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ON THE COMPOSITION FACTORS OF A GROUP WITH THE SAME PRIME GRAPH AS $B_n(5)$

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Abstract. Let G be a finite group. The prime graph of G is a graph whose vertex set is the set of prime divisors of $|G|$ and two distinct primes p and q are joined by an edge, whenever G contains an element of order pq. The prime graph of G is denoted by $\Gamma(G)$. It is proved that some finite groups are uniquely determined by their prime graph. In this paper, we show that if G is a finite group such that $\Gamma(G) = \Gamma(B_n(5))$, where $n \geqslant 6$, then G has a unique nonabelian composition factor isomorphic to $B_n(5)$ or $C_n(5)$.

Keywords: prime graph, simple group, recognition, quasirecognition

MSC 2010: 20D05, 20D60

1. Introduction

If n is an integer, then we denote by $\pi(n)$ the set of all prime divisors of n. If G is a finite group, then $\pi(|G|)$ is denoted by $\pi(G)$. The *spectrum* of a finite group G which is denoted by $\omega(G)$ is the set of its element orders. We construct the *prime graph* of G which is denoted by $\Gamma(G)$ as follows: the vertex set is $\pi(G)$, and two distinct primes p and q are joined by an edge (we write $p \sim q$) if and only if G contains an element of order pq. Let $s(G)$ be the number of connected components of $\Gamma(G)$ and let $\pi_i(G)$, $i = 1, \ldots, s(G)$, be the connected components of $\Gamma(G)$. If $2 \in \pi(G)$ we always suppose that $2 \in \pi_1(G)$. In graph theory a subset of vertices of a graph is called an independent set if its vertices are pairwise non-adjacent. Denote by $t(G)$ the maximal number of primes in $\pi(G)$ pairwise non-adjacent in $\Gamma(G)$. In other words, if $\varrho(G)$ is an independent set with the maximal number of vertices in $\Gamma(G)$,

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then $t(G) = |\varrho(G)|$. Similarly if $p \in \pi(G)$, then let $\varrho(p, G)$ be an independent set with the maximal number of vertices in $\Gamma(G)$ containing p and $t(p, G) = |\varrho(p, G)|$.

A finite group G is called *recognizable by prime graph* if $\Gamma(H) = \Gamma(G)$ implies that $H \cong G$. A nonabelian simple group P is called *quasirecognizable by prime graph* if every finite group whose prime graph equals $\Gamma(P)$ has a unique nonabelian composition factor isomorphic to P (see [11]). Obviously, recognition (quasirecognition) by prime graph implies recognition (quasirecognition) by spectrum, but the converse is not true in general. Moreover, a method of recognition by spectrum cannot be used for recognition by prime graph.

Hagie in [7] determined finite groups G satisfying $\Gamma(G) = \Gamma(S)$, where S is a sporadic simple group. It is proved that if $q = 3^{2n+1}$ $(n > 0)$, then the simple group ${}^2G_2(q)$ is recognizable by its prime graph [11], [27]. A group G is called a CIT group if G is of even order and the centralizer in G of any involution is a 2-group. In [13], finite groups with the same prime graph as a CIT simple group are determined. Also in [14], it is proved that if $p > 11$ is a prime number and $p \not\equiv 1 \pmod{12}$, then $\mathrm{PSL}(2, p)$ is recognizable by its prime graph. In [12] and [18], finite groups with the same prime graph as $\mathrm{PSL}(2, q)$, where q is not prime, are determined. It is proved that simple groups $F_4(q)$, where $q = 2^n > 2$ (see [10]) and ${}^2F_4(q)$ (see [1]) are quasirecognizable by prime graph. Also in [9], it is proved that if p is a prime number which is not a Mersenne or a Fermat prime and $p \neq 11, 13, 19$, and $\Gamma(G) = \Gamma(\mathrm{PGL}(2, p))$, then G has a unique nonabelian composition factor which is isomorphic to $\mathrm{PSL}(2, p)$; while if $p = 13$, then G has a unique nonabelian composition factor which is isomorphic to $\mathrm{PSL}(2, 13)$ or $\mathrm{PSL}(2, 27)$. Then it is proved that for an odd prime p and odd $k > 2$, $\mathrm{PGL}(2, p^k)$ is recognizable by its prime graph [2]. In [15], [16], [17], [19] finite groups with the same prime graph as $L_n(2)$ are obtained. In [3], it is proved that if $p = 2^n + 1 \geqslant 5$ is a prime number, then ${}^2D_p(3)$ is quasirecognizable by prime graph. Also in [4], the authors proved that ${}^2D_{2^m+1}(3)$ is recognizable by prime graph.

In this paper as the main result we show that if G is a finite group such that $\Gamma(G) = \Gamma(B_n(5))$, where $n \geqslant 6$, then G has a unique nonabelian composition factor isomorphic to $B_n(5)$ or $C_n(5)$.

In this paper, all groups are finite and by simple groups we mean nonabelian simple groups. All further unexplained notation is standard and referred to [5]. Throughout the proof we use the classification of finite simple groups. In [23, Tables 2–9], independent sets and independent numbers for all simple groups are listed and we use these results in the proof of the main theorem of this paper.

2. Preliminary results

Lemma 2.1 ([25, Theorem 1]). *Let G be a finite group with $t(G) \geqslant 3$ and $t(2,G) \geqslant 2$. Then the following hold:*

(1) *there exists a finite nonabelian simple group S such that $S \leqslant \overline{G} = G/K \leqslant \mathrm{Aut}(S)$ for the maximal normal soluble subgroup K of G;*
(2) *for every independent subset ϱ of $\pi(G)$ with $|\varrho| \geqslant 3$ at most one prime in ϱ divides the product $|K||\overline{G}/S|$. In particular, $t(S) \geqslant t(G) - 1$;*
(3) *one of the following holds:*
   (a) *every prime $r \in \pi(G)$ non-adjacent to 2 in $\Gamma(G)$ does not divide the product $|K||\overline{G}/S|$; in particular, $t(2,S) \geqslant t(2,G)$;*
   (b) *there exists a prime $r \in \pi(K)$ non-adjacent to 2 in $\Gamma(G)$; in which case $t(G) = 3$, $t(2,G) = 2$, and $S \cong \mathrm{Alt}_7$ or $L_2(q)$ for some odd q.*

Remark 2.2. In Lemma 2.1, for every odd prime $p \in \pi(S)$ we have $t(p,S) \geqslant t(p,G) - 1$.

Lemma 2.3 ([20, Lemma 1]). *Let N be a normal subgroup of G. Assume that G/N is a Frobenius group with Frobenius kernel F and cyclic Frobenius complement C. If $(|N|,|F|) = 1$ and F is not contained in $NC_G(N)/N$, then $p|C| \in \pi_e(G)$, where p is a prime divisor of $|N|$.*

Lemma 2.4 (Zsigmondy Theorem, [28]). *Let p be a prime and let n be a positive integer. Then one of the following holds:*

(i) *there is a primitive prime p' for $p^n - 1$, that is, $p' \mid (p^n - 1)$ but $p' \nmid (p^m - 1)$ for every $1 \leqslant m < n$, (usually p' is denoted by r_n)*
(ii) *$p = 2$, $n = 1$ or 6,*
(iii) *p is a Mersenne prime and $n = 2$.*

Lemma 2.5 ([8]). *Let G be a finite simple group.*

(1) *If $G = C_n(q)$, then G possesses a Frobenius subgroup with kernel of order q^n and cyclic complement of order $(q^n - 1)/(2, q-1)$.*
(2) *If $G = {}^2D_n(q)$ and there exists a primitive prime divisor r of $q^{2n-2} - 1$, then G possesses a Frobenius subgroup with kernel of order q^{2n-2} and cyclic complement of order r.*
(3) *If $G = B_n(q)$ or $D_n(q)$ and there exists a primitive prime divisor r_m of $q^m - 1$ where $m = n$ or $n-1$ such that m is odd, then G possesses a Frobenius subgroup with kernel of order $q^{m(m-1)/2}$ and cyclic complement of order r_m.*

Remark 2.6 ([21]). Let p be a prime number and $(q,p)=1$. Let $k \geqslant 1$ be the smallest positive integer such that $q^k \equiv 1 \pmod p$. Then k is called *the order of* q *with respect to* p and we denote it by $\mathrm{ord}_p(q)$. Obviously by Fermat's little theorem it follows that $\mathrm{ord}_p(q) \mid (p-1)$. Also if $q^n \equiv 1 \pmod p$, then $\mathrm{ord}_p(q) \mid n$. Similarly if $m>1$ is an integer and $(q,m)=1$, we can define $\mathrm{ord}_m(q)$. If a is odd, then $\mathrm{ord}_a(q)$ is denoted by $e(a,q)$, too.

If q is odd, let $e(2,q)=1$ if $q \equiv 1 \pmod 4$ and $e(2,q)=2$ if $q \equiv -1 \pmod 4$.

Lemma 2.7 ([24, Proposition 2.4]). *Let G be a simple group of Lie type, $B_n(q)$ or $C_n(q)$ over a field of characteristic p. Define*

$$\eta(m) = \begin{cases} m & \text{if } m \text{ is odd,} \\ m/2 & \text{otherwise.} \end{cases}$$

Let r, s be odd primes with $r,s \in \pi(G) \setminus \{p\}$. Put $k=e(r,q)$ and $l=e(s,q)$, and suppose that $1 \leqslant \eta(k) \leqslant \eta(l)$. Then r and s are non-adjacent if and only if $\eta(k)+\eta(l)>n$, and k, l satisfy

$$l/k \text{ is not an odd natural number.}$$

Lemma 2.8 ([23, Proposition 2.1]). *Let $G=A_{n-1}(q)$ be a finite simple group of Lie type over a field of characteristic p. Let r and s be odd primes and $r,s \in \pi(G) \setminus \{p\}$. Put $k=e(r,q)$ and $l=e(s,q)$, and suppose that $2 \leqslant k \leqslant l$. Then r and s are non-adjacent if and only if $k+l>n$, and k does not divide l.*

Lemma2.9 ([23, Proposition 2.2]). *Let $G={}^2A_{n-1}(q)$ be a finite simple group of Lie type over a field of characteristic p. Define*

$$\nu(m) = \begin{cases} m & \text{if } m \equiv 0 \pmod 4; \\ m/2 & \text{if } m \equiv 2 \pmod 4; \\ 2m & \text{if } m \equiv 1 \pmod 4. \end{cases}$$

Let r and s be odd primes and $r,s \in \pi(G) \setminus \{p\}$. Put $k=e(r,q)$ and $l=e(s,q)$, and suppose that $2 \leqslant \nu(k) \leqslant \nu(l)$. Then r and s are non-adjacent if and only if $\nu(k)+\nu(l)>n$, and $\nu(k)$ does not divide $\nu(l)$.

Let q be a prime. We denote by $D_n^+(q)$ the simple group $D_n(q)$, and by $D_n^-(q)$ the simple group ${}^2D_n(q)$.

Lemma 2.10 ([24, Proposition 2.5]). *Let $G = D^{\varepsilon}_{n'}(q)$ be a finite simple group of Lie type over a field of characteristic p and let the function $\eta(m)$ be defined as in Lemma 2.7. Let r and s be odd primes and $r, s \in \pi(G) \setminus \{p\}$. Put $k = e(r, q)$ and $l = e(s, q)$, and $1 \leqslant \eta(k) \leqslant \eta(l)$. Then r and s are non-adjacent if and only if $2\eta(k) + 2\eta(l) > 2n - (1 - \varepsilon(-1)^{k+l})$, and k, l satisfy*

$$l/k \text{ is not an odd natural number.}$$

If $\varepsilon = +$, then the chain of equalities:

$$n = l = 2\eta(l) = 2\eta(k) = 2k$$

is not true.

3. Main results

Lemma 2.3 is one of the powerful tools for characterization of finite simple groups by spectrum or prime graph. In the next lemma we get its refinement.

Lemma 3.1. *Let G be a group satisfying the conditions of Lemma 2.1, and let the groups K and S be as in the conclusion of Lemma 2.1. Assume that there exist $p \in \pi(K)$ and $p' \in \pi(S)$ such that $p \nsim p'$ in $\Gamma(G)$, and that S contains a Frobenius subgroup with kernel F and cyclic complement C such that $(|F|, |K|) = 1$. Then $p|C| \in \omega(G)$.*

Proof. We claim that $F \nleqslant KC_G(K)/K$. Since $KC_G(K)/K \trianglelefteq G/K$, so $S \cap KC_G(K)/K \trianglelefteq S$. Let $S \cap KC_G(K)/K = S$. Then $S \leqslant KC_G(K)/K$. So for every $t' \in \pi(S)$ and $t \in \pi(K)$ we have $t' \sim t$, which is a contradiction. Consequently $S \cap KC_G(K)/K = 1$, since S is a simple group. So $F \nleqslant KC_G(K)/K$, since $F \leqslant S$. Therefore $p|C| \in \omega(G)$, by Lemma 2.3. □

Remark 3.2. Let $G = B_n(5)$, where $n \geqslant 6$. By [26, Tables 1a–1c], we have $s(G) = 1$ and $\pi(G) = \pi\Big(5^{n^2}\Big(\prod\limits_{i=1}^{n}(5^{2i} - 1)\Big)\Big)$. In the rest of this section we denote by r_i a primitive prime divisor of $5^i - 1$. By [23, Table 6], we know that $\varrho(2, B_n(5)) = \{2, r_{2n}\}$, $t(B_n(5)) = [\frac{1}{4}(3n + 5)]$ and $\{r_{2i}\colon [\frac{1}{2}(n + 1)] \leqslant i \leqslant n\} \cup \{r_i\colon [\frac{1}{2}n] < i \leqslant n,\ i \equiv 1 \pmod 2\}$ is an independent set of maximal size in $\Gamma(G)$.

Therefore if $n \geqslant 9$ and $A = \{r_{2n}, r_{2(n-1)}, r_{2(n-2)}, r_{2(n-3)}, r_{2(n-4)}\}$, then A is an independent set in $\Gamma(B_n(5))$.

Lemma 3.3. *Let $G = B_n(5)$, where $n \geqslant 12$. If $257 \in \pi(G)$, then $t(257, G) \geqslant 62$. Similarly in each case if n is sufficiently large, then $t(193, G) \geqslant 44$, $t(1201, G) \geqslant 144$, $t(14281, G) \geqslant 82$, $t(1129, G) \geqslant 65$, $t(11551, G) \geqslant 470$, $t(7321, G) \geqslant 450$, $t(12705841, G) \geqslant 158833$ and $t(4466009, G) \geqslant 558247$.*

P r o o f. We know that $e(193, 5) = 192$ and so if $193 \in \pi(G)$, then $n \geqslant 96$. By Remark 3.2, $B = \{r_{2n}, r_{2(n-1)}, \ldots, r_{2(n-47)}\}$ is an independent set of $\Gamma(G)$, since $\frac{1}{2}(n+1) \leqslant n - 47$. Therefore $|B| = 48$. If $r_{2i} \in B$, then $n - 47 \leqslant i \leqslant n$, therefore $i \geqslant n - 95$ and so $\eta(2i) + \eta(192) \geqslant n + 1$. Hence $r_{2i} \nsim 193$ in $\Gamma(G)$ if and only if $i/96$ and $96/i$ are not odd natural numbers. Easily we can see that $96/i$ is an odd number if and only if $i = 32$ or $i = 96$. Now 96 divides at most one element of $\{n - 47, \ldots, n\}$. Therefore at least 44 elements of B are not adjacent to 193.

Similarly to the above, since $e(257, 5) = 256$, $e(1201, 5) = 600$, $e(14281, 5) = 340$, $e(1129, 5) = 282$, $e(11551, 5) = 1925$, $e(7321, 5) = 1830$, $e(12705841, G) = 635292$, and $e(4466009, 5) = 2233004$, we derive $t(257, G) \geqslant 62$, $t(1201, G) \geqslant 144$, $t(14281, G) \geqslant 82$, $t(1129, G) \geqslant 65$, $t(11551, G) \geqslant 470$, $t(7321, G) \geqslant 450$, $t(12705841, G) \geqslant 158833$, and $t(4466009, G) \geqslant 558247$. □

Lemma 3.4. *Let G be a finite simple group of Lie type over $\mathrm{GF}(q)$, where $q = p^\alpha$. Let p' be a prime divisor of $|G|$. In Table 1, we give some upper bounds for $t(p', G)$ for some simple groups G and some prime numbers p'.*

	$A_n(p^\alpha)$	${}^2A_n(p^\alpha)$	$B_n(p^\alpha)$ or $C_n(p^\alpha)$	$D_n(p^\alpha)$r or ${}^2D_n(p^\alpha)$
$(p, p') = (2, 257)$	17	17	13	15
$(p, p') = (3, 193)$	17	17	13	15
$(p, p') = (7, 1201)$	9	9	7	9
$(p, p') = (13, 14281)$	9	9	7	9
$(p, p') = (31, 1129)$	9	9	7	9
$(p, p') = (313, 11551)$	12	12	9	10

Table 1. An upper bound for $t(p', G)$

P r o o f. We determine $t(257, G)$ in case $q = 2^\alpha$, and the proofs of the other cases are similar. Now we consider each case separately.

Case 1. Let $G = A_{n'-1}(q)$, where $q = 2^\alpha$. We know that $e(257, q) \mid 16$, since $e(257, 2) = 16$. If $e(257, q) = 1$, then 257 is adjacent to each prime divisor of $q^i - 1$, where $i \leqslant n' - 2$, by [23, Proposition 4.1], so $t(257, G) \leqslant 3$. Otherwise since $e(257, q) \mid 16$, hence 257 is adjacent to each prime divisor of $q^i - 1$, where $i \leqslant n' - 16$, by Lemma 2.8, so $|\varrho(257, G) \setminus \{257\}| \leqslant 16$ and so $t(257, G) \leqslant 17$.

Case 2. Let $G = {}^2A_{n'-1}(q)$, where $q = 2^\alpha$. If $e(257, q) = 2$, then 257 is adjacent to each prime divisor of $q^i - 1$, where $\nu(i) \leqslant n' - 2$, by [23, Proposition 4.2], so

$t(257,G)\leqslant 3$. Otherwise since $e(257,q)\mid 16$, hence 257 is adjacent to each prime divisor of $q^i-(-1)^i$, where $\nu(i)\leqslant n'-16$, by Lemma 2.9, so $|\varrho(257,G)\setminus\{257\}|\leqslant 16$ and so $t(257,G)\leqslant 17$.

Case 3. Let $G=B_{n'}(q)$, where $q=2^\alpha$. We have $e(257,q)\mid 16$, since $e(257,2)=16$. Therefore 257 is adjacent to each prime divisor of q^i-1, where $\eta(i)\leqslant n'-8$, by Lemma 2.7, so $|\varrho(257,G)\setminus\{257\}|\leqslant 12$ and so $t(257,G)\leqslant 13$.

Case 4. Let $G=D^{\varepsilon}_{n'}(q)$, where $q=2^\alpha$. We know that $e(257,q)\mid 16$. Therefore 257 is adjacent to each prime divisor of q^i-1, where $\eta(i)\leqslant n'-9$, by Lemma 2.10, so $|\varrho(257,G)\setminus\{257\}|\leqslant 14$ and so $t(257,G)\leqslant 15$. □

Lemma 3.5. *If $n'\geqslant 10$, then $t(7321,D^{\varepsilon}_{n'}(11^\alpha))\leqslant 9$. Similarly, $t(12705841,D^{\varepsilon}_{n'}(71^\alpha))\leqslant 9$, $t(4466009,D^{\varepsilon}_{n'}(521^\alpha))\leqslant 9$.*

P r o o f. Similarly to Lemma 3.4, we get the result, since $e(7321,11)\mid 8$. □

Theorem 3.6. *Let G be a finite group such that $\Gamma(G)=\Gamma(B_n(5))$, where $n\geqslant 6$. Then G has a unique nonabelian composition factor isomorphic to $B_n(5)$ or $C_n(5)$.*

P r o o f. We know that $t(B_n(5))\geqslant 5$ and $t(2,B_n(5))=2$. By Lemma 2.1, there exists a nonabelian simple group S such that $S\leqslant\overline{G}=G/K\leqslant\mathrm{Aut}(S)$, where K is the maximal normal soluble subgroup of G.

We know that if $n\geqslant 9$, then $A=\{r_{2n},r_{2(n-1)},r_{2(n-2)},r_{2(n-3)},r_{2(n-4)}\}$ is an independent set of $\Gamma(G)$ and so $|A\cap\pi(S)|\geqslant 4$, by Lemma 2.1. Since $r_{2n}\in\varrho(2,G)$, it follows that $r_{2n}\in\pi(S)$ and $r_{2n}\nsim 2$ in $\Gamma(S)$. By Lemma 2.1 we know that $t(S)\geqslant 4$ and $t(2,S)\geqslant 2$. In the sequel, using [26, Tabs. 1a–1c] we consider each possibility for S such that $t(S)\geqslant 4$.

Case 1. Let $S\cong A_{n'}$.

If $n'\leqslant 16$, then $t(S)\leqslant 3$, which is a contradiction with $t(S)\geqslant 4$. Consequently, $n'\geqslant 17$. Let $n\geqslant 12$. If $x\in\pi(A_{n'})$ is such that $x\nsim 17$, then $n'-17<x\leqslant n'$, by [23, Proposition 1.1]. On the other hand, there exist $[18/2]+[18/3]-[18/6]=12$ elements of $[n'-17,n']$ which are divisible by 2 or by 3. Therefore at most 6 elements of $[n'-17,n']$ are prime numbers. Hence $t(17,S)\leqslant 7$. Therefore by Remark 2.2, $t(17,G)\leqslant 8$. Since $n\geqslant 12$, $[(n+1)/2]\leqslant n-5$ so $H=\{r_{2i}\colon\ n-5\leqslant i\leqslant n\}\cup\{r_i\colon n-5\leqslant i\leqslant n,\ i\equiv 1 \pmod 2\}$, is an independent set of $\Gamma(G)$, by Remark 3.2. We know that $e(17,5)=16$ and easily we can see that 17 is not adjacent to at least 8 elements of H and so $t(17,G)\geqslant 9$, which is a contradiction.

If $n=6$, then $601=r_{2n}\in\pi(S)$, so $n'\geqslant 601$. Therefore $449\in\pi(S)$, which is a contradiction, since $449\notin\pi(B_6(5))$. Similarly we derive that $n\notin\{7,8,9,10,11\}$.

In the rest of the proof, if S is a simple group of Lie type over $\mathrm{GF}(q)$, then let r'_i be a primitive prime divisor of q^i-1.

Case 2. Let $S \cong A_{n'-1}(q)$, where $q = p^\alpha$.

By Lemma 2.1, $t(S) \geqslant t(G) - 1$, so

$$2n' > 3n - 5. \tag{3.1}$$

(a) If $n \geqslant 12$, then (3.1) implies that $n' \geqslant 16$.

(2.1.a) Let $p \neq 5$. By [23, Propositions 3.1, 4.1], every r'_i, where $i \notin \{n'-1, n'\}$, is adjacent to 2 and p in $\Gamma(S)$. Since $r_{2n} \in \pi(S)$ and $2 \nsim r_{2n}$ in $\Gamma(S)$ we obtain $e(r_{2n}, q) \in \{n'-1, n'\}$. Since A is an independent set in $\Gamma(G)$, it follows that $e(r_i, q) \neq e(r_j, q)$ for $r_i, r_j \in A$ and $i \neq j$. We know that $|A \cap \pi(S)| \geqslant 4$, by Lemma 2.1. Hence p is adjacent to at least two elements of $\pi(S) \cap A \setminus \{r_{2n}\}$ in $\Gamma(S)$, since $t(p, S) = 3$. For example, let p be adjacent to $r_{2(n-3)}$ and $r_{2(n-4)}$ in $\Gamma(S)$. Then $r_{2(n-3)} \sim p$ and $r_{2(n-4)} \sim p$ in $\Gamma(G)$. Denote $e(p, 5)$ by a. Since $p \sim r_{2(n-4)}$ by Lemma 2.7 it follows that $n - 4 + \eta(a) \leqslant n$ or $2(n-4)/a$ is odd. Similarly since $p \sim r_{2(n-3)}$ it follows that $n - 3 + \eta(a) \leqslant n$ or $2(n-3)/a$ is odd. So $\eta(a) \leqslant 4$, which implies that $a \in \{1, 2, 3, 4, 6, 8\}$ and so $p \in \{2, 3, 7, 13, 31, 313\}$. Similarly to the above for every r_i and r_j, where $i, j \in \{2(n-1), 2(n-2), 2(n-3), 2(n-4)\}$, and $r_i \sim p \sim r_j$, it follows that $p \in \{2, 3, 7, 13, 31, 313\}$.

Assume that $p = 2$. Since $n' \geqslant 16$ and $e(257, 2^\alpha) \mid 16$, it follows that $257 \in \pi(S)$. Hence by Lemma 3.4, $t(257, S) \leqslant 17$, while by Lemma 3.3, $t(257, G) \geqslant 62$. Therefore by Remark 2.2 we get a contradiction. Similarly for every $p \in \{3, 7, 13, 31, 313\}$, we get a contradiction.

(2.2.a) Let $p = 5$ and so $q = 5^\alpha$. We note that $\pi(S) \subseteq \pi(G)$ and by Lemma 2.4, it follows that $\alpha n' \leqslant 2n$. On the other hand, $2 \nsim r_{2n}$ in $\Gamma(S)$, so $e(r_{2n}, q) \in \{n'-1, n'\}$ by [23, Proposition 4.1]. Therefore $2n = e(r_{2n}, 5)$ divides $n'\alpha$ or $(n'-1)\alpha$. If $2n \mid (n'-1)\alpha$, then $2n \leqslant (n'-1)\alpha < n'\alpha \leqslant 2n$, which is a contradiction. Therefore $2n = \alpha n'$. If $\alpha = 1$, then $2n = n'$ and so $r_{n'-1} = r_{2n-1} \in \pi(S) \subseteq \pi(G)$, which is a contradiction. If $\alpha \geqslant 2$, then $n \geqslant n'$. Now (3.1) implies that $n < 5$, and this is a contradiction.

(b) Let $6 \leqslant n \leqslant 11$.

If $n = 6$, then $\pi(G) = \pi(B_6(5)) = \{2, 3, 5, 7, 11, 13, 31, 71, 313, 521, 601\}$. We know that $p \in \pi(S)$ and so $p \in \pi(G)$. By (3.1) we have $n' \geqslant 7$, so $\pi(p^7 - 1) \subseteq \pi(q^7 - 1) \subseteq \pi(S)$. For every $p \in \pi(G)$, we can easily see that $\pi(p^7 - 1) \nsubseteq \pi(G)$, and so we get a contradiction. For example, if $p = 2$, then $127 \in \pi(2^7 - 1)$ and $127 \notin \pi(G)$.

If $n = 7$, then $\pi(G) = \{2, 3, 5, 7, 11, 13, 29, 31, 71, 313, 449, 521, 601, 19531\}$. By (3.1) we have $n' \geqslant 9$. If $p \in \pi(G) \setminus \{5\}$, then similarly to the previous case we get a contradiction.

Let $p = 5$. Since $\pi(S) \subseteq \pi(G)$, hence $n'\alpha \leqslant 14$. Therefore $9 \leqslant n' \leqslant 14$ and $\alpha = 1$. Now we get a contradiction, since $r_9 \notin \pi(G)$.

Similarly for $8 \leqslant n \leqslant 11$, we get a contradiction.

Case 3. Let $S \cong {}^2A_{n'-1}(q)$, where $q = p^\alpha$.

By Lemma 2.1, $t(S) \geqslant t(G) - 1$, so

$$2n' > 3n - 5. \tag{3.2}$$

(a) Let $n \geqslant 12$. Then (3.2) implies $n' \geqslant 16$.

(3.1.a) Let $p \neq 5$. Every $r'_i \in \pi(S)$, where $\nu(i) \notin \{n'-1, n'\}$, is adjacent to 2 and p in $\Gamma(S)$, by [23, Propositions 3.1, 4.2]. We know that $2 \nsim r_{2n}$ in $\Gamma(S)$, therefore $\nu(e(r_{2n}, q)) \in \{n'-1, n'\}$, by [23, Proposition 4.2]. Also we know that $\nu(e(r_i, q)) \neq \nu(e(r_j, q))$ for $r_i, r_j \in A$ and $i \neq j$, since A is an independent set in $\Gamma(G)$. Therefore p is adjacent to at least two elements of $\pi(S) \cap A \setminus \{r_{2n}\}$ in $\Gamma(S)$, since $t(p, S) = 3$. Denote $e(p, 5)$ by a. Similarly to Case 2, it follows that $p \in \{2, 3, 7, 13, 31, 313\}$.

If $p = 3$, then by Lemma 3.4, $t(193, S) \leqslant 17$, while by Lemma 3.3, $t(193, G) \geqslant 44$. Now by Remark 2.2, we get a contradiction.

If $p = 7$, then by Lemma 3.4, $t(1201, S) \leqslant 9$, while by Lemma 3.3, $t(1201, G) \geqslant 144$, which is a contradiction.

Similarly for every $p \in \{2, 3, 7, 13, 31, 313\}$, we get a contradiction.

(3.2.a) Let $p = 5$. By Lemma 2.4, it follows that $2\alpha n' \leqslant 2n$ or $2\alpha(n'-1) \leqslant 2n$, since $\pi(S) \subseteq \pi(G)$. We know that $2 \nsim r_{2n}$ in $\Gamma(S)$. By [23, Proposition 4.2], $\nu(e(r_{2n}, q)) \in \{n'-1, n'\}$. Therefore $2n = e(r_{2n}, 5) \mid 2\alpha n'$ or $2n = e(r_{2n}, 5) \mid 2\alpha(n'-1)$. So we consider the following two cases:

1. Let $2n = 2\alpha n'$, so $n \geqslant n'$. Now (3.2) implies that $n < 5$, which is a contradiction.

2. Let $2n = 2\alpha(n'-1)$. Then $n \geqslant n'-1$ and by (3.2) we have $n < 7$, which is a contradiction.

(b) Let $6 \leqslant n \leqslant 11$.

If $n = 6$, then $\pi(G) = \pi(B_6(5))$. We note that $p \in \pi(S) \subseteq \pi(G)$. By (3.2) we have $n' \geqslant 7$. Since $r_{2n} = 601 \nsim 2$ in $\Gamma(S)$, using [23, Proposition 4.2] we conclude that $\nu(e(r_{2n}, q)) \in \{n'-1, n'\}$ and so $601 = r_{2n} \in \{r'_{(n'-\varepsilon)/2}, r'_{n'-1}, r'_{2(n'-1)}, r'_{n'}, r'_{2n'}\}$, where $\varepsilon = 0$ if n' is even and $\varepsilon = 1$ if n' is odd.

Let $p = 2$. If $r'_{n'} = 601$, then $25 \mid n'\alpha$, since $e(601, 2) = 25$. We consider the following cases:

1. If n' is even, then $(2^{25} - 1) \mid (2^{n'\alpha} - (-1)^{n'})$. So $1801 \in \pi(S)$, which is a contradiction.

2. Let n' be odd. If α is odd, then $(2^{25} + 1) \mid (2^{n'\alpha} - (-1)^{n'})$. Therefore $4051 \in \pi(S)$, which is a contradiction. Let α be even. If $n' = 7$, then $S \cong {}^2A_6(2^\alpha)$. We know that $25 \mid 7\alpha$, so $(2^{25} - 1) \mid |S|$. Hence $1801 \in \pi(S)$, which is a contradiction. Hence $n' \geqslant 9$ and so $257 \in \pi(2^{16} - 1) \subseteq \pi(q^8 - 1) \subseteq \pi(S)$, which is a contradiction.

Similarly $601 \notin \{r'_{(n'-\varepsilon)/2}, r'_{n'-1}, r'_{2(n'-1)}, r'_{2n'}\}$, where $\varepsilon = 0$ if n' is even and $\varepsilon = 1$ if n' is odd.

Let $p=3$. Since $e(601,3)=75$, similarly we get a contradiction.

Let $p=5$. Since $\pi(S)\subseteq\pi(G)$, hence $2n'\alpha\leqslant 12$ or $2(n'-1)\alpha\leqslant 12$. Therefore $n'=7$ and $\alpha=1$, since $n'\geqslant 7$, so $S\cong {}^2A_6(5)$. We know that $601\in\pi(S)$, which is a contradiction.

Let $p=7$. Since $n'\geqslant 7$, hence $\pi(p^6-1)\subseteq\pi(S)$. Therefore $43\in\pi(S)$, which is a contradiction. Similarly for every $p\in\{11,13,31,71,313,521,601\}$, we get a contradiction.

Finally, for $7\leqslant n\leqslant 11$, we can get a contradiction similarly and we omit the proof for these cases.

Case 4. Let $S\cong D^{\varepsilon}_{n'}(q)$, where $q=p^{\alpha}$.

By Lemma 2.1, $t(S)\geqslant t(G)-1$, so

$$3n' > 3n-7. \tag{3.3}$$

(a) Let $n\geqslant 12$. Since $t(S)\geqslant t(G)-1$, we see that (3) implies that if $\varepsilon=+$, then $n'\geqslant 11$ and if $\varepsilon=-$, then $n'\geqslant 10$.

We note that $B=A\cup\{r_{2(n-5)}\}$ is an independent set in $\Gamma(G)$, since $n\geqslant 12$.

(4.1.a) Let $p\neq 5$. We know that every $r'_i\in\pi(S)$, where $\eta(i)\notin\{n'-1,n'\}$, is adjacent to 2 and p in $\Gamma(S)$, by [23, Propositions 3.1, 4.4]. For every $r_i,r_j\in B$, where $i\neq j$ we have $\eta(e(r_i,q))\neq\eta(e(r_j,q))$, since B is an independent set in $\Gamma(G)$. Since $2\nsim r_{2n}$ in $\Gamma(S)$, we obtain $\eta(e(r_{2n},q))\in\{n'-1,n'\}$. Therefore p is adjacent to at least two elements of $\pi(S)\cap B\setminus\{r_{2n}\}$ in $\Gamma(S)$. If $a=e(p,5)$, then similarly to Case 2 we conclude that $p\in\{2,3,7,11,13,31,71,313,521\}$.

If $p=13$, then by Lemma 3.4, $t(14281,S)\leqslant 9$, while by Lemma 3.3, $t(14281,G)\geqslant 82$. Therefore by Remark 2.2, we get a contradiction.

If $p=11$, then by Lemma 3.5, $t(7321,S)\leqslant 9$, while by Lemma 3.3, $t(7321,G)\geqslant 450$. So we get a contradiction.

Similarly for every $p\in\{2,3,7,11,13,31,71,313,521\}$ we get a contradiction.

(4.2.a) Let $p=5$. Then $r_{2n}\in\pi(S)$ and $2\nsim r_{2n}$, since $r_{2n}\in\varrho(2,G)$.

- Let $S\cong {}^2D_{n'}(5^{\alpha})$. By [23, Proposition 4.4], $r_{2n}\in\{r'_{2n'},r'_{2(n'-1)}\}$. Therefore $2n=e(r_{2n},5)\mid 2\alpha n'$ or $2n=e(r_{2n},5)\mid 2\alpha(n'-1)$. On the other hand, $\pi(S)\subseteq\pi(G)$ and by Lemma 2.4, it follows that $2\alpha n'\leqslant 2n$. Hence $2n=2\alpha n'$ and so $n'=n/\alpha$. Therefore by (3.3), $\alpha=1$, since $n'\geqslant 10$. Therefore $q=5$. Now we consider two subcases:

1. If n is odd, then $r_n\notin\pi(S)$ so $r_n\in\pi(K)\cup\pi(\overline{G}/S)$. Since $\pi(\mathrm{Out}(S))=\{2\}$, we have $r_n\in\pi(K)$. Then by Lemma 2.5, S contains a Frobenius subgroup with kernel F of order $5^{2(n-1)}$ and a cyclic complement C of order $r_{2(n-1)}$, where $r_{2(n-1)}$ is a primitive prime divisor of $5^{2(n-1)}-1$. Since $r_{2n}\in\pi(S)$ and $r_n\nsim r_{2n}$ in $\Gamma(G)$, then by Lemma 3.1, $r_n\sim r_{2(n-1)}$, which is a contradiction, by Lemma 2.7.

2. If n is even, then $r_{n+2} \in \pi(5^{2(n+2)/2} - 1) \subseteq \pi(S)$. Similarly it follows that $r_{n-2} \in \pi(S)$. Now using Lemma 2.7, we conclude that $r_{n-2} \sim r_{n+2}$ in $\Gamma(G)$ and using Lemma 2.10, it follows that $r_{n-2} \nsim r_{n+2}$ in $\Gamma(S)$. Since $\pi(\overline{G}/S) = \{2\}$ it follows that $r_{n-2} \in \pi(K)$ or $r_{n+2} \in \pi(K)$. By Lemma 2.5, S contains a Frobenius subgroup of the form $5^{2n-2} : r_{2n-2}$. We note that $r_{n-1} \in \pi(S)$, $r_{n-1} \nsim r_{n-2}$ and $r_{n-1} \nsim r_{n+2}$ in $\Gamma(G)$. Therefore by Lemma 3.1, we have $r_{2n-2} \sim r_{n-2}$ or $r_{2n-2} \sim r_{n+2}$, which is a contradiction with Lemma 2.7.

• Let $S \cong D_{n'}(5^\alpha)$. By [23, Proposition 4.4], $e(r_{2n}, q) \in \{2(n'-1), n'-1, n'\}$. Therefore $2n = e(r_{2n}, 5)$ divides $2\alpha(n'-1)$, $\alpha(n'-1)$, or $\alpha n'$. On the other hand, we note that $\pi(S) \subseteq \pi(G)$ and by Lemma 2.4 it follows that $2\alpha(n'-1) \leqslant 2n$. So $2n = 2\alpha(n'-1)$. If $\alpha \geqslant 2$, then by (3.3) we have $3n' > 3\alpha(n'-1) - 7 \geqslant 6n' - 13$, which is a contradiction, since $n' \geqslant 7$. Therefore $\alpha = 1$, $n' = n+1$ and so $S \cong D_{n+1}(5)$. Consequently, if n is even, then $r_{n+1} = r_{n'} \in \pi(S)$ and $r_{n+1} \notin \pi(G)$, which is a contradiction. Let n be odd. If $4 \mid (n-1)$, then $r_{2(n-1)} \nsim r_4$ in $\Gamma(G)$ by Lemma 2.7. But $r_{2(n-1)} \sim r_4$ in $\Gamma(S)$ by Lemma 2.10, which is a contradiction. If $4 \mid (n-3)$, then similarly to the above $r_{2(n-3)} \nsim r_8$ in $\Gamma(G)$ and $r_{2(n-3)} \sim r_8$ in $\Gamma(S)$ by Lemmas 2.7 and 2.10, which is a contradiction.

(b) Let $6 \leqslant n \leqslant 11$.

• Let $S \cong {}^2D_{n'}(q)$.

If $n = 6$, then $p \in \pi(S) \subseteq \pi(B_6(5))$. By (3.3), we have $n' \geqslant 4$. Let $p = 2$. Since $n' \geqslant 4$, we can easily see that $(q^8 - 1) \mid |S|$ and so $(p^8 - 1) \mid |S|$. So $17 \in \pi(S)$, which is a contradiction. Similarly $p \neq 3$.

Let $p = 5$. Since $\pi(S) \subseteq \pi(G)$, we have $2n'\alpha \leqslant 12$. Therefore $4 \leqslant n' \leqslant 6$ and $\alpha = 1$. We know that $601 \in \pi(S)$. Then $n' = 6$, since $e(601, 5) = 12$. So $S \cong {}^2D_6(5)$. We know that $r_8 \sim r_4$ in $\Gamma(G)$ and $r_8 \nsim r_4$ in $\Gamma(S)$, by Lemma 2.7 and Lemma 2.10. Therefore $r_4 \in \pi(\overline{G}/S) \cup \pi(K)$ or $r_8 \in \pi(\overline{G}/S) \cup \pi(K)$. We know that $\pi(\overline{G}/S) = \{2\}$. Therefore $r_4 \in \pi(K)$ or $r_8 \in \pi(K)$. By Lemma 2.5, ${}^2D_6(5)$ contains a Frobenius subgroup of the form $5^{10} : r_{10}$. We know that $r_5 \in \pi(S)$ and $r_5 \nsim r_4$ and $r_5 \nsim r_8$ in $\Gamma(B_6(5))$. Therefore by Lemma 3.1, $r_4 \sim r_{10}$ or $r_8 \sim r_{10}$, which is a contradiction.

Let $p = 7$. Since $n' \geqslant 4$, we have $\pi(p^6 - 1) \subseteq \pi(S)$. Therefore $43 \in \pi(S)$, which is a contradiction. Similarly for every $p \in \{11, 13, 31, 71, 313, 521, 601\}$ we get a contradiction.

Similarly to the above for $7 \leqslant n \leqslant 11$, we get a contradiction.

• Let $S \cong D_{n'}(q)$.

If $n = 6$, then $p \in \pi(B_6(5))$. By (3.3), we have $n' \geqslant 4$. Since $r_{2n} \nsim 2$ in $\Gamma(S)$, hence $601 = r_{2n} \in \{r'_{n'}, r'_{n'-1}, r'_{2(n'-1)}\}$, by [23, Proposition 4.4].

Let $p = 2$. If $r'_{n'} = 601$, then $25 \mid n'\alpha$, since $e(601, 2) = 25$. Therefore $1801 \in \pi(2^{25} - 1) \subseteq \pi(S)$, which is a contradiction. Similarly $601 \notin \{r'_{n'-1}, r'_{2(n'-1)}\}$.

Let $p = 3$. We have $e(601, 3) = 75$ and similarly to the above, we get a contradiction.

Let $p = 5$. Since $\pi(S) \subseteq \pi(G)$, it follows that $2(n' - 1)\alpha \leqslant 12$. Therefore we consider the following cases:

1. Let $\alpha = 2$ and $n' = 4$, so $S \cong D_4(5^2)$. Therefore $r_5 \in \pi(G)$ and $r_5 \notin \pi(S)$. So $r_5 \in \pi(\overline{G}/S) \cup \pi(K)$. Since $\pi(\mathrm{Out}(S)) = \{2\}$, we have $r_5 \in \pi(K)$. By Lemma 2.5, $D_4(5^2)$ contains a Frobenius subgroup of the form $5^6 : r_6$. We know that $r_{12} \in \pi(S)$ and $r_{12} \nsim r_5$ in $\Gamma(B_6(5))$. Therefore by Lemma 3.1, $r_5 \sim r_6$ in $\Gamma(B_6(5))$, which is a contradiction.

2. Let $\alpha = 1$ and $4 \leqslant n' \leqslant 7$. We know that $601 \in \pi(S)$ and $e(601, 5) = 12$, hence $n' = 7$. So $S \cong D_7(5)$. Therefore $r_7 \in \pi(S)$ and $r_7 \notin \pi(G)$, which is a contradiction.

Let $p = 7$. Since $n' \geqslant 4$, we have $\pi(p^6 - 1) \subseteq \pi(S)$. Therefore $43 \in \pi(S)$, which is a contradiction. Similarly for every $p \in \{11, 13, 31, 71, 313, 521, 601\}$, we get a contradiction.

If $n = 7$, then $\pi(G) = \{2, 3, 5, 7, 11, 13, 29, 31, 71, 313, 449, 521, 601, 19531\}$. Since $t(S) \geqslant t(G) - 1$, we have $n' \geqslant 6$. If $p \in \pi(G) \setminus \{5\}$, then similarly to the previous case, we get a contradiction.

Let $p = 5$. Since $\pi(S) \subseteq \pi(G)$, we have $2(n' - 1)\alpha \leqslant 14$. Therefore $6 \leqslant n' \leqslant 8$ and $\alpha = 1$. We know that $29 \in \pi(S)$ and $e(29, 5) = 14$, so $n' = 8$. Then $S \cong D_8(5)$. Now by Lemmas 2.10 and 2.7, $r_3 \sim r_5$ in $\Gamma(S)$ and $r_3 \nsim r_5$ in $\Gamma(G)$, which is a contradiction.

Similarly to the above for $8 \leqslant n \leqslant 11$, we get a contradiction.

Case 5. Let $S \cong C_{n'}(q)$, where $q = p^\alpha$.

By Lemma 2.1, $t(S) \geqslant t(G) - 1$, so

$$3n' > 3n - 8. \tag{3.4}$$

(a) Let $n \geqslant 12$. Then (4) implies that $n' \geqslant 10$.

(5.1.a) Let $p \neq 5$. By [23, Propositions 3.1, 4.3], every $r'_i \in \pi(S)$, where $i \notin \{2n', n'\}$, is adjacent to 2 and p in $\Gamma(S)$. We obtain $e(r_{2n}, q) \in \{2n', n'\}$, since $r_{2n} \in \varrho(2, G)$. Since A is an independent set in $\Gamma(G)$, it follows that $\eta(e(r_i, q)) \neq \eta(e(r_j, q))$ for $r_i, r_j \in A$ and $i \neq j$. Therefore p is adjacent to at least two elements of $\pi(S) \cap A \setminus \{r_{2n}\}$ in $\Gamma(S)$. So similarly to Case 2, $p \in \{2, 3, 7, 13, 31, 313\}$.

If $p = 31$, then by Lemma 3.4, $t(1129, S) \leqslant 7$, while by Lemma 3.3, $t(1129, G) \geqslant 65$. Therefore by Remark 2.2, we get a contradiction.

Similarly for every $p \in \{2, 3, 7, 13, 31, 313\}$ we get a contradiction.

In the same manner we prove that S cannot be isomorphic to $B_{n'}(q)$, where $q = p^\alpha$, $p \neq 5$, and $n' \geqslant 10$.

(5.2.a) Let $p = 5$. We know that $r_{2n} \in \pi(S)$ and $2 \nsim r_{2n}$ in $\Gamma(S)$. By [23, Proposition 4.3], $e(r_{2n}, q) \in \{2n', n'\}$. Therefore, $2n = e(r_{2n}, 5) \mid 2\alpha n'$ or $2n =$

$e(r_{2n},5) \mid \alpha n'$. On the other hand, $2\alpha n' \leqslant 2n$, by Lemma 2.4. So $2\alpha n' = 2n$, and by (3.4), $\alpha = 1$, since $n' \geqslant 10$. Then $S \cong C_n(5)$. We note that $\Gamma(C_n(5)) = \Gamma(B_n(5))$ (see [24, Proposition 2.4]).

(b) Let $6 \leqslant n \leqslant 11$.

If $n = 6$, then $p \in \pi(B_6(5))$. By (3.4), we have $n' \geqslant 4$.

Let $p = 5$. Since $\pi(S) \subseteq \pi(G)$, so $2n'\alpha \leqslant 12$. Therefore $4 \leqslant n' \leqslant 6$ and $\alpha = 1$. We know that $601 \in \pi(S)$ and $e(601, 5) = 12$, so $n' = 6$. Then $S \cong C_6(5)$.

If $p = 2$, then $17 \in \pi(2^8 - 1) \subseteq \pi(S)$, which is a contradiction. Similarly for every $p \in \{3, 7, 11, 13, 31, 71, 313, 521, 601\}$, we get a contradiction.

Similarly to the above for $7 \leqslant n \leqslant 11$, we can prove that $S \cong C_n(5)$.

Similarly to the above discussion it follows that $S \cong B_n(5)$.

Case 6. Let $S \cong F_4(q)$, where $q = p^\alpha$.

We know that $t(S) \leqslant 5$. If $n > 7$, then $t(G) \geqslant 7$, which is a contradiction, by Lemma 2.1.

If $n = 6$, then $p \in \pi(B_6(5))$.

Let $p = 5$. Since $\pi(S) \subseteq \pi(G)$, we have $12\alpha \leqslant 12$. Therefore $\alpha = 1$ and $S \cong F_4(5)$. We know that $r_{10} \in \pi(G)$ and $r_{10} \notin \pi(S)$. So $r_{10} \in \pi(\overline{G}/S) \cup \pi(K)$. Therefore $r_{10} \in \pi(K)$, since $\mathrm{Out}(S) = 1$. By [22], $B_4(5) \leqslant F_4(5)$ and by Lemma 2.5, $B_4(5)$ contains a Frobenius subgroup of the form $5^3 : r_3$. We know that $r_{12} \in \pi(S)$ and $r_{12} \nsim r_{10}$ in $\Gamma(G)$. Therefore by Lemma 3.1, $r_3 \sim r_{10}$, which is a contradiction.

If $p = 2$, then $17 \in \pi(2^8 - 1) \subseteq \pi(S)$, which is a contradiction. Similarly for every $p \in \{3, 7, 11, 13, 31, 71, 313, 521, 601\}$, we get a contradiction.

If $n = 7$, then in a similar manner, we get a contradiction.

Case 7. Let $S \cong E_6(q)$, where $q = p^\alpha$.

We know that $t(S) = 5$. If $n > 7$, then $t(G) \geqslant 7$, which is a contradiction, by Lemma 2.1.

If $n = 6$, then $p \in \pi(B_6(5))$. Similarly to Case 6, if $p \neq 5$, then we get a contradiction.

Let $p = 5$. Since $\pi(S) \subseteq \pi(G)$, hence $12\alpha \leqslant 12$. Therefore $\alpha = 1$ and $S \cong E_6(5)$. Now by [22], $F_4(5) \leqslant E_6(5)$ and using the previous case we get a contradiction.

If $n = 7$, then similarly we get a contradiction.

In the same manner we can prove that S is not isomorphic to ${}^2E_6(q)$.

Case 8. Let $S \cong E_7(q)$, where $q = p^\alpha$.

We know that $t(S) = 8$. If $n \geqslant 12$, then $t(G) \geqslant 10$, which is a contradiction, by Lemma 2.1. We know that $19 \in \pi(S)$, therefore $n \geqslant 9$. Also $p \in \pi(G)$. If $n = 9$, then

$$\pi(G) = \{2, 3, 5, 7, 11, 13, 17, 19, 29, 31, 71, 313, 449, 521, 601, 829, 5167, 11489, 19531\}.$$

Let $p = 5$. Since $\pi(S) \subseteq \pi(G)$, we have $18\alpha \leqslant 18$, and so $\alpha = 1$ and $S \cong E_7(5)$. We know that $r_{16} \in \pi(G)$ and $r_{16} \notin \pi(S)$. So $r_{16} \in \pi(\overline{G}/S) \cup \pi(K)$. Therefore $r_{16} \in \pi(K)$, since $\pi(\mathrm{Out}(S)) = \{2\}$. By [22], $C_4(5) \leqslant A_7(5) \leqslant E_7(5)$ and by Lemma 2.5, $C_4(5)$ contains a Frobenius subgroup of the form $5^4 : (5^4 - 1)/2$. We know that $r_{18} \in \pi(S)$ and $r_{18} \nsim r_{16}$ in $\Gamma(B_9(5))$. Therefore by Lemma 3.1, $r_4 \sim r_{16}$ in $\Gamma(G)$, which is a contradiction.

If $p = 2$, then $73 \in \pi(2^{18} - 1) \subseteq \pi(S)$, which is a contradiction. Similarly for every $p \in \pi(G)$, we get a contradiction.

Similarly to the above for $n = 10$ and $n = 11$, we get a contradiction.

Case 9. Let $S \cong E_8(q)$, where $q = p^\alpha$.

We know that $t(S) = 12$. So by Lemma 2.1 we have $n \leqslant 16$. We know that $19 \in \pi(S)$, so $n \geqslant 9$. Therefore $9 \leqslant n \leqslant 16$ and $p \in \pi(G)$.

Let $n = 16$. For every $p \in \pi(G) \setminus \{5\}$, we get a contradiction, since $\pi(p^{30} - 1) \nsubseteq \pi(B_{16}(5))$. For example, if $p = 2$, then $151 \in \pi(2^{30} - 1) \subseteq \pi(S)$ and $151 \notin \pi(B_{16}(5))$.

Let $p = 5$. Since $\pi(S) \subseteq \pi(G)$, so $30\alpha \leqslant 32$. Therefore $\alpha = 1$ and $S \cong E_8(5)$. We know that $r_{13} \in \pi(G)$ and $r_{13} \notin \pi(S)$. So $r_{13} \in \pi(\overline{G}/S) \cup \pi(K)$. Therefore $r_{13} \in \pi(K)$, since $\mathrm{Out}(S) = 1$. Using [22], we have $D_8(5) \leqslant E_8(5)$ and $D_8(5)$ contains a Frobenius subgroup $5^{21} : r_7$. Now $r_{30} \nsim r_7$ and by Lemma 3.1, we have $r_{13} \sim r_7$, which is a contradiction, by Lemma 2.7. For other cases we easily get a contradiction.

Case 10. Let $S \cong {}^2B_2(q)$, where $q = 2^{2n'+1}$.

We know that $t(S) = 4$. Therefore $n = 6$. Then $A = \{r_5, r_6, r_8, r_{10}, r_{12}\}$ is an independent set in $\Gamma(G)$. At least 4 elements of A belong to $\pi(S)$. Since $t(S) = 4$ and $2 \in \varrho(S)$, it follows that one of the elements of A must be equal to 2, which is a contradiction.

Case 11. Let $S \cong {}^2G_2(q)$, where $q = 3^{2n'+1}$.

We know that $t(S) = 5$. Therefore $n = 6$ or $n = 7$.

If $n = 7$, then $A = \{r_5, r_7, r_8, r_{10}, r_{12}, r_{14}\}$ is an independent set in $\Gamma(G)$. On the other hand, for each independent set $\varrho(S)$ we have $|\varrho(S) \setminus \{3\}| = 4$, by [23, Table 9]. So we get a contradiction since $|A \cap \pi(S)| \geqslant 5$.

Let $n = 6$, we know that $r_{2n} = 601 \in \pi(S)$. So $601 \mid (q - 1)$ or $601 \mid (q^3 + 1)$.

If $601 \mid (q - 1)$, then $75 \mid (2n' + 1)$, since $e(601, 3) = 75$. Therefore $4561 \in \pi(3^{75} - 1) \subseteq \pi(q - 1) \subseteq \pi(S)$, which is a contradiction. Similarly, if $601 \mid (q^3 + 1)$, we get a contradiction.

Case 12. Let $S \cong {}^2F_4(q)$, where $q = 2^{2n'+1} \geqslant 32$.

We know that $t(S) = 5$, so $n = 6$ or $n = 7$.

Let $n = 7$. We know that $29 = r_{2n} \in \pi(S)$. So 29 divides $q - 1$, $q^3 + 1$, $q^4 - 1$, or $q^6 + 1$. If $29 \mid (q - 1)$, then $28 \mid (2n' + 1)$, since $e(29, 2) = 28$. Therefore $127 \in \pi(2^{28} - 1) \subseteq \pi(q - 1) \subseteq \pi(S)$, which is a contradiction. Similarly for other cases, we get a contradiction.

Similarly to the above for $n = 6$, we get a contradiction.

Case 13. Let S be a sporadic group.

If $n \geqslant 16$, then $t(G) \geqslant 13$, which is a contradiction by Lemma 2.1, since $t(S) \leqslant 11$. For $6 \leqslant n \leqslant 15$ we can easily see that $r_{2n} \notin \pi(S)$, which is a contradiction. □

Theorem 3.7. *If $\Gamma(G) = \Gamma(B_n(5))$, where $n \geqslant 6$, then there exists a nonabelian simple group S such that $S \leqslant G/K \leqslant \mathrm{Aut}(S)$, and one of the following holds:*

(1) *$S \cong B_n(5)$ and K is a $\{2, 3\}$-group.*
(2) *$S \cong C_n(5)$, where n is odd, and K is an elementary abelian r_m-group such that $m \mid n$.*
(2) *$S \cong C_n(5)$, where n is even, and K is an elementary abelian r_m-group such that $\eta(m) \leqslant n/2$ or n/m is odd.*

Proof. By Lemma 2.1, we know that $S \leqslant G/K \leqslant \mathrm{Aut}(S)$, where K is the maximal normal soluble subgroup of G. By Theorem 3.6, $S \cong B_n(5)$ or $S \cong C_n(5)$. Assume that there exists p such that $p \mid |K|$. We claim that without loss of generality we can consider K as an elementary abelian p-group for $p \in \pi(G)$. Since K is soluble, there is $p \in \pi(G)$ such that $O^p(K) \neq K$. Then $K/O^p(K)$ is a nontrivial p-group. Let $\hat{K} = K/O^p(K)$ and $\hat{G} = G/O^p(K)$, since $O^p(K)$ is a characteristic subgroup of K and $K \lhd G$. If the Frattini subgroup of $\hat{K}$ is denoted by $\Phi(\hat{K})$, then $\hat{K}/\Phi(\hat{K})$ is an elementary abelian p-group and we have

$$\frac{G}{K} \cong \frac{\hat{G}}{\hat{K}} \cong \frac{\hat{G}/\Phi(\hat{G})}{\hat{K}/\Phi(\hat{K})}.$$

Therefore without loss of generality we can assume that K is an elementary abelian p-group. Since by [6] we know that $B_n(5)$ and $C_n(5)$ act unisingularly we conclude that $p \neq 5$.

We claim that if $n \geqslant 6$ is odd, then for each element $t \in \pi(B_n(5)) = \pi(C_n(5))$ we have $t \nsim r_n$ or $t \nsim r_{2n}$. If $t = 2$, then $2 \nsim r_n$ or $2 \nsim r_{2n}$ by [23, Proposition 2.4]. Let $t \neq 2$ and denote $e(t, 5)$ by a. If $t \sim r_n$ and $t \sim r_{2n}$, then by Lemma 2.7, n/a and $2n/a$ are odd, which is a contradiction.

Also we claim that if $n \geqslant 6$ is even, then for each element $t \in \pi(B_n(5)) = \pi(C_n(5))$ we have $t \nsim r_{2(n-1)}$ or $t \nsim r_{2n}$. Let $e(t, 5) = a$. Let $t \sim r_{2(n-1)}$ and $t \sim r_{2n}$. Since $t \sim r_{2(n-1)}$, it follows that $n - 1 + \eta(a) \leqslant n$ or $2(n-1)/a$ is odd, by Lemma 2.7. Similarly, since $t \sim r_{2n}$, it follows that $2n/a$ is odd, by Lemma 2.7. Therefore $a = 1$ or 2 and $2n/a$ is odd, which is a contradiction, since n is even.

- Let $S \cong B_n(5)$.

If n is odd, then S contains a Frobenius subgroup with kernel of order $5^{n(n-1)/2}$ and a cyclic complement of order r_n, by Lemma 2.5. By assumption, $S \leqslant G/K$, and so

G/K contains a Frobenius subgroup T/K of the form $5^{n(n-1)/2} : r_n$. If $p \nsim r_n$, then since $p \neq 5$, by Lemma 3.1, it follows that $p \sim r_n$, which is a contradiction. Therefore $p \sim r_n$, and so $p \nsim r_{2n}$, by the above discussion. Also we know that $B_{n-2}(5) \leqslant B_n(5)$, by [22], and so $B_{n-2}(5) \leqslant G/K$. Similarly G/K contains a Frobenius subgroup of the form $5^{(n-2)(n-3)/2} : r_{n-2}$, by Lemma 2.5. Since $p \neq 5$ and $p \nsim r_{2n}$ it follows that $p \sim r_{n-2}$, by Lemma 3.1. Let $e(p, 5) = m$. Since $p \sim r_n$ it follows that n/m is odd, by Lemma 2.7. Similarly since $p \sim r_{n-2}$ it follows that $n - 2 + \eta(m) \leqslant n$ or $(n-2)/m$ is odd. Consequently, $m = 1$ and so $p = 2$, since m is odd. Therefore K is a 2-group.

Let n be even. We note that G/K contains a Frobenius subgroup of the form $5^{(n-1)(n-2)/2} : r_{n-1}$, by Lemma 2.5. By the above discussion, $p \nsim r_{2(n-1)}$ or $p \nsim r_{2n}$. Therefore since $p \neq 5$, by Lemma 3.1, we conclude that $p \sim r_{n-1}$. Also we know that $B_{n-2}(5) \leqslant B_n(5)$, by [22]. Similarly G/K contains a Frobenius subgroup of the form $5^{(n-3)(n-4)/2} : r_{n-3}$, by Lemma 2.5. Similarly $p \sim r_{n-3}$, by Lemma 3.1. Let $e(p, 5) = m$. Since $p \sim r_{n-1}$, it follows that $n - 1 + \eta(m) \leqslant n$ or $(n-1)/m$ is odd, by Lemma 2.7. Similarly since $p \sim r_{n-3}$ it follows that $n - 3 + \eta(m) \leqslant n$ or $(n-3)/m$ is odd. Consequently, $m \in \{1, 2, 3\}$, so $p \in \{2, 3, 31\}$.

Let $p = 31$. We know that ${}^2D_n(5) \leqslant B_n(5)$, by [22], and by Lemma 2.5, ${}^2D_n(5)$ contains a Frobenius subgroup of the form $5^{2(n-1)} : r_{2(n-1)}$. We know that $p \nsim r_{2(n-1)}$ or $p \nsim r_{2n}$. Since $p \neq 5$ by Lemma 3.1, $31 = p \sim r_{2(n-1)}$, which is a contradiction by Lemma 2.7. Therefore $p = 3$ or $p = 2$, so K is a $\{2, 3\}$-group.

- Let $S \cong C_n(5)$.

By Lemma 2.5, $C_n(5)$ contains a Frobenius subgroup of the form $5^n : (5^n - 1)/2$. By assumption, $S \leqslant G/K$. Then G/K contains a Frobenius subgroup T/K of the form $5^{n(n-1)/2} : r_n$. Now using Lemma 3.1 similarly to the above, $p \sim r_n$. Let $p = r_m$. If n is odd, then $m \mid n$, by Lemma 2.7; and if n is even then $\eta(m) \leqslant n/2$ or n/m is odd, by Lemma 2.7. □

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