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AN IDENTITY WITH GENERALIZED DERIVATIONS ON LIE IDEALS, RIGHT IDEALS AND BANACH ALGEBRAS

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Abstract. Let R be a prime ring of characteristic different from 2, U the Utumi quotient ring of R, $C = Z(U)$ the extended centroid of R, L a non-central Lie ideal of R, F a non-zero generalized derivation of R. Suppose that $[F(u), u]F(u) = 0$ for all $u \in L$, then one of the following holds:

(1) there exists $\alpha \in C$ such that $F(x) = \alpha x$ for all $x \in R$;
(2) R satisfies the standard identity s_4 and there exist $a \in U$ and $\alpha \in C$ such that $F(x) = ax + xa + \alpha x$ for all $x \in R$.

We also extend the result to the one-sided case. Finally, as an application we obtain some range inclusion results of continuous or spectrally bounded generalized derivations on Banach algebras.

Keywords: prime rings, differential identities, generalized derivations, Banach algebra

MSC 2010: 16N60, 16W25, 47B47, 47B48

1. Introduction

Let R be a prime ring with center $Z(R)$ and extended centroid C. Many results in literature indicate that the global structure of a ring R is often tightly connected to the behaviour of additive mappings defined on R. A well known result of Posner [25] states that if d is a derivation of R such that $[d(x), x] \in Z(R)$ for any $x \in R$, then either $d = 0$ or R is commutative. Later in [18] Lanski proved that if d is a nonzero derivation of R such that $[d(x), x] \in Z(R)$ for all $x \in L$, a non-central Lie ideal of R, then $\operatorname{char}(R) = 2$ and R satisfies the standard identity S_4.

In [5] the first author proved that if the characteristic of the ring is different from 2, then the annihilator of the set $A = \{d(u)u - ud(u), u \in L\}$, with L a non-central Lie ideal of R, is zero. Moreover, as a consequence of the main result in [6], it follows that the centralizer $C(A)$ of the set $\{d(u)u - ud(u), u \in L\}$ is trivial, that is $C(A) = Z(R)$. These facts in a prime ring are natural tests which indicate that the set $\{d(u)u - ud(u), u \in L\}$ is rather large in R.

Here we will consider a similar situation in the case the derivation d is replaced by the generalized derivations F. Our purpose is to investigate the set $\{[F(x), x]F(x), x \in S\}$, where S is either a Lie ideal or a right ideal of a prime ring R. More specifically, an additive map $F\colon\ R \longrightarrow R$ is said to be a generalized derivation if there is a derivation d of R such that, for all $x, y \in R$, $F(xy) = F(x)y + xd(y)$. A significative example is a map of the form $F(x) = ax + xb$, for some $a, b \in R$; such generalized derivations are called inner. Generalized derivations have been primarily studied on operator algebras. Therefore any investigation from the algebraic point of view might be interesting (see for example [19]). Here our purpose is to prove the following theorem:

Theorem. *Let R be a prime ring of characteristic different from 2, U the Utumi quotient ring of R, $C = Z(U)$ the extended centroid of R, L a non-central Lie ideal of R, F a non-zero generalized derivation of R. Suppose that $[F(u), u]F(u) = 0$ for all $u \in L$. Then one of the following assertions holds:*

(1) *there exists $\alpha \in C$ such that $F(x) = \alpha x$ for all $x \in R$;*
(2) *R satisfies the standard identity s_4 and there exist $a \in U$ and $\alpha \in C$ such that $F(x) = ax + xa + \alpha x$ for all $x \in R$.*

As a consequence we also prove the following:

Theorem. *Let R be a non-commutative prime ring of characteristic different from 2, U the Utumi quotient ring of R, $C = Z(U)$ the extended centroid of R, I a non-zero right ideal of R, F a non-zero generalized derivation of R. Suppose that $[F(x), x]F(x) = 0$ for all $x \in I$. Then one of the following assertions holds:*

(1) *$[I, I]I = (0)$;*
(2) *there exist $a, b \in U$ and $\alpha, \beta \in C$ such that $F(x) = ax + xb$ for all $x \in R$, with $(a - \alpha)I = (0)$ and $(b - \beta)I = (0)$.*

In the last section of this paper we will consider R as a Banach algebra with Jacobson radical $\mathrm{rad}(R)$. The classical result of Singer and Wermer in [27] says that any continuous derivation on a commutative Banach algebra has the range in the Jacobson radical of the algebra. Singer and Wermer also formulated the conjecture that the continuity assumption can be removed. In 1988 Thomas verified this conjecture in [28].

Of course the same result of Singer and Wermer does not hold in noncommutative Banach algebras (because of inner derivations). Hence in this context a very interesting question is how to obtain the noncommutative version of the Singer-Wermer theorem. A first answer to this problem was obtained by Sinclair in [26]. He proved that every continuous derivation of a Banach algebra leaves primitive ideals of the algebra invariant. Since then many authors obtained more information about derivations satisfying certain suitable conditions in Banach algebras.

In [22] Mathieu and Murphy proved the result that if d is a continuous derivation on an arbitrary Banach algebra such that $[d(r), r] \in Z(R)$ for all $r \in R$, then d maps into the radical. Later in [23] Mathieu and Runde removed the continuity assumption using the classical result of Posner on centralizing derivations of prime rings in [25] and Thomas' theorem in [28]: they showed that if d is a derivation which satisfies $[d(r), r] \in Z(R)$ for all r in a Banach algebra R, then d has its range in the radical of the algebra.

Continuing along this line, in [16] it is proved that if d is a continuous linear Jordan derivation in a Banach algebra R such that $[d(x), x]d(x)[d(x), x] \in \mathrm{rad}(R)$ for all $x \in R$, then d maps into $\mathrm{rad}(R)$. Then in [17] the same conclusion is obtained in the case $d(x)[d(x), x]d(x) \in \mathrm{rad}(R)$ for all $x \in R$.

More recently in [24], Park proves that if d is a derivation of a non-commutative Banach algebra R such that $[[d(x), x], d(x)] \in \mathrm{rad}(R)$ for all $x \in R$, then again d maps into $\mathrm{rad}(R)$.

Here we will continue the investigation about the relationship between the structure of an algebra R and the behaviour of generalized derivations defined on R.

Then we apply our first result on prime rings to the study of analogous conditions for continuous or spectrally bounded generalized derivations on Banach algebras. More precisely, we will prove:

Theorem. *Let R be a non-commutative Banach algebra, F a continuous generalized derivation of R such that $F(x) = ax + d(x)$ for some element $a \in R$ and d a derivation of R. If $[F(x), x]F(x) \in \mathrm{rad}(R)$ for all $x \in R$, then $d(R) \subseteq \mathrm{rad}(R)$, $[a, R] \subseteq \mathrm{rad}(R)$.*

Theorem. *Let R be a Banach algebra, $F = L_a + d$ a spectrally bounded generalized derivation of R for some element $a \in R$ and d a derivation of R. If $[F(x), x]F(x) \in \mathrm{rad}(R)$ for all $x \in R$ then $d(R) \subseteq \mathrm{rad}(R)$ and $[a, R] \subseteq \mathrm{rad}(R)$.*

Before starting with the proofs, we fix some well known facts. In all that follows let R be a non commutative prime ring, U its Utumi quotient ring and $C = Z(U)$ the center of U. We refer the reader to [2] for the definitions and the related properties of these objects. Moreover, we denote by s_4 the standard polynomial in 4 non-commuting variables. In particular, we make use of the following facts:

Fact 1. If I is a two-sided ideal of R, then R, I and U satisfy the same generalized polynomial identities with coefficients in U ([4]).

Fact 2. Every derivation d of R can be uniquely extended to a derivation of U (see Proposition 2.5.1 in [2]).

Fact 3. We denote by $\mathrm{Der}(U)$ the set of all derivations on U. By a derivation word we mean an additive map Δ of the form $\Delta = d_1 d_2 \dots d_m$, with each $d_i \in \mathrm{Der}(U)$. Then a differential polynomial is a generalized polynomial, with coefficents in U, of the form $\Phi({}^{\Delta_j} x_i)$ involving noncommutative indeterminates x_i on which the derivation words Δ_j act as unary operations. The differential polynomial $\Phi({}^{\Delta_j} x_i)$ is said to be a differential identity on a subset T of U if it vanishes for any assignment of values from T to its indeterminates x_i.

Let D_{int} be the C-subspace of $\mathrm{Der}(U)$ consisting of all inner derivations on U and let d be a non-zero derivation on R. By Theorem 2 in [15] we have the following result (see also Theorem 1 in [20]): If $\Phi(x_1, \dots, x_n, {}^d x_1, \dots, {}^d x_n)$ is a differential identity on R, then one of the following assertions holds:

(1) either $d \in D_{\text{int}}$;

(2) or R satisfies the generalized polynomial identity $\Phi(x_1, \dots, x_n, y_1, \dots, y_n)$.

Fact 4. If I is a two-sided ideal of R, then R, I and U satisfy the same differential identities ([20]).

We refer the reader to Chapter 7 in [2] for a complete and detailed description of the theory of generalized polynomial identities involving derivations.

Fact 5. If one assumes that either R does not satisfy s_4 or $\mathrm{char}\,(R) \neq 2$, then there exists a non-zero two-sided ideal I of R such that $0 \neq [I, R] \subseteq L$. In particular, if R is a simple ring it follows that $[R, R] \subseteq L$.

This follows from pp. 4–5 in [11], Lemma 2 and Proposition 1 in [8].

2. The case of inner generalized derivations on prime rings

In this section we study the case when the generalized derivation F is inner defined as follows: $F(x) = ax + xb$ for all $x \in R$, where a, b are fixed elements of U.

In all that follows we denote

$$P(x_1, x_2) = [a[x_1, x_2] + [x_1, x_2]b, [x_1, x_2]](a[x_1, x_2] + [x_1, x_2]b)$$

and assume that R satisfies the generalized identity $P(x_1, x_2)$.

In order to prove the main proposition, we also need the following

Remark 1. Notice that in case F is an inner generalized derivation, then we may write $F(x) = ax + xb$ for all $x \in R$ and by the main assumption of the paper we have that

$$[a[r_1, r_2] + [r_1, r_2]b, [r_1, r_2]](a[r_1, r_2] + [r_1, r_2]b) = 0$$

for all $r_1, r_2 \in R$. Moreover, for any inner automorphism φ of R we have that

$$[\varphi(a)[r_1,r_2]+[r_1,r_2]\varphi(b),[r_1,r_2]](\varphi(a)[r_1,r_2]+[r_1,r_2]\varphi(b))=0$$

for all $r_1, r_2 \in R$. Clearly a (or b, $a+b$, $a-b$) is central in R if and only if $\varphi(a)$ (or $\varphi(b)$, $\varphi(a+b)$, $\varphi(a-b)$, respectively) is central in R. Hence, to prove our result, if necessary we may replace a, b respectively with $\varphi(a)$, $\varphi(b)$.

Lemma 1. *Let F be a infinite field and $n \geqslant 2$. If $A_1, \ldots, A_k$ are not scalar matrices in $M_n(F)$ then there exists an invertible matrix $Q \in M_n(F)$ such that each of the matrices $QA_1Q^{-1}, \ldots, QA_kQ^{-1}$ has no zero entries (for the Proof see* [7] *Lemma* 1.5*)*

Lemma 2. *Let $R = M_m(K)$ be the ring of $m \times m$ matrices over the field K of characteristic different from 2, with $m > 1$, $q \in R$ such that $[uq,u]uq = 0$ for all $u \in [R,R]$. Then $q \in Z(R)$.*

P r o o f. Assume q is a non-scalar matrix and prove that a contradiction follows. By Remark 1 and Lemma 1, we may assume that q has no zero entries. Say $q = \sum\limits_{ij} q_{ij}e_{ij}$, where $q_{ij} \in K$, and e_{ij} are the usual matrix units. Let $u = [r_1,r_2] = [e_{ij},e_{ji}] = e_{ii} - e_{jj}$ for any $i \neq j$. Thus

$$X = [(e_{ii}-e_{jj})q, e_{ii}-e_{jj}](e_{ii}-e_{jj})q = 0$$

and, in particular, the (j,i)-entry of the matrix X is zero. By calculation it follows that $-2q_{ji}q_{ii} = 0$, which is a contradiction. Thus we conclude that q must be a central matrix in R. □

Lemma 3. *Let $R = M_m(K)$ be the ring of $m \times m$ matrices over the field F of characteristic different from 2, with $m \geqslant 3$, $q \in R$ and $\alpha \in Z(R)$ such that $[q,u^2](qu+uq+\alpha u) = 0$ for all $u \in [R,R]$. Then $q \in Z(R)$.*

P r o o f. As above, let $q = \sum\limits_{ij} q_{ij}e_{ij}$, where $q_{ij} \in K$, and e_{ij} are the usual matrix units. Let $u = [r_1,r_2] = [e_{ij},e_{ji}] = e_{ii} - e_{jj}$ for any $i \neq j$. Applying the main assumption of this lemma we have that

$$[q, e_{ii}+e_{jj}](q(e_{ii}-e_{jj}) + (e_{ii}-e_{jj})q + \alpha(e_{ii}-e_{jj}))$$

and both the right and left multiplying by e_{kk} for any $k \neq i,j$ yields

$$q_{ki}q_{ik} - q_{kj}q_{jk} = 0. \tag{1}$$

By Remark 1 we know that q and $\varphi(q)$ possess the same properties for all $\varphi \in \mathrm{Aut}\,(R)$. In particular, let $\varphi(x) = (1+e_{kj})x(1-e_{kj})$, $\chi(x) = (1-e_{kj})x(1+e_{kj})$ for all $k \neq i, j$, and denote $\varphi(q) = \sum c_{rs}e_{rs}$, $\chi(q) = p_{rs}e_{rs}$, for suitable elements c_{rs} and p_{rs} of K. By applying (1) we have

$$c_{ki}c_{ik} - c_{kj}c_{jk} = 0,$$

that is

$$q_{ji}q_{ik} - q_{jj}q_{jk} + q_{kk}q_{jk} + q_{jk}^2 = 0 \tag{2}$$

and also

$$p_{ki}p_{ik} - p_{kj}p_{jk} = 0,$$

that is

$$-q_{ji}q_{ik} + q_{jj}q_{jk} - q_{kk}q_{jk} + q_{jk}^2 = 0. \tag{3}$$

Comparing (2) with (3) we get $q_{jk} = 0$, that is, q is a diagonal matrix in R. Consider now the inner automorphism of R induced by the invertible matrix $P = I + e_{ij}$ for any $i \neq j$: $\lambda(x) = PxP^{-1}$. By calculation we have that $\lambda(q) = q + e_{ij}q - qe_{ij} - e_{ij}qe_{ij}$ and by the previous argument we also have that $\lambda(q)$ is a diagonal matrix. In particular, the (i,j)-entry of $\lambda(q)$ is zero, that is $q_{ii} = q_{jj}$. By the arbitrariness of $i \neq j$, we have that q is a central matrix in R. $\square$

Lemma 4. *Let $R = M_m(K)$ be the ring of $m \times m$ matrices over the infinite field K of characteristic different from 2, with $m \geqslant 2$, $a, b \in R$ such that $[au+ub, u](au+ub) = 0$ for all $u \in [R,R]$. Then either $a, b \in Z(R)$ or $m = 2$ and $b - a \in Z(R)$.*

P r o o f. Assume that neither a nor $b-a$ is a scalar matrix. By Remark 1 and Lemma 1, we may assume that a and $b-a$ have no zero entries. Say $a = \sum_{ij} a_{ij}e_{ij}$ and $b - a = \sum_{ij} c_{ij}e_{ij}$, where $a_{ij}, c_{ij} \in K$, and e_{ij} are the usual matrix units. Let $u = [r_1, r_2] = [e_{ij}, e_{jj}] = e_{ij}$ for any $i \neq j$. Thus by our assumption $e_{ij}(b-a)e_{ij}ae_{ij} = 0$, that is $c_{ji}a_{ji} = 0$, a contradiction. Therefore either $a \in Z(R)$ or $b - a \in Z(R)$. In any case the conclusion follows respectively from Lemma 2 or Lemma 3. $\square$

Proposition 1. *Let R be a prime ring of characteristic different from 2, $a, b \in R$ such that $[au+ub, u](au+ub) = 0$ for all $u \in [R,R]$. Then either $a, b \in Z(R)$ or R satisfies $s_4(x_1, \ldots, x_4)$, the standard identity of degree 4, and $b - a \in Z(R)$.*

P r o o f. Since R satisfies the generalized polynomial identity

$$P(x_1, x_2) = [a[x_1, x_2] + [x_1, x_2]b, [x_1, x_2]](a[x_1, x_2] + [x_1, x_2]b)$$

by a theorem due to Beidar (Theorem 2 in [1]) this generalized polynomial identity is also satisfied by U. In case C is infinite, we have $P(r_1, r_2) = 0$ for all $r_1, r_2 \in U \otimes_C \overline{C}$, where $\overline{C}$ is the algebraic closure of C. Since both U and $U \otimes_C \overline{C}$ are centrally closed ([9], Theorems 2.5 and 3.5), we may replace R by U or $U \otimes_C \overline{C}$ according as C is finite or infinite. Thus we may assume that R is centrally closed over C which is either finite or algebraically closed. If $a, b \in C$, then we are done, thus we may assume that either $a \notin C$ or $b \notin C$. In this case, by [4], $P(x_1, x_2)$ is a non-trivial generalized polynomial identity for R. Hence, by Martindale's theorem [21], R is a primitive ring having a non-zero socle H with C as the associated division ring. In light of Jacobson's theorem ([13], page 75) R is isomorphic to a dense ring of linear transformations on some vector space V over C.

Assume first that V is finite-dimensional over C. Then the density of R on V implies that $R \cong M_m(C)$, the ring of all $m \times m$ matrices over C. Since R is not commutative we assume $m \geqslant 2$.

If we assume that C is infinite, we are done by Lemma 4.

Now let K be an infinite field which is an extension of the field C and let $\overline{R} = M_m(K) \cong R \otimes_C K$. Notice that R satisfies $s_4(x_1, \ldots, x_4)$ if and only $\overline{R}$ satisfies $s_4(x_1, \ldots, x_4)$. As above we consider the generalized polynomial $P(x_1, x_2)$ and remark that it is multi-homogeneous of multi-degree $(2, 2)$ in the indeterminates x_1, x_2. Hence the complete linearization of $P(x_1, x_2)$ is a multilinear generalized polynomial $\Theta(x_1, x_2, y_1, y_2)$ in 4 indeterminates, moreover, $\Theta(x_1, x_2, x_1, x_2) = 4P(x_1, x_2)$. Clearly the multilinear polynomial $\Theta(x_1, x_2, y_1, y_2)$ is a generalized polynomial identity for R and $\overline{R}$ as well. Since $\mathrm{char}(C) \neq 2$ we obtain $P(r_1, r_2) = 0$ for all $r_1, r_2 \in \overline{R}$, and the conclusion follows from the first argument.

Assume next that V is infinite-dimensional over C. As in Lemma 2 in [29], the set $[R, R]$ is dense on R and so from $P(r_1, r_2) = 0$ for all $r_1, r_2 \in R$ we have $[ar + rb, r](ar + rb) = 0$ for all $r \in R$. Due to the infinite-dimensionality, R cannot satisfy any polynomial identity. In particular, the non-zero ideal H cannot satisfy $s_4(x_1, \ldots, x_4)$. Suppose that either $a \notin C$ or $b \notin C$, then at least one of them doesn't centralize the non zero ideal H of R, and we will prove that this leads to a contradiction.

Hence we are supposing that there exist $h_1, h_2 \in H$ such that either $[a, h_1] \neq 0$ or $[b, h_2] \neq 0$ and there exist $h_3, h_4, h_5, h_6 \in H$ such that $s_4(h_3, \ldots, h_6) \neq 0$.

Let $e^2 = e$ be any non-trivial idempotent element of H. For $r = exe$, with any $x \in R$, we have that $[aexe + exeb, exe](aexe + exeb) = 0$. By left and right multiplying with $(1-e)$ we obtain $(1-e)a(exe)^3b(1-e)=0$. Since eRe is a central simple algebra, we have that either $(1-e)ae=0$ or $eb(1-e)=0$. If $(1-e)ae=0$ then $ae=eae$ and $bae=beae$. On the other hand, if $eb(1-e)=0$, we get $eb=ebe$, and so $eba=abea$. In either case we notice that the ring eRe satisfies the generalized identity

$$[(eae)X+X(ebe),X]((eae)X+X(ebe)).$$

By Litoff's theorem in [10] there exists $e^2=e\in H$ such that $h_1,h_2,h_3,h_4,h_5,h_6\in eRe$, moreover, eRe is a central simple algebra finite dimensional over its center. Since $s_4(h_3,\ldots,h_6)\neq 0$, we have $eRe\cong M_t(C)$ for $t\geqslant 3$. By the finite dimensional case, we have that $eae,ebe\in Z(eRe)$, but this contradicts the choices of h_1,h_2 in eRe. □

3. The results on Lie ideals and right ideals

In the following we will make use of the result of Kharchenko [15] about the differential identities on a prime ring R (see Facts 1–4). We refer to Chapter 7 in [2] for a complete and detailed description of the theory of generalized polynomial identities involving derivations.

We first prove

Theorem 1. *Let R be a prime ring of characteristic different from 2, U the Utumi quotient ring of R, $C=Z(U)$ the extended centroid of R, L a non-central Lie ideal of R, F a non-zero generalized derivations of R. Suppose that $[F(u),u]F(u)=0$ for all $u\in L$. Then one of the following assertions holds:*

(1) *there exists $\alpha\in C$ such that $F(x)=\alpha x$ for all $x\in R$;*

(2) *R satisfies the standard identity s_4 and there exist $a\in U$ and $\alpha\in C$ such that $F(x)=ax+xa+\alpha x$, for all $x\in R$.*

Proof. By Theorem 3 in [19] every generalized derivation g on a dense right ideal of R can be uniquely extended to the Utumi quotient ring U of R, and thus we can think of any generalized derivation of R to be defined on the whole U and to be of the form $g(x)=ax+d(x)$ for some $a\in U$ and a derivation d on U. Thus we will assume in all that follows that there exist $a\in U$ and a derivation d on U such that $F(x)=ax+d(x)$. We note that we may assume that R is not commutative, since L is not central. Moreover, since $\operatorname{char}(R)\neq 2$, there exists a non-central two-sided ideal I of R such that $[I,I]\subseteq L$ (see Fact 5). Therefore $[F(u),u]F(u)=0$ for all $u\in[I,I]$. Moreover, by [20] R and I satisfy the same differential polynomial identities, that is $[F(u),u]F(u)=0$ for all $u\in[R,R]$.

By assumption R satisfies the differential identity

$$(4)\quad [a[x_1,x_2]+[d(x_1),x_2]+[x_1,d(x_2)],[x_1,x_2]](a[x_1,x_2]+[d(x_1),x_2]+[x_1,d(x_2)]).$$

First suppose that $0 \neq d$ is not an inner derivation on U. By Kharchenko's theorem [15] R satisfies the polynomial identity

$$(5)\qquad [a[x_1,x_2]+[y_1,x_2]+[x_1,y_2],[x_1,x_2]](a[x_1,x_2]+[y_1,x_2]+[x_1,y_2]),$$

in particular, R satisfies the blended component

$$[a[x_1,x_2],[x_1,x_2]]a[x_1,x_2]$$

and by Proposition 1 we have that $a \in C$ and by (5) R satisfies the following polynomial identity with coefficient in C:

$$[[x_1,y_2],[x_1,x_2]](a[x_1,x_2]+[x_1,y_2]).$$

Since R satisfies a polynomial identity, there exists $M_m(K)$, the ring of all matrices over a suitable field K, such that R and $M_m(K)$ satisfy the same polynomial identities (see [12], Theorem 2 p. 54 and Lemma 1 p. 89). Suppose $m \geqslant 2$ and choose $x_1 = e_{11}$, $y_2 = e_{12}$, $x_2 = e_{21}$. Thus we obtain

$$[e_{12},-e_{21}](-ae_{21}+e_{12}) = 0$$

and right multiplying by e_{22} yields the contradiction $-e_{12} = 0$. Hence $m = 1$ and R is commutative.

Notice that in case $d = 0$, R satisfies $[a[x_1,x_2],[x_1,x_2]](a[x_1,x_2])$ and the same conclusion as above holds.

Finally we consider $0 \neq d$ is an inner derivation of U. Thus there exists $q \in U$ such that $F(x) = ax + [q,x] = (a+q)x + x(-q)$ for all $x \in R$. In this case by Proposition 1 we have that one of the following assertions holds:

(1) either $a, q \in C$, and in this case $F(x) = ax$ for all $x \in R$;

(2) or R satisfies $s_4(x_1,\ldots,x_4)$ and $a + q = -q + \gamma$ for a suitable $\gamma \in C$, that is $F(x) = -qx - xq + \gamma x$ for all $x \in R$. □

As a reduction of the previous Theorem, we may also prove

Theorem 2. *Let R be a non-commutative prime ring of characteristic different from 2, U the Utumi quotient ring of R, $C = Z(U)$ the extended centroid of R, F a non-zero generalized derivation of R. Suppose that $[F(x), x]F(x) = 0$ for all $x \in R$. Then there exists $\alpha \in C$ such that $F(x) = \alpha x$ for all $x \in R$.*

Proof. By Theorem 1, we have to consider the only case when R satisfies the standard identity s_4 and there exist $a \in U$ and $\alpha \in C$ such that $F(x) = ax + xa + \alpha x$ for all $x \in R$. Hence R is a PI-ring, thus there exists a suitable field K such that R and the matrix ring $M_2(K)$ satisfy the same generalized polynomial identities. In particular, because of the form of F, $M_2(K)$ satisfies

$$[a, x^2](ax + xa + \alpha x). \tag{6}$$

Denote $a = \sum a_{ij}e_{ij}$ for suitable $a_{ij} \in K$. For $x = e_{ii}$ in (6), both the right and left multiplying by e_{jj}, for $j \neq i$, yields $a_{ji}a_{ij} = 0$. Consider now the inner automorphisms of R induced by the invertible matrices $P = I + e_{ij}$ and $Q = I - e_{ij}$ for any $i \neq j$: $\lambda(x) = PxP^{-1}$ and $\chi(x) = QxQ^{-1}$, respectively. By calculation we have that $\lambda(a) = a + e_{ij}a - ae_{ij} - e_{ij}ae_{ij}$ and denote $\lambda(a) = \sum a'_{ij}e_{ij}$ for suitable $a'_{ij} \in K$. Since $[\lambda(a), x^2](\lambda(a)x + x\lambda(a) + \alpha x) = 0$ for all $x \in M_2(K)$ by the previous argument we also have that $a'_{ji}a'_{ij} = 0$. By calculation we have

$$a_{ji}(a_{jj} - a_{ii} - a_{ji}) = 0 \tag{7}$$

and analogously by applying the same argument to $\chi(a) = a - e_{ij}a + ae_{ij} - e_{ij}ae_{ij}$,

$$a_{ji}(-a_{jj} + a_{ii} - a_{ji}) = 0. \tag{8}$$

Hence comparing (7) with (8) we obtain $a_{ji} = 0$, that is, a is a diagonal matrix.

Finally, since also $\lambda(a)$ must be a diagonal matrix, in particular the (i, j)-entry of $\lambda(a)$ is zero, that is $a_{ii} = a_{jj}$. By the arbitrariness of $i \neq j$, we have that q is a central matrix in $M_2(K)$ as well as in R. □

Remark 2. Since $F(x) = ax + d(x)$ for suitable $a \in U$ and a derivation d of R, we point out that one can rewrite the conclusions of the previous theorem as follows: either R is commutative or there exists $q \in U$ sucht that d is the inner derivation induced by q, $a \in C$ and $q \in C$, that is $d = 0$.

We conclude this section with

Theorem 3. *Let R be a non-commutative prime ring of characteristic different from 2, U the Utumi quotient ring of R, $C = Z(U)$ the extended centroid of R, I a non-zero right ideal of R, F a non-zero generalized derivation of R. Suppose that $[F(x), x]F(x) = 0$ for all $x \in I$. Then one of the following assertions holds:*

(1) $[I, I]I = (0)$;

(2) *there exist $a, b \in U$ and $\alpha, \beta \in C$ such that $F(x) = ax + xb$ for all $x \in R$, with $(a - \alpha)I = (0)$ and $(b - \beta)I = (0)$.*

Proof. As remarked above, by Theorem 3 in [19] we will assume in all that follows that there exist $c \in U$ and a derivation d on U such that $F(x) = cx + d(x)$, and divide the proof into two cases. $\square$

3.1. d is an inner derivation of U. In this case there exists $q \in U$ such that $d(x) = [q, x]$ for all $x \in R$, and by the hypothesis we have that I satisfies

$$[ax + xb, x](ax + xb) \tag{9}$$

where $a = c + q$ and $b = -q$. Assume that the conclusion does not hold that is, there exist $c_1, c_2, c_3, c_4, c_5 \in I$ such that $[c_1, c_2]c_3 \neq 0$, $ac_4 \notin CI$ and $bc_4 \notin CI$. We will prove that this leads to a contradiction. Notice that for all $x_0 \in I$ and for all $y \in R$, starting from (9), we have that

$$[ax_0y + x_0yb, x_0y](ax_0y + x_0yb) \tag{10}$$

is a generalized polynomial identity for R. Since for $x_0 = c_4$, ax_0 and x_0 are linearly C-independent, (10) is a non-trivial generalized polynomial identity for R (see [4]). Since R is GPI, U has a non-zero socle H with non-zero right ideal $J = IH$. Note that H is simple, $J = JH$ and J satisfies the same basic conditions as I. Now without loss of generality we just replace R by H and I by J. Moreover, $R = H$ is a regular ring, hence there exists $e = e^2 \in I$ such that $c_1R + c_2R + c_3R + c_4R + c_5R = eR$, with $c_i = ec_i$ for each $i = 1, \ldots, 5$. Therefore eR satisfies (9), in particular, for all $x \in R$ we have

$$[aex(1-e) + ex(1-e)b, ex(1-e)](aex(1-e) + ex(1-e)b) = 0$$

and the left multiplying by $(1 - e)$ yields easily that $(1 - e)ae = 0$, that is $ae \in eR$. Thus $F(x) = ax + xb \in eR$ for all $x \in eR$, that is $F(eR) \subseteq eR$. Let $\varrho = eR$, $\bar{\varrho} = \varrho/\varrho \cap l_R(\varrho)$, with $l_R(\varrho)$ the left annihilator of ϱ in R. Therefore $\bar{\varrho}$ satisfies the generalized polynomial identity (9). By Theorem 2 we have that one of the following assertions holds:

- ▷ either $[a, eR]e = (0)$ and $[b, eR]e = (0)$, which implies that there exist $\alpha, \beta \in C$ such that $(a-\alpha)e = (0)$ and $(b-\beta)e = 0$, and this contradicts the choices of $c_4, c_5 \in eR$;
- ▷ or $[\overline{x_1}, \overline{x_2}]$ is a polynomial identity for $\bar{\varrho}$, that is $[x_1, x_2]x_3$ is a polynomial identity for eR, which contradicts the choices of $c_1, c_2, c_3 \in eR$.

3.2. d is not an inner derivation of U. Starting from the main hypothesis we have that for all $x_0 \in I$, R satisfies

$$[cx_0y + d(x_0)y + x_0d(y), x_0y](cx_0y + d(x_0)y + x_0d(y)). \tag{11}$$

In view of Kharchenko's result in [15] and by (11) R satisfies

$$[cx_0y + d(x_0)y + x_0t, x_0y](cx_0y + d(x_0)y + x_0t) \tag{12}$$

and in particular R satisfies the blended component

$$[x_0t, x_0y](cx_0y + d(x_0)y + x_0t) \tag{13}$$

which is a non-trivial generalized polynomial identity for R, since we may assume there is at least one element $x_0 \in I - C$ (see [4]). As above, without loss of generality we just replace R by H and I by J. Moreover, also here we assume that the conclusion does not hold, more precisely, there exist $h_1, h_2, h_3 \in I$ such that $[h_1, h_2]h_3 \neq 0$. By the regularity of R, there exists $g = g^2 \in I$ such that $h_1R + h_2R + h_3R = gR$, with $h_i = gh_i$ for each $i = 1, \ldots, 3$. Since gR satisfies (13), in particular for all $y \in R$, we have that

$$[gt, gy(1-g)](cgy + d(g)y(1-g) + gt), \tag{14}$$

that is $gy(1-g)(cg + d(g))y(1-e) = 0$, which implies $(1-g)(cg + d(g)) = 0$, that is $cg + d(g) = g(cg + d(g)) \in gR$. Hence $F(gR) \subseteq gR$. Let now $\varrho = gR$, $\bar{\varrho} = \varrho/\varrho \cap l_R(\varrho)$, with $l_R(\varrho)$ the left annihilator of ϱ in R. Therefore $\bar{\varrho}$ satisfies the generalized polynomial identity (12). By Theorem 2, and since d is not inner, we have that $[\overline{x_1}, \overline{x_2}]$ is a polynomial identity for $\bar{\varrho}$, that is, $[x_1, x_2]x_3$ is a polynomial identity for eR, which contradicts the choices of $h_1, h_2, h_3 \in gR$. □

4. The results in Banach algebras

Here R will denote a complex Banach algebra. Let us introduce some well known and elementary definitions for the sake of completeness.

By a Banach algebra we shall mean a complex normed algebra R whose underlying vector space is a Banach space. By $\mathrm{rad}(R)$ we denote the Jacobson radical of R. Without loss of generality we assume R to be unital. In fact any Banach agebra R without a unity can be embedded into a unital Banach algebra $R_I = R \oplus \mathbb{C}$ as an ideal of codimension one. In particular, we may identify R with the ideal $\{(x,0)\colon\ x \in R\}$ in R_I via the isometric isomorphism $x \to (x,0)$.

Our first result in this section concerns continuous generalized derivations on Banach algebras:

Theorem 4. *Let R be a non-commutative Banach algebra, F a continuous generalized derivation of R such that $F(x) = ax + d(x)$ for some element $a \in R$ and d a derivation of R. If $[F(x),x]F(x) \in \mathrm{rad}(R)$ for all $x \in R$, then $d(R) \subseteq \mathrm{rad}(R)$, $[a,R] \subseteq \mathrm{rad}(R)$.*

Proof. Under the assumption that F is continuous, and since it is well known that the left multiplication map is also continuous, we have that the derivation d is continuous. In [26] Sinclair proved that any continuous derivation of a Banach algebra leaves the primitive ideals invariant. Therefore, for any primitive ideal P of R, it follows that $F(P) \subseteq aP + d(P) \subseteq P$, that is, also the continuous generalized derivation F leaves the primitive ideals invariant. Denote $R/P = \overline{R}$ for any primitive ideals P. Hence we may introduce the generalized derivation $F_P\colon\ \overline{R} \to \overline{R}$ by $F_P(\bar{x}) = F_P(x+P) \subseteq F(x) + P = ax + d(x) + P$ for all $x \in R$ and $\bar{x} = x + P$. Moreover, by $[F(r),r]F(r) \in \mathrm{rad}(R)$ for all $r \in R$, it follows that $[F_P(\bar{r}),\bar{r}]F_P(\bar{r}) = \bar{0}$ for all $\bar{r} \in \overline{R}$. Since $\overline{R}$ is primitive, a fortiori it is prime. Thus by Theorem 2 and Remark 2, one of the following assertions holds:

- ▷ either $\overline{R}$ is commutative, that is $[R,R] \subseteq P$;
- ▷ or $[a,R] \subseteq P$ and $d = \bar{0}$, more precisely, d is inner in $\overline{R}$, induced by an element $\bar{q} \in \overline{R}$ and $\bar{q} \in Z(\overline{R})$, that is $d(R) \subseteq P$.

Now let P be a primitive ideal such that $\overline{R}$ is commutative. Singer and Wermer in [27] proved that any continuous linear derivation on a commutative Banach algebra maps the algebra into its radical. Moreover, by a result of Johnson and Sinclair [14] any linear derivation on a semisimple Banach algebra is continuous. Hence there are no non-zero linear continuous derivations on commutative semisimple Banach algebras. Therefore $d = \bar{0}$ in $\overline{R}$, and since $[R,R] \subseteq P$ follows by the commutativity of $\overline{R}$, we also have $[a,R] + d(R) \subseteq P$.

Hence in any case $d(R) \subseteq P$ and $[a, R] + d(R) \subseteq P$ for all primitive ideals P of R. Since the radical $\mathrm{rad}(R)$ of R is the intersection of all primitive ideals, we get the required conclusion. $\square$

In the special case when R is a semisimple Banach algebra we may prove

Theorem 5. *Let R be a non-commutative semisimple Banach algebra, F a generalized derivation of R such that $F(x) = ax + d(x)$ for some element $a \in R$ and d a derivation of R. If $[F(x), x]F(x) = 0$ for all $x \in R$, then $d(R) = 0$ and $[a, R] = 0$.*

Proof. We may prove the result in the same way as Theorem 4 and we omit the proof for brevity. Just let us remark that at the beginning of the proof one has to use the fact that the derivation d is continuous in a semisimple Banach algebra (see [26]). Hence, since any left multiplication map is continuous, also F is continuous. Finally, we use the fact that $\mathrm{rad}(R) = 0$, since R is semisimple. $\square$

The last result of this paper has the same flavour as Theorem 4. Now we replace the assumption concerning the continuity of the generalized derivation F by the one that F is spectrally bounded. Here we denote by $I(R)$ the set of invertible elements in R. The spectrum of an element x is the subset given by $\sigma(x) = \{\lambda \in \mathbb{C} \colon x - \lambda e \notin I(R)\}$, where e denotes the unity of R. The spectral radius $r(x)$ of an element x is defined as $r(x) = \sup\{|\lambda| \colon \lambda \in \sigma(x)\}$, provided $\sigma(x)$ is not empty. Finally, a linear map $f \colon R \to R$ is called spectrally bounded if there exists a constant $\alpha \geqslant 0$ such that $r(f(x)) \leqslant \alpha r(x)$ for all $x \in R$. In order to prove our final theorem we will use some results concerning spectrally bounded derivations and generalized derivations contained in [3], more precisely, we need the following facts:

Fact 6. Every spectrally bounded derivation on a unital Banach algebra maps the algebra into the radical (Theorem 2.5 in [3]).

Fact 7. Every spectrally bounded generalized derivation leaves each primitive ideal invariant (Lemma 2.7 in [3]).

Fact 8. Let $F = L_a + d$ be a generalized derivation on a unital Banach algebra R, where L_a is the left multiplication (by the element a) map and d some derivation of R. Then F is spectrally bounded if and only if both L_a and d are spectrally bounded (Theorem 2.8 in [3]).

Now we may prove

Theorem 6. *Let R be a Banach algebra, $F = L_a + d$ a spectrally bounded generalized derivation of R for some element $a \in R$ and a derivation d of R. If $[F(x), x]F(x) \in \operatorname{rad}(R)$ for all $x \in R$ then $d(R) \subseteq \operatorname{rad}(R)$ and $[a, R] \subseteq \operatorname{rad}(R)$.*

Proof. Since F is spectrally bounded, by Fact 8, L_a and d are spectrally bounded.

Combining this with Fact 6 we have that $d(R) \subseteq \operatorname{rad}(R)$. Moreover, by Fact 7, F leaves each primitive ideal invariant. Thus it follows that for any primitive ideal P of R we may introduce generalized derivations $F_P\colon \overline{R} \to \overline{R}$ by $F_P(\bar{x}) = F_P(x+P) \subseteq F_P(x) + P \subseteq ax + d(x) + P \subseteq ax + P$ for all $x \in R$ and $\bar{x} = x + P$. As above, since $[F(r), r]F(r) \in \operatorname{rad}(R)$ for all $r \in R$, it follows that $[F_P(\bar{r}), \bar{r}]F_P(\bar{r}) = \bar{0}$ for all $\bar{r} \in \overline{R}$. By Theorem 2 and Remark 2, one has that

- ▷ either $\overline{R}$ is commutative, that is $[R, R] \subseteq P$;
- ▷ or $d = \bar{0}$, more precisely d is inner in $\overline{R}$, induced by an element $\bar{q} \in \overline{R}$ and $\bar{q} \in Z(\overline{R})$, that is $d(R) + [a, R] \subseteq P$.

Now let P be a primitive ideal such that $\overline{R}$ is commutative. As remarked in the proof of Theorem 4, by combining the results in [27] and [14], we have that there are no non-zero linear continuous derivations on commutative semisimple Banach algebras. Therefore $d = \bar{0}$ in $\overline{R}$, and since $[R, R] \subseteq P$ follows by the commutativity of $\overline{R}$, we also have $[a, R] \subseteq P$.

Hence in any case $d(R) \subseteq P$ and $[a, R] \subseteq P$ for all primitive ideals P of R. Since the radical $\operatorname{rad}(R)$ of R is the intersection of all primitive ideals, we get the required conclusion. □

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