(H(t,x)\setminus P)\setminus H_k(t,x)=\emptyset,\quad (H_k(t,x)\setminus P)\setminus H(t,x)=\emptyset \quad \text{for } (t,x)\in\mathcal{D},\ k\geqslant k_0. \tag{8.63}$$

Obviously, for $(t,x)\in\mathcal{D}$ and $k\in\mathbb{N}$ we get

$$\begin{aligned}H(t,x)\div H_k(t,x)&=H(t,x)\setminus H_k(t,x)\cup H_k(t,x)\setminus H(t,x)\\&=[(H(t,x)\setminus P)\setminus H_k(t,x)]\cup[(H(t,x)\cap P)\setminus H_k(t,x)]\\&\quad\cup[(H_k(t,x)\setminus P)\setminus H(t,x)]\cup[(H_k(t,x)\cap P)\setminus H(t,x)].\end{aligned}$$

Therefore, by virtue of (8.57) and (8.63), the last relation yields

$$\begin{aligned}&\iint_{H(t,x)\div H_k(t,x)}|\sigma_k(s,\eta)|\,\mathrm{d}s\,\mathrm{d}\eta\\&\quad=\iint_{(H(t,x)\cap P)\setminus H_k(t,x)}|\sigma_k(s,\eta)|\,\mathrm{d}s\,\mathrm{d}\eta+\iint_{(H_k(t,x)\cap P)\setminus H(t,x)}|\sigma_k(s,\eta)|\,\mathrm{d}s\,\mathrm{d}\eta\\&\quad\leqslant\iint_P|\sigma_k(s,\eta)|\,\mathrm{d}s\,\mathrm{d}\eta\\&\quad\leqslant\iint_P\omega(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta\quad\text{for } (t,x)\in\mathcal{D},\ k\geqslant k_0,\end{aligned}$$

which, together with (8.60) and (8.61), guarantees the relation (8.58).

On the other hand, it is clear that

$$\left|\iint_{H_k(t,x)} \sigma_k(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta - \iint_{H(t,x)} \sigma_k(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta\right|$$

$$= \left|\iint_{H_k(t,x)\setminus H(t,x)} \sigma_k(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta - \iint_{H(t,x)\setminus H_k(t,x)} \sigma_k(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta\right|$$

$$\leqslant \iint_{H(t,x)\div H_k(t,x)} |\sigma_k(s,\eta)|\,\mathrm{d}s\,\mathrm{d}\eta \quad \text{for } (t,x)\in\mathcal{D},\ k\in\mathbb{N}.$$

Consequently, the validity of the condition (8.59) follows immediately from the above-proved relation (8.58). □

Proof of Corollary 8.2. We shall show that the assumptions of Corollary 8.1 are satisfied. Indeed, according to Lemma 8.2, the assumptions (8.11), (8.17), and (8.18) guarantee the validity of the condition (8.12). It remains to verify that the condition (8.15) holds.

Let $\varepsilon > 0$ be arbitrary. Clearly, there is a number $\delta > 0$ such that

$$\left|\int_{x_1}^{x_2} \psi(\eta)\,\mathrm{d}\eta\right| < \varepsilon \quad \text{for } x_1, x_2 \in [c,d],\ |x_2 - x_1| < \delta.$$

Moreover, the assumption (8.18) yields the existence of $k_0 \in \mathbb{N}$ with the property (8.62). Consequently, we have

$$\left|\int_{h_k(t)}^{h(t)} \psi(\eta)\,\mathrm{d}\eta\right| < \varepsilon \quad \text{for } t\in[a,b],\ k \geqslant k_0,$$

and thus the condition (8.15) holds. □

In order to prove Corollary 8.4, we need the following lemmas.

Lemma 8.3. *Let* $f \in L(\mathcal{D};\mathbb{R})$, $w \in C^*(\mathcal{D};\mathbb{R})$, *and* $h \in CD([a,b];[c,d])$. *Then the relation*

$$\iint_{H(t,x)} f(s,\eta)w(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta = z(t,x)w(t,x) - \int_{h^{-1}(x)}^{t} z(s,x)w'_{[1]}(s,x)\,\mathrm{d}s$$
$$- \int_{h(t)}^{x} z(t,\eta)w'_{[2]}(t,\eta)\,\mathrm{d}\eta$$
$$+ \iint_{H(t,x)} z(s,\eta)w''_{[1,2]}(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta \quad \text{for } (t,x)\in\mathcal{D}$$

holds, where the mapping H *is defined by the formula* (2.1) *and*

$$z(t,x) = \iint_{H(t,x)} f(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta \quad \text{for } (t,x)\in\mathcal{D}.$$

P r o o f. Put

$$\chi(t,x)=\begin{cases}1 & \text{for } (t,x)\in\mathcal{D},\ x\geqslant h(t),\\ 0 & \text{for } (t,x)\in\mathcal{D},\ x<h(t)\end{cases}$$

and

$$z_0(t,x)=\int_a^t\int_c^x \chi(s,\eta)f(s,\eta)\,\mathrm{d}\eta\,\mathrm{d}s \quad \text{for } (t,x)\in\mathcal{D}.$$

Clearly,

$$z_0(t,x)=\begin{cases}z(t,x) & \text{if } x\geqslant h(t),\\ 0 & \text{if } x<h(t).\end{cases}$$

It can be verified by direct calculation that for any $(t,x)\in\mathcal{D}$ such that $x\geqslant h(t)$ we have

$$\begin{aligned}\iint_{H(t,x)} f(s,\eta)w(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta &= \int_a^t\int_c^x \chi(s,\eta)f(s,\eta)w(s,\eta)\,\mathrm{d}\eta\,\mathrm{d}s\\ &= z_0(t,x)w(t,x)-\int_a^t z_0(s,x)w'_{[1]}(s,x)\,\mathrm{d}s\\ &\quad -\int_c^x z_0(t,\eta)w'_{[2]}(t,\eta)\,\mathrm{d}\eta+\int_a^t\int_c^x z_0(s,\eta)w''_{[1,2]}(s,\eta)\,\mathrm{d}\eta\,\mathrm{d}s\\ &= z(t,x)w(t,x)-\int_{h^{-1}(x)}^t z(s,x)w'_{[1]}(s,x)\,\mathrm{d}s\\ &\quad -\int_{h(t)}^x z(t,\eta)w'_{[2]}(t,\eta)\,\mathrm{d}\eta+\iint_{H(t,x)} z(s,\eta)w''_{[1,2]}(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta.\end{aligned}$$

By analogy, for any $(t,x)\in\mathcal{D}$ with the property $x\leqslant h(t)$ we get

$$\begin{aligned}\iint_{H(t,x)} f(s,\eta)w(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta &= \int_t^b\int_x^d (1-\chi(s,\eta))f(s,\eta)w(s,\eta)\,\mathrm{d}\eta\,\mathrm{d}s\\ &= z(t,x)w(t,x)+\int_t^{h^{-1}(x)} z(s,x)w'_{[1]}(s,x)\,\mathrm{d}s\\ &\quad +\int_x^{h(t)} z(t,\eta)w'_{[2]}(t,\eta)\,\mathrm{d}\eta+\iint_{H(t,x)} z(s,\eta)w''_{[1,2]}(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta.\end{aligned}$$

Consequently, the assertion of the lemma holds. □

Using the previous statement, we prove the following Krasnosel'skij-Krein type lemma.

Lemma 8.4. *Let $h \in CD([a,b];[c,d])$, $p, p_k \in L(\mathcal{D};\mathbb{R})$, and $\alpha, \alpha_k\colon \mathcal{D} \to \mathbb{R}$ be measurable and essentially bounded functions ($k \in \mathbb{N}$). Assume that the relations* (8.19) *and* (8.20) *with H given by* (2.1) *are satisfied, and*

$$\lim_{k\to+\infty} \operatorname{ess\,sup}\{|\alpha_k(t,x) - \alpha(t,x)|\colon\ (t,x) \in \mathcal{D}\} = 0. \tag{8.64}$$

Then

$$\lim_{k\to+\infty} \iint_{H(t,x)} [p_k(s,\eta)\alpha_k(s,\eta) - p(s,\eta)\alpha(s,\eta)]\,\mathrm{d}s\,\mathrm{d}\eta = 0 \tag{8.65}$$

uniformly on $\mathcal{D}$.

P r o o f. We can assume without loss of generality that

$$|p(t,x)| \leqslant \omega(t,x) \quad \text{for a.e. } (t,x) \in \mathcal{D}. \tag{8.66}$$

Let $\varepsilon > 0$ be arbitrary. According to the assumption (8.64), there exists $k_0 \in \mathbb{N}$ such that

$$\iint_{\mathcal{D}} \omega(t,x)|\alpha_k(t,x) - \alpha(t,x)|\,\mathrm{d}t\,\mathrm{d}x < \frac{\varepsilon}{4} \quad \text{for } k \geqslant k_0. \tag{8.67}$$

Since the function α is measurable and essentially bounded, there exists a function $w \in C(\mathcal{D};\mathbb{R})$ which has continuous all derivatives up to the second order and such that

$$\iint_{\mathcal{D}} \omega(t,x)|\alpha(t,x) - w(t,x)|\,\mathrm{d}t\,\mathrm{d}x < \frac{\varepsilon}{4}. \tag{8.68}$$

For any $k \in \mathbb{N}$, we put

$$z_k(t,x) = \iint_{H(t,x)} [p_k(s,\eta) - p(s,\eta)]\,\mathrm{d}s\,\mathrm{d}\eta \quad \text{for } (t,x) \in \mathcal{D}.$$

Clearly, the condition (8.20) can be rewritten in the form

$$\lim_{k\to+\infty} \|z_k\|_C = 0. \tag{8.69}$$

Lemma 8.3 yields that

$$\begin{aligned}\iint_{H(t,x)} [p_k(s,\eta) - p(s,\eta)]w(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta &= z_k(t,x)w(t,x)\\ &\quad - \int_{h^{-1}(x)}^{t} z_k(s,x)w'_{[1]}(s,x)\,\mathrm{d}s - \int_{h(t)}^{x} z_k(t,\eta)w'_{[2]}(t,\eta)\,\mathrm{d}\eta\\ &\quad + \iint_{H(t,x)} z_k(s,\eta)w''_{[1,2]}(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta \quad \text{for } (t,x) \in \mathcal{D},\ k \in \mathbb{N}.\end{aligned}$$

Consequently, by using the relation (8.69), we get

$$\lim_{k\to+\infty}\iint_{H(t,x)}[p_k(s,\eta)-p(s,\eta)]w(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta=0\quad\text{uniformly on }\mathcal{D}.$$

Hence, there exists a number $k_1\geqslant k_0$ such that

$$\left|\iint_{H(t,x)}[p_k(s,\eta)-p(s,\eta)]w(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta\right|<\frac{\varepsilon}{4}\quad\text{for }(t,x)\in\mathcal{D},\ k\geqslant k_1. \tag{8.70}$$

On the other hand, it is clear that

$$\begin{aligned}\iint_{H(t,x)}&[p_k(s,\eta)\alpha_k(s,\eta)-p(s,\eta)\alpha(s,\eta)]\,\mathrm{d}s\,\mathrm{d}\eta\\&=\iint_{H(t,x)}p_k(s,\eta)[\alpha_k(s,\eta)-\alpha(s,\eta)]\,\mathrm{d}s\,\mathrm{d}\eta\\&\quad+\iint_{H(t,x)}[p_k(s,\eta)-p(s,\eta)]w(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta\\&\quad+\iint_{H(t,x)}[p_k(s,\eta)-p(s,\eta)][\alpha(s,\eta)-w(s,\eta)]\,\mathrm{d}s\,\mathrm{d}\eta\end{aligned}$$

for $(t,x)\in\mathcal{D}$, $k\in\mathbb{N}$. Therefore, in view of the relations (8.19), (8.66)–(8.68), and (8.70), we get

$$\begin{aligned}\left|\iint_{H(t,x)}[p_k(s,\eta)\alpha_k(s,\eta)-p(s,\eta)\alpha(s,\eta)]\,\mathrm{d}s\,\mathrm{d}\eta\right|&\\\leqslant\iint_{\mathcal{D}}\omega(s,\eta)\,|\alpha_k(s,\eta)-\alpha(s,\eta)|\,\mathrm{d}s\,\mathrm{d}\eta&\\+\left|\iint_{H(t,x)}[p_k(s,\eta)-p(s,\eta)]w(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta\right|&\\+2\iint_{\mathcal{D}}\omega(s,\eta)\,|\alpha(s,\eta)-w(s,\eta)|\,\mathrm{d}s\,\mathrm{d}\eta<\varepsilon\quad\text{for }(t,x)\in\mathcal{D},\ k\geqslant k_1,&\end{aligned}$$

and thus the relation (8.65) holds. □

P r o o f of Corollary 8.4. Let the operator ℓ be defined by the formula (6.1). Put

$$\ell_k(v)(t,x)=p_k(t,x)v(\tau_k(t,x),\mu_k(t,x))\qquad\text{for a. e. }(t,x)\in\mathcal{D}\text{ and all }v\in C(\mathcal{D};\mathbb{R}),\ k\in\mathbb{N}. \tag{8.71}$$

We will show that the condition (8.17) is satisfied. Indeed, let $y\in C^*(\mathcal{D};\mathbb{R})$ be arbitrary. It is clear that the conditions (8.21) and (8.22) guarantee the validity of the relation (8.64), where

$$\alpha_k(t,x)=y(\tau_k(t,x),\mu_k(t,x)),\quad \alpha(t,x)=y(\tau(t,x),\mu(t,x))$$

for a. e. $(t,x) \in \mathcal{D}$ and all $k \in \mathbb{N}$. Therefore, Lemma 8.4 guarantees the validity of the condition (8.65) and thus the condition (8.17) holds. On the other hand, by virtue of the relation (8.19), the condition (8.11) is satisfied.

Consequently, the assertion of the corollary follows from Corollary 8.2. □

P r o o f of Corollary 8.5. We first mention that, according to Corollary 7.3, the problems (1.2)–(1.4) and (1.2_k)–(1.4_k) have unique solutions u and u_k, respectively.

Let the operators ℓ and ℓ_k be defined by the formulas

$$\ell(v)(t,x) = p(t,x)v(t,x) \quad \text{for a.e.} \quad (t,x) \in \mathcal{D} \text{ and all } v \in C(\mathcal{D};\mathbb{R}), \tag{8.72}$$

and

$$\ell_k(v)(t,x) = p_k(t,x)v(t,x) \quad \text{for a.e. } (t,x) \in \mathcal{D} \text{ and all } v \in C(\mathcal{D};\mathbb{R}),\ k \in \mathbb{N}, \tag{8.73}$$

respectively. Clearly,

$$\|\ell_k\| = \|p_k\|_L \quad \text{for } k \in \mathbb{N}. \tag{8.74}$$

Therefore, the assumptions (8.5)–(8.8) of Theorem 8.1 are satisfied. In order to apply Theorem 8.1, it remains to show that the conditions (8.2) and (8.4) are fulfilled.

It is easy to see that

$$\begin{aligned}
&\left| \iint_{H_k(t,x)} [p_k(s,\eta) - p(s,\eta)]\,\mathrm{d}s\,\mathrm{d}\eta \right| \\
&\qquad \leqslant \left| \iint_{H_k(t,x)} p_k(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta - \iint_{H(t,x)} p(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta \right| \\
&\qquad\quad + \iint_{H(t,x)\div H_k(t,x)} |p(s,\eta)|\,\mathrm{d}s\,\mathrm{d}\eta \quad \text{for } (t,x) \in \mathcal{D},\ k \in \mathbb{N}.
\end{aligned}$$

Therefore, the conditions (8.23) and (8.24) guarantee that

$$\lim_{k\to+\infty} \varrho_k \|f_k\|_C = 0, \tag{8.75}$$

where

$$f_k(t,x) = \iint_{H_k(t,x)} [p_k(s,\eta) - p(s,\eta)]\,\mathrm{d}s\,\mathrm{d}\eta \quad \text{for } (t,x) \in \mathcal{D},\ k \in \mathbb{N}. \tag{8.76}$$

Observe that for an arbitrary $y \in C(\mathcal{D};\mathbb{R})$ we have

$$\begin{aligned}
&\left| \iint_{H_k(t,x)} \ell_k(y)(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta - \iint_{H(t,x)} \ell(y)(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta \right| \\
&\qquad \leqslant \left| \iint_{H_k(t,x)} [p_k(s,\eta) - p(s,\eta)]y(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta \right| \\
&\qquad\quad + \iint_{H(t,x)\div H_k(t,x)} |p(s,\eta)y(s,\eta)|\,\mathrm{d}s\,\mathrm{d}\eta \quad \text{for } (t,x) \in \mathcal{D},\ k \in \mathbb{N}.
\end{aligned} \tag{8.77}$$

Moreover, for an arbitrary $y \in C^*(\mathcal{D};\mathbb{R})$, Lemma 8.3 guarantees that

$$\begin{aligned}(8.78)\qquad \iint_{H_k(t,x)} [p_k(s,\eta) - p(s,\eta)]y(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta &= f_k(t,x)y(t,x)\\ &- \int_{h_k^{-1}(x)}^{t} f_k(s,x)y'_{[1]}(s,x)\,\mathrm{d}s - \int_{h_k(t)}^{x} f_k(t,\eta)y'_{[2]}(t,\eta)\,\mathrm{d}\eta\\ &+ \iint_{H_k(t,x)} f_k(s,\eta)y''_{[1,2]}(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta \quad \text{for } (t,x)\in\mathcal{D},\ k\in\mathbb{N}.\end{aligned}$$

Let $k \in \mathbb{N}$ and $y \in M(\ell_k, h_k)$ be arbitrary. Then, by virtue of Notation 8.1 and Lemma 3.4, we get

$$(8.79)\qquad |y(t,x)| = \left|\iint_{H_k(t,x)} p_k(s,\eta)z(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta\right| \leqslant \varrho_k \quad \text{for } (t,x)\in\mathcal{D},$$

$$(8.80)\qquad |y'_{[1]}(t,x)| = \left|\int_{h_k(t)}^{x} p_k(t,\eta)z(t,\eta)\,\mathrm{d}\eta\right| \leqslant \int_c^d |p_k(t,\eta)|\,\mathrm{d}\eta$$

for a.e. $t\in[a,b]$ and all $x\in[c,d]$,

$$(8.81)\qquad |y'_{[2]}(t,x)| = \left|\int_{h_k^{-1}(x)}^{t} p_k(s,x)z(s,x)\,\mathrm{d}s\right| \leqslant \int_a^b |p_k(s,x)|\,\mathrm{d}s$$

for all $t\in[a,b]$ and a.e. $x\in[c,d]$,

and

$$(8.82)\qquad |y''_{[1,2]}(t,x)| = |p_k(t,x)z(t,x)| \leqslant |p_k(t,x)| \quad \text{for a.e. } (t,x)\in\mathcal{D}.$$

By virtue of relations (8.79)–(8.82), it follows from the inequalities (8.77) and (8.78) that

$$\begin{aligned}&\left|\iint_{H_k(t,x)} \ell_k(y)(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta - \iint_{H(t,x)} \ell(y)(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta\right|\\ &\qquad\leqslant 4\varrho_k\|f_k\|_C + \varrho_k \iint_{H(t,x)\div H_k(t,x)} |p(s,\eta)|\,\mathrm{d}s\,\mathrm{d}\eta \quad \text{for } (t,x)\in\mathcal{D},\ k\in\mathbb{N}.\end{aligned}$$

Therefore, according to the relations (8.24) and (8.75), the condition (8.2) holds, where the numbers λ_k are given by the formula (8.3).

Now let $y \in C^*(\mathcal{D};\mathbb{R})$ be arbitrary. Put

$$\begin{aligned}(8.83)\qquad \varrho_0 = \|y\|_C &+ \max\left\{\int_a^b |y'_{[1]}(s,x)|\,\mathrm{d}s\colon\ x\in[c,d]\right\}\\ &+ \max\left\{\int_c^d |y'_{[2]}(t,\eta)|\,\mathrm{d}\eta\colon\ t\in[a,b]\right\} + \|y''_{[1,2]}\|_L.\end{aligned}$$

Then the inequalities (8.77) and (8.78) imply that

$$\left| \iint_{H_k(t,x)} \ell_k(y)(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta - \iint_{H(t,x)} \ell(y)(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta \right| \\ \leqslant \varrho_0 \left[\|f_k\|_C + \iint_{H(t,x)\div H_k(t,x)} |p(s,\eta)|\,\mathrm{d}s\,\mathrm{d}\eta \right] \quad \text{for } (t,x) \in \mathcal{D},\ k \in \mathbb{N}.$$

According to the relations (8.24) and (8.75), the last inequality yields the validity of the condition (8.4).

Consequently, the assertion of the corollary follows from Theorem 8.1. □

P r o o f of Corollary 8.6. We will show that all the assumptions of Corollary 8.5 are satisfied. It follows from the condition (8.26) that

$$\sup\{\|p_k\|_L\colon\ k \in \mathbb{N}\} < +\infty. \tag{8.84}$$

Therefore, in view of the relations (8.14) and (8.16), the assumptions (8.6) and (8.8) of Corollary 8.5 are satisfied. Moreover, analogously to the proof of Corollary 8.2 it can be shown that the conditions (8.14) and (8.18) yield the validity of the relation (8.15) and thus the assumption (8.7) of Corollary 8.5 holds. Furthermore, by virtue of the relations (8.18) and (8.84), Lemma 8.2 guarantees that the condition (8.24) holds.

On the other hand, it is clear that

$$\left| \iint_{H_k(t,x)} p_k(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta - \iint_{H(t,x)} p(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta \right| \leqslant \|p_k - p\|_L \\ + \left| \iint_{H_k(t,x)} p(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta - \iint_{H(t,x)} p(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta \right| \quad \text{for } (t,x) \in \mathcal{D},\ k \in \mathbb{N}, \tag{8.85}$$

and

$$\left| \iint_{H_k(t,x)} q_k(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta - \iint_{H(t,x)} q(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta \right| \leqslant \|q_k - q\|_L \\ + \left| \iint_{H_k(t,x)} q(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta - \iint_{H(t,x)} q(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta \right| \quad \text{for } (t,x) \in \mathcal{D},\ k \in \mathbb{N}. \tag{8.86}$$

Therefore, by virtue of the conditions (8.18), (8.26), (8.27), (8.84), and Lemma 8.2, the relations (8.85) and (8.86) imply the validity of the assumptions (8.5) and (8.23) of Corollary 8.5. □

Example 9.1. Let $p \in L(\mathcal{D};\mathbb{R}_+)$ and $h \in CD([a,b];[c,d])$ be such that the relations

$$\iint_{H(b,d)} p(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta = 1, \qquad \iint_{H(a,c)} p(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta \leqslant 1$$

are fulfilled, where the mapping H is defined by the formula (2.1). Let, moreover, the operator ℓ be defined by the formula

$$\ell(v)(t,x) = p(t,x)v(b,d) \quad \text{for a.e. } (t,x) \in \mathcal{D} \text{ and all } v \in C(\mathcal{D};\mathbb{R}).$$

Then the condition (7.2) with $\alpha = 1$ is satisfied for every $m \in \mathbb{N}$ and $v \in C(\mathcal{D};\mathbb{R})$. Moreover,

$$\int_a^b \int_{h(s)}^d p_j(s,\eta)\,\mathrm{d}\eta\,\mathrm{d}s = 1, \quad \int_a^b \int_c^{h(s)} p_j(s,\eta)\,\mathrm{d}\eta\,\mathrm{d}s \leqslant 1 \quad \text{for } j \in \mathbb{N},$$

where the functions p_j are given by the formula (7.4).

On the other hand, the homogeneous problem (1.1_0), (1.3_0), (1.4_0) has a nontrivial solution

$$u(t,x) = \iint_{H(t,x)} p(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta \quad \text{for } (t,x) \in \mathcal{D}.$$

This example shows that the assumption $\alpha \in [0,1[$ in Theorem 7.1 cannot be replaced by the assumption $\alpha \in [0,1]$, and the strict inequality (7.3) in Corollary 7.1 cannot be replaced by the nonstrict one.

Example 9.2. Let $\mathcal{D} = [0,1] \times [0,1]$,

$$r_k(t) = k\sin(k^2 t), \quad f_k(t) = k\cos(k^2 t) \quad \text{for } t \in [-1,1],\ k \in \mathbb{N}, \tag{9.1}$$

$$y_k(t) = k\mathrm{e}^{-\cos(k^2 t)/k} \int_0^t \mathrm{e}^{\cos(k^2 s)/k} \cos(k^2 s)\,\mathrm{d}s \quad \text{for } t \in [-1,1],\ k \in \mathbb{N}, \tag{9.2}$$

and

$$z_k(t) = \int_0^t y_k(s)\,\mathrm{d}s \quad \text{for } t \in [-1,1],\ k \in \mathbb{N}. \tag{9.3}$$

It is not difficult to verify that for every $k \in \mathbb{N}$ we have

$$y_k'(t) = r_k(t)y_k(t) + f_k(t) \quad \text{for } t \in [-1,1],\ k \in \mathbb{N}, \tag{9.4}$$

$$|y_k(t)| \leqslant 1 + |t|\mathrm{e}^2 \quad \text{for } t \in [-1,1],\ k \in \mathbb{N}, \tag{9.5}$$

and

$$\lim_{k\to+\infty} y_k(t) = \frac{t}{2} \quad \text{for } t\in[-1,1], \tag{9.6}$$

because

$$\begin{aligned} y_k(t) = \frac{1}{k}\,\sin(k^2t) + \frac{1}{2}\,\mathrm{e}^{-\cos(k^2t)/k}\int_0^t \mathrm{e}^{\cos(k^2s)/k}\,\mathrm{d}s \\ - \frac{1}{2}\,\mathrm{e}^{-\cos(k^2t)/k}\int_0^t \mathrm{e}^{\cos(k^2s)/k}\cos(2k^2s)\,\mathrm{d}s \quad \text{for } t\in[-1,1],\ k\in\mathbb{N}. \end{aligned}$$

Obviously, the relations (9.2)–(9.6) yield the equality

$$z_k''(t) = -r_k'(t)z_k(t) + w_k'(t) + f_k(t) \quad \text{for } t\in[-1,1],\ k\in\mathbb{N},$$

where

$$w_k(t) = r_k(t)z_k(t) \quad \text{for } t\in[-1,1],\ k\in\mathbb{N}, \tag{9.7}$$

and, moreover, the relation

$$\lim_{k\to+\infty} z_k(t) = \frac{t^2}{4} \quad \text{uniformly on } [-1,1]. \tag{9.8}$$

Furthermore, it follows from (9.1) that

$$\lim_{k\to+\infty}\int_0^t r_k(s)\,\mathrm{d}s = 0 \quad \text{uniformly on } [-1,1], \tag{9.9}$$

$$\lim_{k\to+\infty}\int_0^t f_k(s)\,\mathrm{d}s = 0 \quad \text{uniformly on } [-1,1]. \tag{9.10}$$

The relations (9.3) and (9.7) give

$$\int_0^t w_k(s)\,\mathrm{d}s = z_k(t)\int_0^t r_k(s)\,\mathrm{d}s - \int_0^t y_k(s)\Big(\int_0^t r_k(\xi)\,\mathrm{d}\xi\Big)\,\mathrm{d}s$$

for $t\in[-1,1]$ and $k\in\mathbb{N}$ and thus, by using (9.5), (9.8), (9.9), and the Krasnosel'skij-Krein lemma, we get

$$\lim_{k\to+\infty}\int_0^t w_k(s)\,\mathrm{d}s = 0 \quad \text{uniformly on } [-1,1]. \tag{9.11}$$

Now, let $p \equiv 0$ and $q \equiv 0$ on $\mathcal{D}$, $g \equiv 0$, $\varphi \equiv 0$, and $\psi \equiv 0$ on $[0,1]$,

$$\tau(t,x) = t, \quad \mu(t,x) = x \quad \text{for } (t,x) \in \mathcal{D},$$

and

$$h(t) = 1 - t \quad \text{for } t \in [0,1].$$

Moreover, for any $k \in \mathbb{N}$, we put $g_k \equiv 0$, $\varphi_k \equiv 0$, and $\psi_k \equiv 0$ on $[0,1]$,

$$\begin{gathered} p_k(t,x) = -r_k'(t+x-1) \quad \text{for } (t,x) \in \mathcal{D}, \\ q_k(t,x) = w_k'(t+x-1) + f_k(t+x-1) \quad \text{for } (t,x) \in \mathcal{D}, \\ \tau_k(t,x) = t, \quad \mu_k(t,x) = x \quad \text{for } (t,x) \in \mathcal{D}, \end{gathered}$$

and

$$h_k(t) = 1 - t \quad \text{for } t \in [0,1].$$

It can be easily verified by direct calculation that

$$\begin{aligned} \iint_{H(t,x)} p_k(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta &= -\int_{1-t}^{x}\int_{1-\eta}^{t} r_k'(s+\eta-1)\,\mathrm{d}s\,\mathrm{d}\eta \\ &= -\int_{1-t}^{x} r_k(t+\eta-1)\,\mathrm{d}\eta = -\int_{0}^{t+x-1} r_k(\xi)\,\mathrm{d}\xi \quad \text{for } (t,x) \in \mathcal{D},\ k \in \mathbb{N}, \\ \iint_{H_k(t,x)} w_k'(s+\eta-1)\,\mathrm{d}s\,\mathrm{d}\eta &= \int_{1-t}^{x}\int_{1-\eta}^{t} w_k'(s+\eta-1)\,\mathrm{d}s\,\mathrm{d}\eta \\ &= \int_{1-t}^{x} w_k(t+\eta-1)\,\mathrm{d}\eta = \int_{0}^{t+x-1} w_k(\xi)\,\mathrm{d}\xi \quad \text{for } (t,x) \in \mathcal{D},\ k \in \mathbb{N}, \end{aligned}$$

and

$$\begin{aligned} \iint_{H_k(t,x)} f_k(s+\eta-1)\,\mathrm{d}s\,\mathrm{d}\eta &= \int_{1-t}^{x}\int_{1-\eta}^{t} f_k(s+\eta-1)\,\mathrm{d}s\,\mathrm{d}\eta \\ &= \int_{1-t}^{x}\left(\int_{0}^{t+\eta-1} f_k(\xi)\,\mathrm{d}\xi\right)\mathrm{d}\eta \quad \text{for } (t,x) \in \mathcal{D},\ k \in \mathbb{N}. \end{aligned}$$

Therefore, by virtue of the conditions (9.9)–(9.11), the relations (8.13) and (8.20) hold.

Consequently, the assumptions of Corollary 8.4 are satisfied except the condition (8.19). Let the operators ℓ and ℓ_k be defined by the formulas (6.1) and (8.71), respectively. Then, in view of Lemma 8.3, it is easy to verify that the assumptions of Corollary 8.1 are fulfilled except the condition (8.11).

On the other hand,

$$u(t,x) = 0 \quad \text{for } (t,x) \in \mathcal{D}$$

and

$$u_k(t,x) = z_k(t+x-1) \quad \text{for } (t,x) \in \mathcal{D},\ k \in \mathbb{N}$$

are solutions to the problems (1.1′), (1.3), (1.4) and ($1.1'_k$), (1.3_k), (1.4_k), respectively, as well as solutions to the problems (1.1), (1.3), (1.4) and (1.1_k), (1.3_k), (1.4_k), respectively. However, in view of the condition (9.8), we get

$$\lim_{k\to+\infty} u_k(t,x) = \lim_{k\to+\infty} z_k(t+x-1) = \frac{(t+x-1)^2}{4} \quad \text{for } (t,x) \in \mathcal{D}$$

and thus the relation (8.10) does not hold.

This example shows that the assumption (8.11) in Corollary 8.1 and the assumption (8.19) in Corollary 8.4 are essential and cannot be omitted.

References

[1] *D. Bielawski*: On the set of solutions of boundary value problems for hyperbolic differential equations. J. Math. Anal. Appl. *253* (2001), 334–340.

[2] *C. Carathéodory*: Vorlesungen über Reelle Funktionen. Leipzig und Berlin: B. G. Teubner, 1918. (In German.)

[3] *K. Deimling*: Absolutely continuous solutions of Cauchy's problem for $u_{xy} = f(x,y,u,u_x,u_y)$. Ann. Mat. Pura Appl., IV. Ser. *89* (1971), 381–391.

[4] *K. Deimling*: Das Picard-Problem für $u_{xy} = f(x,y,u,u_x,u_y)$ unter Carathéodory-Voraussetzungen. Math. Z. *114* (1970), 303–312. (In German.)

[5] *N. Dunford, J. T. Schwartz*: Linear Operators. I. General Theory. New York and London: Interscience Publishers. XIV, 1958.

[6] *M. Grigolia*: On the existence and uniqueness of solutions of the Goursat problem for systems of functional partial differential equations of hyperbolic type. Mem. Differ. Equ. Math. Phys. *16* (1999), 154–158.

[7] *M. P. Grigolia*: On a generalized characteristic boundary value problem for hyperbolic systems. Differ. Uravn. *21* (1985), 678–686.

[8] *S. Kharibegashvili*: Goursat and Darboux type problems for linear hyperbolic partial differential equations and systems. Mem. Differ. Equ. Math. Phys. *4* (1995), 127 p.

[9] *T. Kiguradze*: Some boundary value problems for systems of linear partial differential equations of hyperbolic type. Mem. Differ. Equ. Math. Phys. *1* (1994), 144 p.

[10] *T. I. Kiguradze*: On periodic boundary value problems for linear hyperbolic equations. I. Differ. Equations *29* (1993), 231–245; translation from Differ. Uravn. *29* (1993), 281–297.

[11] *T. I. Kiguradze*: Periodic boundary value problems for hyperbolic equations. II. Differ. Equations *29* (1993), 542–549; translation from Differ. Uravn. *29* (1993), 637–645.

[12] *V. Lakshmikantham, S. G. Pandit*: The method of upper, lower solutions and hyperbolic partial differential equations. J. Math. Anal. Appl. *105* (1985), 466–477.

[13] *Yu. A. Mitropol'skij, L. B. Urmancheva*: On two-point problem for systems of hyperbolic equations. Ukr. Math. J. *42* (1990), 1492–1498; translation from Ukr. Mat. Zh. *42* (1990), 1657–1663.

[14] *I. P. Natanson*: Theory of Functions of Real Variable. Nauka, Moscow, 1974. (In Russian.)

[15] *H. H. Schaefer*: Normed tensor products of Banach lattices. Proc. internat. Sympos. partial diff. Equ. Geometry normed lin. Spaces II. Isr. J. Math. *13* (1972), 400–415.

[16] *J. Šremr*: Absolutely continuous functions of two variables in the sense of Carathéodory. Electron J. Differ. Equ. *2010 (2010)*, 11 p.

[17] *G. P. Tolstov*: On the mixed second derivative. Mat. Sb., N. Ser. *24* (1949), 27–51. (In Russian.)

[18] *F. G. Tricomi*: Lezioni Sulle Equazioni a Derivate Parziali. Corso di Analisi Superiore. Editrice Gheroni, Torino, 1954. (In Italian.)

[19] *O. Vejvoda*: Periodic solutions of a linear and weakly nonlinear wave equation in one dimension. I. Czech. Math. J. *14* (1964), 341–382.

[20] *O. Vejvoda et al.*: Partial Differential Equations. SNTL, Praha, 1981.

[21] *S. Walczak*: Absolutely continuous functions of several variables and their application to differential equations. Bull. Pol. Acad. Sci., Math. *35* (1987), 733–744.

[22] *W. Walter*: Differential and Integral Inequalities. Translated by Lisa Rosenblatt and Lawrence Shampine. Ergebnisse der Mathematik und ihrer Grenzgebiege. Band 55. Berlin-Heidelberg-New York: Springer-Verlag, 1970.

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