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GLOBAL CLASSICAL SOLUTIONS TO A KIND OF MIXED INITIAL-BOUNDARY VALUE PROBLEM FOR INHOMOGENEOUS QUASILINEAR HYPERBOLIC SYSTEMS

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Abstract. In this paper, the mixed initial-boundary value problem for inhomogeneous quasilinear strictly hyperbolic systems with nonlinear boundary conditions in the first quadrant $\{(t,x)\colon\ t \geqslant 0, x \geqslant 0\}$ is investigated. Under the assumption that the right-hand side satisfies a matching condition and the system is strictly hyperbolic and weakly linearly degenerate, we obtain the global existence and uniqueness of a C^1 solution and its L^1 stability with certain small initial and boundary data.

Keywords: quasilinear hyperbolic system, mixed initial-boundary value problem, global classical solution, weak linear degeneracy, matching conditon

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1. Introduction and main results

Consider the first order strictly quasilinear hyperbolic system

$$\frac{\partial u}{\partial t} + A(u)\frac{\partial u}{\partial x} = B(u), \tag{1.1}$$

where $u = (u_1, \ldots, u_n)^\top$ is the unknown vector function of (t,x), $A(u)$ is an $n \times n$ matrix with suitably smooth elements $a_{ij}(u)$ $(i,j = 1, \ldots, n)$, and $B(u) = (B_1(u), \ldots, B_n(u))^\top$ is a vector function of u with suitably smooth elements.

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By the definition of strict hyperbolicity, for any given u on the domain under consideration, $A(u)$ has n distinct real eigenvalues $\lambda_1(u), \ldots, \lambda_n(u)$. We furthermore suppose that

$$\lambda_1(0) < \ldots < \lambda_m(0) < 0 < \lambda_{m+1}(0) < \ldots < \lambda_n(0). \tag{1.2}$$

Let $l_i(u) = (l_{i1}(u), \ldots, l_{in}(u))$ and $r_i(u) = (r_{i1}(u), \ldots, r_{in}(u))^\top)$ be respectively a left and right eigenvector corresponding to $\lambda_i(u)$ $(i = 1, \ldots, n)$:

$$l_i(u)A(u) = \lambda_i(u)l_i(u) \quad \text{and} \quad A(u)r_i(u) = \lambda_i(u)r_i(u). \tag{1.3}$$

We have

$$\det|l_{ij}(u)| \neq 0 \quad (\text{equivalently, } \det|r_{ij}(u)| \neq 0). \tag{1.4}$$

Without loss of generality, we suppose that

$$l_i(u)r_j(u) \equiv \delta_{ij} \quad (i, j = 1, \ldots, n), \tag{1.5}$$

$$r_i(u)^\top r_i(u) \equiv 1 \quad (i = 1, \ldots, n), \tag{1.6}$$

where δ_{ij} stands for Kronecker's symbol.

All $\lambda_i(u)$, $l_{ij}(u)$ and $r_{ij}(u)$ $(i, j = 1, \ldots, n)$ are supposed to have the same regularity as $a_{ij}(u)$, $(i, j = 1, \ldots, n)$.

Under the assumption that

$$B(u) \equiv 0, \tag{1.7}$$

for the Cauchy problem of system (1.1) with the initial data

$$t = 0\colon\ u = \varphi(x), \quad x \in \mathbb{R}, \tag{1.8}$$

where $\varphi(x)$ is a C^1 vector value function with bounded C^1 norm, many results concerning global existence and blow-up of classical solutions have been obtained (see [1], [5]–[6], and [16]). In particular, by means of the concept of weak linear degeneracy, for small initial data with certain decaying properties, the global existence and the blow-up phenomenon of a C^1 solution to the Cauchy problem (1.1) and (1.8) have been completely studied (see [9] and [14]–[15]). In virtue of two basic L^1 estimates, Zhou [20] furthermore relaxed the restrictions on the initial data and then showed that the Cauchy problem (1.1) and (1.8) for the weakly linearly degenerate and strictly hyperbolic system admits a unique C^1 solution which also satisfies L^1 stability.

In order to consider the effect of nonlinear boundary conditions on the global regularity of a classical solution of system (1.1), Li & Wang [12] investigated the mixed initial-boundary value problem for system (1.1) on a half-unbounded domain. The result obtained by Li & Wang indicates that the interaction of nonlinear boundary conditions with nonlinear hyperbolic waves does not cause any negative effect on the global regularity of the C^1 solution, provided that the C^1 norms of initial and boundary data both decaying at infinity are small enough. Recently, Zhou & Yang [21] also relaxed the restrictions on the initial and boundary data and then established results on the global C^1 solution for linearly degenerate and weakly linearly degenerate system, respectively.

For inhomogeneous quasilinear hyperbolic system (1.1), under the assumptions that $B(u)$ satisfies the so-called matching condition and the system is strictly hyperbolic and weakly linearly degenerate, Kong [7] and Li [8] established the corresponding results for the Cauchy problem (1.1) and (1.8). Recently, Wu [19] extended the results established by Zhou [20] to the inhomogeneous case. Chen [3] also obtained the corresponding inhomogeneous result in [12].

In this paper, we are going to re-prove the global existence result with less restrictions on the initial and boundary data. In particular, the supreme norms of the derivatives of the initial and boundary data are not assumed to be small. We shall also obtain global L^1 stability results in this situation.

Moreover, for the mixed initial-boundary value problem on a bounded domain $\{(t,x)\colon\ t\geqslant 0,\ 0\leqslant x\leqslant L\}$, the results on the global regularity can be found in [4], [9]–[11], and [18].

On the domain

$$D \overset{\text{def}}{=} \{(t,x)\colon\ t\geqslant 0,\ x\geqslant 0\} \tag{1.9}$$

we consider the mixed initial-boundary value problem for system (1.1) with the initial data

$$t=0\colon\ u=\varphi(x),\quad x\geqslant 0 \tag{1.10}$$

and the boundary condition

$$x=0\colon\ v_s=f_s(\alpha(t),v_1,\ldots,v_m)+h_s(t)\quad (s=m+1,\ldots,n), \tag{1.11}$$

where

$$v_i=l_i(u)u\quad (i=1\ldots,n), \tag{1.12}$$

$h_s(t)$ $(s = m+1, \dots, n)$ are given C^1 functions of t and

$$\alpha(t) \stackrel{\text{def}}{=} (\alpha_1(t), \dots, \alpha_k(t)).$$

Let

$$h(t) \stackrel{\text{def}}{=} (h_{m+1}(t), \dots, h_n(t)),$$

and let us define

$$|u| \stackrel{\text{def}}{=} \left(\sum_{k=1}^{n} u_k^2\right)^{1/2}$$

for any vector value function $u = (u_1, \dots, u_n)^\top$. Without loss of generality, we suppose that

$$f_s(\alpha(t), 0, \dots, 0) \equiv 0 \quad (s = m+1, \dots, n). \tag{1.13}$$

We also remark that in a neighborhood of $u = 0$, the boundary condition (1.11) takes the same form independently of the choice of left eigenvectors.

To state our results precisely, we shall first recall the concept of weak linear degeneracy (see [9], [14]) and matching condition (see [7], [8], [19]) as follows.

Definition 1.1. The ith characteristic $\lambda_i(u)$ is weakly linearly degenerate if along the ith characteristic trajectory $u = u^{(i)}(s)$ passing through $u = 0$, defined by

$$\frac{\mathrm{d}u}{\mathrm{d}s} = r_i(u), \tag{1.14}$$

$$s = 0\colon\ u = 0, \tag{1.15}$$

we have

$$\nabla\lambda_i(u) r_i(u) \equiv 0, \quad \forall\, |u| \text{ small}, \tag{1.16}$$

namely,

$$\lambda_i(u^{(i)}(s)) \equiv \lambda_i(0), \quad \forall\, |s| \text{ small}. \tag{1.17}$$

If all characteristics $\lambda_i(u)$ $(i = 1, \dots, n)$ are weakly linearly degenerate, then the system (1.1) is said to be weakly linearly degenerate.

Definition 1.2. $B(u)$ satisfies the matching condition if along all characteristic trajectories passing through $u = 0$ we have $B(u) \equiv 0$, $\forall\, |u|$ small, i.e.,

$$B(u^{(i)}(s)) \equiv 0,\ \ \forall\, |s| \text{ small} \quad (i = 1, \dots, n). \tag{1.18}$$

In this case, it is easy to see that

$$B(0) = 0, \quad \nabla B(0) = 0. \tag{1.19}$$

Our main results can be summarized as follows:

Theorem 1.1. *Suppose that in a neighborhood of* $u = 0$ *we have* $A(u) \in C^2$, *system* (1.1) *is strictly hyperbolic and weakly linearly degenerate, and* $B(u) \in C^2$ *satisfies the matching condition. Suppose furthermore that* φ, α, f_s, *and* $h_s \in C^1$ $(s = m+1, \dots, n)$. *Suppose finally that the conditions of* C^1 *compatibility are satisfied at the point* $(0,0)$ *and assumptions* (1.2) *and* (1.13) *hold. Let*

$$M = \max\Big\{\sup_{x\geqslant 0}|\varphi'(x)|,\ \sup_{t\geqslant 0}|\alpha'(t)|,\ \sup_{t\geqslant 0}|h'(t)|\Big\}. \tag{1.20}$$

Then there exists a positive constant ε *independent of* M *such that the mixed initial-boundary value problem* (1.1) *and* (1.10)–(1.11) *admits a unique global* C^1 *solution* $u = u(t,x)$ *on the domain* D *provided that*

$$\int_0^\infty |\varphi'(x)|\,\mathrm{d}x \leqslant \varepsilon, \quad \int_0^\infty |\varphi(x)|\,\mathrm{d}x \leqslant \frac{\varepsilon}{M+1} \tag{1.21}$$

and

$$\int_0^\infty (|h'(t)| + |\alpha'(t)|)\,\mathrm{d}t \leqslant \varepsilon, \quad \int_0^\infty (|h(t)| + |\alpha(t)|)\,\mathrm{d}t \leqslant \frac{\varepsilon}{M+1}. \tag{1.22}$$

Theorem 1.2. *Under the assumptions of Theorem* 1.1, *suppose furthermore that* $B(u) \in C^3$. *If* $u^{(1)}$, $u^{(2)}$ *are two solutions given by Theorem* 1.1 *with initial data* $\varphi^{(1)}$, $\varphi^{(2)}$, *boundary conditions* $\alpha^{(1)}$, $h^{(1)}$ *and* $\alpha^{(2)}$, $h^{(2)}$, *respectively, then we have*

$$\begin{aligned}\int_0^\infty & |u^{(1)}(t,x) - u^{(2)}(t,x)|\,\mathrm{d}x \\ &\leqslant C\Big(\int_0^\infty |\varphi^{(1)}(x) - \varphi^{(2)}(x)|\,\mathrm{d}x + \int_0^\infty |h^{(1)}(t) - h^{(2)}(t)|\,\mathrm{d}t \\ &\qquad + \int_0^\infty |\alpha^{(1)}(t) - \alpha^{(2)}(t)|\,\mathrm{d}t\Big), \quad \forall\, t \geqslant 0,\end{aligned} \tag{1.23}$$

where C *is a positive constant independent of* M *and* t.

Under the assumption that (1.7) holds, for the weak solution to the Cauchy problem (1.1) and (1.8) for general quasilinear hyperbolic systems, Bressan et al. [2] and Liu & Yang [17] both obtained global L^1 stability with respect to time t.

This paper is organized as follows: In Section 2, we recall and generalize John's formula on the decomposition of waves. Section 3 is devoted to two basic lemmas concerning the L^1 estimate. In Sections 4 and 5, we prove Theorem 1.1 and Theorem 1.2, respectively.

Suppose that $A(u) \in C^2$. By Lemma 2.5 in [14] (see also [9]), there exists an invertible C^3 transformation $u = u(\tilde{u})$ ($u(0) = 0$) such that in the $\tilde{u}$-space, for each $i = 1, \ldots, n$, the ith characteristic trajectory passing through $\tilde{u} = 0$ coincides with the $\tilde{u}_i$-axis at least for $|\tilde{u}_i|$ small, namely

$$\tilde{r}_i(\tilde{u}_i e_i) \equiv e_i, \quad \forall\, |\tilde{u}_i| \text{ small} \quad (i = 1, \ldots, n), \tag{2.1}$$

where

$$e_i = (0, \ldots, 0, \overset{(i)}{1}, 0, \ldots, 0)^\top \in \mathbb{R}^n \quad (i = 1, \ldots, n). \tag{2.2}$$

Such a transformation is called a normalized transformation and the corresponding unknown variables $\tilde{u} = (\tilde{u}_1, \ldots, \tilde{u}_n)^\top$ are called normalized variables or normalized coordinates.

Let v_i $(i = 1, \ldots, n)$ be defined by (1.12) and

$$w_i = l_i(u) u_x \quad (i = 1, \ldots, n), \tag{2.3}$$

$$\beta_i = l_i(u) B(u) \quad (i = 1, \ldots, n). \tag{2.4}$$

By (1.5), we have

$$u = \sum_{k=1}^{n} v_k r_k(u) \tag{2.5}$$

and

$$u_x = \sum_{k=1}^{n} w_k r_k(u). \tag{2.6}$$

Let

$$\frac{\mathrm{d}}{\mathrm{d}_i t} = \frac{\partial}{\partial t} + \lambda_i(u) \frac{\partial}{\partial x} \tag{2.7}$$

be the directional derivative with respect to time t along the ith characteristic. We have (see [19])

$$\frac{\mathrm{d}u}{\mathrm{d}_i t} = \sum_{\substack{k=1 \\ k \neq i}}^{n} (\lambda_i(u) - \lambda_k(u)) w_k r_k(u) + \sum_{h=1}^{n} \beta_h r_h(u). \tag{2.8}$$

It is easy to see, in normalized coordinates, that

$$\frac{\mathrm{d}u_i}{\mathrm{d}_i t} = \sum_{j,k=1}^{n} \varrho_{ijk}(u) u_j w_k + \sum_{j=1}^{n} \Big(\sum_{h=1}^{n} \bar{\varrho}_{ijh}(u) \beta_h(u) \Big) u_j + r_{ii}(u)\beta_i, \tag{2.9}$$

where

$$\varrho_{ijk} = \begin{cases} (\lambda_i(u) - \lambda_k(u)) \displaystyle\int_0^1 \frac{\partial r_{ki}(su_1, \dots, su_{k-1}, u_k, su_{k+1}, \dots, su_n)}{\partial u_j} \,\mathrm{d}s, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad k \neq i,\ j \neq k, \\ 0 \quad \text{otherwise}, \end{cases} \tag{2.10}$$

$$\bar{\varrho}_{ijh} = \begin{cases} \displaystyle\int_0^1 \frac{\partial r_{hi}(su_1, \dots, su_{h-1}, u_h, su_{h+1}, \dots, su_n)}{\partial u_j} \,\mathrm{d}s, \\ \qquad\qquad\qquad\qquad\qquad\qquad h \neq i,\ j \neq h, \\ 0 \quad \text{otherwise}. \end{cases} \tag{2.11}$$

From (2.6) and (2.9), we get

$$\frac{\partial u_i}{\partial t} + \frac{\partial(\lambda(u) u_i)}{\partial x} = \sum_{j,k=1}^{n} F_{ijk}(u) u_j w_k + \sum_{j=1}^{n} \Big(\sum_{h=1}^{n} \bar{\varrho}_{ijh}(u) \beta_h(u) \Big) u_j + r_{ii}(u)\beta_i, \tag{2.12}$$

or equivalently,

$$\begin{aligned} &\mathrm{d}[u_i(\mathrm{d}x - \lambda_i(u)\,\mathrm{d}t)] \\ &\quad = \Big[\sum_{j,k=1}^{n} F_{ijk}(u) u_j w_k + \sum_{j=1}^{n} \Big(\sum_{h=1}^{n} \bar{\varrho}_{ijh}(u) \beta_h(u) \Big) u_j + r_{ii}(u)\beta_i \Big] \mathrm{d}t \wedge \mathrm{d}x, \end{aligned} \tag{2.13}$$

where

$$F_{ijk}(u) = \varrho_{ijk}(u) + \nabla\lambda_i(u) r_k(u) \delta_{ij}. \tag{2.14}$$

By (2.10), it obviously follows that

$$F_{ijj}(u) \equiv 0, \quad \forall\, j \neq i. \tag{2.15}$$

Moreover, when system (1.1) is weakly linearly degenerate, then in normalized coordinates we have

$$F_{ijj}(u_j e_j) \equiv 0, \quad \forall\, |u_j|,\ \forall\, i, j \in \{1, \dots, n\}. \tag{2.16}$$

On the other hand, we have

$$\frac{\mathrm{d}w_i}{\mathrm{d}_i t} = \sum_{j,k=1}^{n} \gamma_{ijk}(u) w_j w_k + \sum_{j=1}^{n}\Big(\sum_{h=1}^{n} B_{ijh}(u)\beta_h(u) + \nu_{ij}(u)\Big) w_j, \tag{2.17}$$

where

$$\gamma_{ijk}(u) = \tfrac{1}{2}\{(\lambda_j(u) - \lambda_k(u)) l_i(u)\nabla r_k(u) r_j(u) - \nabla\lambda_k(u) r_j(u)\delta_{ik} + (j|k)\}, \tag{2.18}$$

$$B_{ijh}(u) = -l_i(u)\nabla r_j(u) r_h(u), \tag{2.19}$$

and

$$\nu_{ij}(u) = l_i(u)\nabla B(u) r_j(u), \tag{2.20}$$

with $(j|k)$ standing for all terms obtained by changing j and k in the previous terms. Hence, we have

$$\gamma_{ijj}(u) \equiv 0, \quad \forall\, j \neq i. \tag{2.21}$$

When system (1.1) is weakly linearly degenerate, in normalized coordinates we get

$$\gamma_{iii}(u_i e_i) \equiv 0, \ \ \forall\, |u_i| \quad (i = 1, \dots, n) \tag{2.22}$$

and

$$B_{ijj}(u_j e_j) \equiv 0, \quad \forall\, |u_j| \ \ (i, j = 1, \dots, n). \tag{2.23}$$

Similarly to (2.13), we obtain

$$\frac{\partial w_i}{\partial t} + \frac{\partial(\lambda_i(u) w_i)}{\partial x} = \sum_{j,k=1}^{n} \Gamma_{ijk}(u) w_j w_k + \sum_{j=1}^{n}\Big(\sum_{h=1}^{n} B_{ijh}(u)\beta_h(u) + \nu_{ij}(u)\Big) w_j, \tag{2.24}$$

or equivalently,

$$\begin{aligned} &\mathrm{d}[w_i(\mathrm{d}x - \lambda_i(u)\,\mathrm{d}t)] \\ &\quad = \Big[\sum_{j,k=1}^{n} \Gamma_{ijk}(u) w_j w_k + \sum_{j=1}^{n}\Big(\sum_{h=1}^{n} B_{ijh}(u)\beta_h(u) + \nu_{ij}(u)\Big) w_j\Big] \mathrm{d}t \wedge \mathrm{d}x, \end{aligned} \tag{2.25}$$

where

$$\Gamma_{ijk}(u) = \tfrac{1}{2}(\lambda_j(u) - \lambda_k(u)) l_i(u)[\nabla r_k(u) r_j(u) - \nabla r_j(u) r_k(u)]. \tag{2.26}$$

Thus, we have

$$\Gamma_{ijj}(u) \equiv 0, \quad \forall\, i, j \in \{1, \dots, n\}. \tag{2.27}$$

As for (2.9) and (2.17), we have

Lemma 2.1 ([19]). *Suppose that $A(u) \in C^2$ in a neighborhood of $u = 0$ and system (1.1) is strictly hyperbolic. Suppose furthermore that $B(u) \in C^2$ satisfies the matching condition. Then, in normalized coordinates, we have*

$$\sum_{j=1}^{n}\Big(\sum_{h=1}^{n} \bar{\varrho}_{ijh}(u)\beta_h(u)\Big)u_j + r_{ii}(u)\beta_i(u) = \sum_{j,k=1}^{n} P_{ijk}(u)u_j u_k, \ \ \forall\,|u| \text{ small} \quad (i = 1, \dots, n), \tag{2.28}$$

where $P_{ijk}(u)$ are C^1 functions with respect to their arguments in a neighborhood of $u = 0$ and

$$P_{ijj}(u_j e_j) \equiv 0, \ \ \forall\,|u_j| \text{ small} \quad (i, j = 1, \dots, n). \tag{2.29}$$

In addition, we get

$$\sum_{j=1}^{n}\Big(\sum_{h=1}^{n} B_{ijh}(u)\beta_h(u) + \nu_{ij}(u)\Big)w_j = \sum_{j,k=1}^{n} Q_{ijk}(u)u_k w_j, \ \ \forall\,|u| \text{ small} \quad (i = 1, \dots, n), \tag{2.30}$$

where $Q_{ijk}(u)$ are C^1 functions with respect to their arguments in a neighborhood of $u = 0$ and

$$Q_{ijj}(u_j e_j) \equiv 0, \ \ \forall\,|u_j| \text{ small} \quad (i, j = 1, \dots, n). \tag{2.31}$$

By Lemma 2.1, (2.9) and (2.12) can be rewritten as

$$\frac{\mathrm{d}u_i}{\mathrm{d}_i t} = \sum_{j,k=1}^{n} \varrho_{ijk}(u)u_j w_k + \sum_{j,k=1}^{n} P_{ijk}(u)u_j u_k, \tag{2.32}$$

$$\frac{\partial u_i}{\partial t} + \frac{\partial(\lambda_i(u)u_i)}{\partial x} = \sum_{j,k=1}^{n} F_{ijk}(u)u_j w_k + \sum_{j,k=1}^{n} P_{ijk}(u)u_j u_k. \tag{2.33}$$

By Lemma 2.1, (2.17) and (2.24) can also be rewritten as

$$\frac{\mathrm{d}w_i}{\mathrm{d}_i t} = \sum_{j,k=1}^{n} \gamma_{ijk}(u)w_j w_k + \sum_{j,k=1}^{n} Q_{ijk}(u)u_k w_j, \tag{2.34}$$

$$\frac{\partial w_i}{\partial t} + \frac{\partial(\lambda_i(u)w_i)}{\partial x} = \sum_{j,\,k=1}^{n} \Gamma_{ijk}(u)w_j w_k + \sum_{j,k=1}^{n} Q_{ijk}(u)u_k w_j. \tag{2.35}$$

We next present a formula on the decomposition of waves for the difference of two solutions $u^{(1)}$, $u^{(2)}$ to system (1.1) (see [19], [20]). Let

$$w_i^{(1)} = l_i(u^{(1)})u_x^{(1)}, \quad w_i^{(2)} = l_i(u^{(2)})u_x^{(2)} \quad (i = 1, \dots, n). \tag{2.36}$$

By (1.5), we have

$$u_x^{(1)} = \sum_{i=1}^{n} w_i^{(1)} r_i(u^{(1)}), \quad u_x^{(2)} = \sum_{i=1}^{n} w_i^{(2)} r_i(u^{(2)}). \tag{2.37}$$

Let

$$u^{(0)} = u^{(1)} - u^{(2)}, \tag{2.38}$$

$$\xi_i^{(1)} = l_i(u^{(1)})u^{(0)}, \quad \xi_i^{(2)} = l_i(u^{(2)})u^{(0)}, \tag{2.39}$$

then we get

$$u^{(0)} = \sum_{j=1}^{n} \xi_j^{(1)} r_j(u^{(1)}) = \sum_{j=1}^{n} \xi_j^{(2)} r_j(u^{(2)}). \tag{2.40}$$

Thus,

$$\begin{aligned}
&\xi_{it}^{(1)} + (\lambda_i(u^{(1)})\xi_i^{(1)})_x \\
&= \sum_{j,k=1}^{n} \Theta_{ijk}(u^{(1)})\xi_j^{(1)} w_k^{(1)} - (\lambda_i(u^{(1)}) - \lambda_i(u^{(2)}) l_i(u^{(1)}) r_i(u^{(2)}) w_i^{(2)} \\
&\quad + \sum_{j \neq i} (\lambda_i(u^{(1)}) - \lambda_j(u^{(2)})) l_i(u^{(1)})(r_j(u^{(1)}) - r_j(u^{(2)})) w_j^{(2)} \\
&\quad - \sum_{j=1}^{n} l_i(u^{(1)}) \nabla r_j(u^{(1)}) B(u^{(1)}) \xi_j^{(1)} + l_i(u^{(1)})(B(u^{(1)}) - B(u^{(2)}))
\end{aligned} \tag{2.41}$$

and

$$\begin{aligned}
&\xi_{it}^{(2)} + (\lambda_i(u^{(2)})\xi_i^{(2)})_x \\
&= \sum_{j,k=1}^{n} \Theta_{ijk}(u^{(2)})\xi_j^{(2)} w_k^{(2)} + (\lambda_i(u^{(1)}) - \lambda_i(u^{(2)})) l_i(u^{(2)}) r_i(u^{(1)}) w_i^{(1)} \\
&\quad - \sum_{j \neq i} (\lambda_i(u^{(2)}) - \lambda_j(u^{(1)})) l_i(u^{(2)})(r_j(u^{(1)}) - r_j(u^{(2)})) w_j^{(1)} \\
&\quad - \sum_{j=1}^{n} l_i(u^{(2)}) \nabla r_j(u^{(2)}) B(u^{(2)}) \xi_j^{(2)} - l_i(u^{(2)})(B(u^{(1)}) - B(u^{(2)})),
\end{aligned} \tag{2.42}$$

where

$$\Theta_{ijk}(u) = (\lambda_k(u) - \lambda_i(u))l_i(u)\nabla r_j(u)r_k(u) + \nabla\lambda_i(u)r_k(u)\delta_{ij}. \tag{2.43}$$

We have, in normalized coordinates, that

$$\Theta_{ijj}(u_je_j) \equiv 0, \;\; \forall\, |u_j| \text{ small}, \;\; \forall\, j \neq i. \tag{2.44}$$

When system (1.1) is weakly linearly degenerate, in normalized coordinates we get

$$\Theta_{ijj}(u_je_j) \equiv 0, \;\; \forall\, |u_j| \text{ small} \quad (i, j = 1, \ldots, n). \tag{2.45}$$

3. Two basic L^1 estimates

In this section, we present two basic L^1 estimates (see [21]).

Lemma 3.1. *Let $\varphi = \varphi(t,x) \in C^1$ satisfy*

$$\varphi_t + (\lambda(t,x)\varphi)_x = F, \quad 0 \leqslant t \leqslant T, \;\; x \geqslant 0, \tag{3.1}$$

$$t = 0\colon\; \varphi = \varphi_0, \;\; x \geqslant 0, \tag{3.2}$$

where $\lambda \in C^1$. Then for any $t \in [0,T]$,

(i) *if $\lambda \geqslant 0$, we have*

$$\|\varphi(t,\cdot)\|_{L^1(\mathbb{R}^+)} \leqslant \|\varphi_0\|_{L^1(\mathbb{R}^+)} + \int_0^T \|F(t,\cdot)\|_{L^1(\mathbb{R}^+)}\,\mathrm{d}t + \int_0^T \lambda(t,0)|\varphi(t,0)|\,\mathrm{d}t, \tag{3.3}$$

(ii) *if $\lambda \leqslant 0$, we have*

$$\begin{aligned} \|\varphi(t,\cdot)\|_{L^1(\mathbb{R}^+)} &+ \int_0^T (-\lambda(t,0))|\varphi(t,0)|\,\mathrm{d}t \\ &\leqslant \|\varphi_0\|_{L^1(\mathbb{R}^+)} + \int_0^T \|F(t,\cdot)\|_{L^1(\mathbb{R}^+)}\,\mathrm{d}t, \end{aligned} \tag{3.4}$$

where $\|\varphi(t,\cdot)\|_{L^1(\mathbb{R}^+)} = \int_0^\infty |\varphi(t,x)|\,\mathrm{d}x$.

Lemma 3.2. *Let $\varphi = \varphi(t,x)$ and $\psi = \psi(t,x)$ be C^1 functions satisfying*

$$\varphi_t + (\lambda(t,x)\varphi)_x = F, \quad 0 \leqslant t \leqslant T, \ \ x \geqslant 0, \tag{3.5}$$

$$t = 0\colon \ \varphi = \varphi_0(x), \ \ x \geqslant 0 \tag{3.6}$$

and

$$\psi_t + (\mu(t,x)\psi)_x = G, \quad 0 \leqslant t \leqslant T, \ \ x \geqslant 0, \tag{3.7}$$

$$t = 0\colon \ \psi = \psi_0(x), \ \ x \geqslant 0, \tag{3.8}$$

where $\lambda, \mu \in C^1$ such that there exists a positive constant δ_0 independent of T verifying

$$\mu(t,x) - \lambda(t,x) \geqslant \delta_0, \quad 0 \leqslant t \leqslant T, \ \ x \geqslant 0. \tag{3.9}$$

Then, for any $t \in [0,T]$,

(i) *if $\lambda \geqslant 0$ and $\mu > 0$, we have*

$$\begin{aligned} \delta_0 \int_0^T \int_0^\infty & |\varphi(t,x)| \cdot |\psi(t,x)| \, \mathrm{d}x \, \mathrm{d}t \\ & \leqslant 4\Big(\|\varphi_0\|_{L^1(\mathbb{R}^+)} + \int_0^T \|F(t,\cdot)\|_{L^1(\mathbb{R}^+)} \, \mathrm{d}t + \int_0^T \lambda(t,0)|\varphi(t,0)| \, \mathrm{d}t \Big) \\ & \quad \times \Big(\|\psi_0\|_{L^1(\mathbb{R}^+)} + \int_0^T \|G(t,\cdot)\|_{L^1(\mathbb{R}^+)} \, \mathrm{d}t + \int_0^T \mu(t,0)|\psi(t,0)| \, \mathrm{d}t \Big), \end{aligned} \tag{3.10}$$

(ii) *if $\lambda \leqslant 0$ and $\mu \geqslant 0$, we have*

$$\begin{aligned} \delta_0 \int_0^T \int_0^\infty & |\varphi(t,x)| \cdot |\psi(t,x)| \, \mathrm{d}x \, \mathrm{d}t \\ & \leqslant 4\Big(\|\varphi_0\|_{L^1(\mathbb{R}^+)} + \int_0^T \|F(t,\cdot)\|_{L^1(\mathbb{R}^+)} \, \mathrm{d}t \Big) \\ & \quad \times \Big(\|\psi_0\|_{L^1(\mathbb{R}^+)} + \int_0^T \|G(t,\cdot)\|_{L^1(\mathbb{R}^+)} \, \mathrm{d}t + \int_0^T \mu(t,0)|\psi(t,0)| \, \mathrm{d}t \Big), \end{aligned} \tag{3.11}$$

(iii) *if $\lambda < 0$ and $\mu \leqslant 0$, we have*

$$\begin{aligned} \delta_0 \int_0^T \int_0^\infty & |\varphi(t,x)| \cdot |\psi(t,x)| \, \mathrm{d}x \, \mathrm{d}t \\ & \leqslant 4\Big(\|\varphi_0\|_{L^1(\mathbb{R}^+)} + \int_0^T \|F(t,\cdot)\|_{L^1(\mathbb{R}^+)} \, \mathrm{d}t \Big) \\ & \quad \times \Big(\|\psi_0\|_{L^1(\mathbb{R}^+)} + \int_0^T \|G(t,\cdot)\|_{L^1(\mathbb{R}^+)} \, \mathrm{d}t \Big). \end{aligned} \tag{3.12}$$

4. Proof of Theorem 1.1

By the existence and uniqueness of the local C^1 solution to mixed initial-boundary value problem (1.1) and (1.10)–(1.11), in order to prove Theorem 1.1, it suffices to establish uniform a priori estimates for the C^0 norms of u and $\partial u/\partial x$ on the existence domain of the C^1 solution $u = u(t,x)$ (see [13]).

By (1.2), there exist positive constants δ and δ_0 small enough such that

$$\lambda_{j+1}(u) - \lambda_j(\bar{u}) \geqslant \delta_0, \ \ \forall\, |u|, |\bar{u}| \leqslant \delta \quad (j = 1, \ldots, n-1). \tag{4.1}$$

For the time being it is supposed that on the existence domain of the C^1 solution $u = u(t,x)$ to the mixed initial-boundary value problem (1.1) and (1.10)–(1.11) we have

$$|u(t,x)| \leqslant \delta. \tag{4.2}$$

At the end of the proof of Lemma 4.2, we shall explain that this hypothesis is reasonable. Thus, in order to prove Theorem 1.1, we only need to establish uniform a priori estimates for the supreme norms of u and $w = (w_1, \ldots, w_n)^\top$ defined by (2.3) on any given time interval $[0,T]$.

By the assumptions $\int_0^\infty |\varphi'(x)|\,\mathrm{d}x < \infty$ and $\int_0^\infty |\varphi(x)|\,\mathrm{d}x < \infty$, it is easy to see that

$$\lim_{x\to\infty} \varphi(x) = 0. \tag{4.3}$$

Due to finite propagation speed of waves, we have

$$\lim_{x\to\infty} u(t,x) = 0, \quad \forall\, t \in [0,T]. \tag{4.4}$$

Thus, it follows that

$$u(t,x) = -\int_x^\infty u_y(t,y)\,\mathrm{d}y. \tag{4.5}$$

Differentiating the boundary condition (1.11) with respect to time t and taking into account (1.2) and (4.2), for $\delta > 0$ small enough we have (see [12])

$$\begin{aligned} x = 0\colon\ w_s = {}& \sum_{r=1}^{m} f_{sr}(\alpha(t),u) w_r + \sum_{i=1}^{k} \overline{f}_{si}(\alpha(t),u)\alpha_i'(t) \\ & + \sum_{\bar{s}=m+1}^{n} \tilde{f}_{s\bar{s}}(\alpha(t),u) h_{\bar{s}}'(t) \quad (s = m+1, \ldots, n), \end{aligned} \tag{4.6}$$

where f_{sr}, $\overline{f}_{si}$, and $\tilde{f}_{s\bar{s}}$ are continuous functions with respect to their arguments.

By Lemma 2.5 in [14], there exists a normalized transformation. Without loss of generality, we assume that $u = (u_1, \dots, u_n)^\top$ are already normalized variables. To consider the mixed initial-boundary value problem in normalized coordinates, we need the following lemma.

Lemma 4.1 ([12]). *The boundary condition* (1.11) *and the assumption* (1.22) *keep the same form under any given smooth invertible transformation* $u = u(\tilde{u})$ $(u(0) = 0)$.

Moreover, in normalized coordinates, it is easy to see that

$$v_i = l_i(u)u = l_i(0)u + o(|u|) = u_i + o(|u|) \quad (i = 1, \dots, n). \tag{4.7}$$

Substituting (4.7) into (1.11), the Implicit Function Theorem implies, for sufficiently small $|u|$, that

$$\begin{aligned} x = 0\colon\ u_s &= g_s(\alpha(t), h_{m+1}(t), \dots, h_n(t), u_1, \dots, u_m) \\ &\stackrel{\text{def}}{=} \overline{g}_s(\overline{\alpha}(t), u_1, \dots, u_m) + \bar{h}_s(t) \quad (s = m+1, \dots, n), \end{aligned} \tag{4.8}$$

where $g_s \in C^1$ $(s = m+1, \dots, n)$,

$$\overline{\alpha}(t) = (\alpha(t), h_{m+1}(t), \dots, h_n(t)), \tag{4.9}$$

$$\bar{h}_s(t) = g_s(\overline{\alpha}(t), 0, \dots, 0) \quad (s = m+1, \dots, n), \tag{4.10}$$

and then

$$\overline{g}_s(\overline{\alpha}(t), 0, \dots, 0) \equiv 0 \quad (s = m+1, \dots, n). \tag{4.11}$$

It furthermore follows from (1.13) that

$$g_s(\alpha(t), 0, \dots, 0) \equiv 0 \quad (s = m+1, \dots, n). \tag{4.12}$$

Let

$$W_b(T) = \max_{r=1,\dots,m} \int_0^T (-\lambda_r(u(t,0)))|w_r(t,0)|\,\mathrm{d}t, \tag{4.13}$$

$$U_b(T) = \max_{r=1,\ldots,m} \int_0^T (-\lambda_r(u(t,0)))|v_r(t,0)|\,\mathrm{d}t, \tag{4.14}$$

$$U_\infty(T) = \sup_{0\leqslant t\leqslant T} \sup_{x\geqslant 0} |u(t,x)|, \tag{4.15}$$

$$W_\infty(T) = \sup_{0\leqslant t\leqslant T} \sup_{x\geqslant 0} |w(t,x)|, \tag{4.16}$$

$$W_1(T) = \sup_{0\leqslant t\leqslant T} \int_0^\infty |w(t,x)|\,\mathrm{d}x, \tag{4.17}$$

$$\tilde{U}_1(T) = \max_{i=1,\ldots,n} \max_{j\neq i} \sup_{C_j} \int_{C_j} |u_i|\,\mathrm{d}t, \tag{4.18}$$

$$\widetilde{W}_1(T) = \max_{i=1,\ldots,n} \max_{j\neq i} \sup_{C_j} \int_{C_j} |w_i|\,\mathrm{d}t, \tag{4.19}$$

where C_j stands for any given jth characteristic on the domain $D(T)$.

Hence, we conclude from (4.5) that

$$|u(t,x)| \leqslant \int_0^\infty |u_x(t,x)|\,\mathrm{d}x \leqslant CW_1(T). \tag{4.20}$$

Here and hereafter, C will denote a generic positive constant independent of ε, M, and T; the meaning of C may change from line to line.

Lemma 4.2. *There exists a positive constant C independent of ε, M, and T such that*

$$W_b(T), W_1(T), \widetilde{W}_1(T) \leqslant C\varepsilon, \tag{4.21}$$

$$U_b(T), \tilde{U}_1(T) \leqslant C\frac{\varepsilon}{M+1}, \tag{4.22}$$

$$W_\infty(T) \leqslant CM. \tag{4.23}$$

P r o o f. We introduce

$$Q_U(T) = \sum_{j=1}^n \sum_{i\neq j} \int_0^T \int_0^\infty |u_i(t,x)| \cdot |u_j(t,x)|\,\mathrm{d}x\,\mathrm{d}t, \tag{4.24}$$

$$Q_W(T) = \sum_{j=1}^n \sum_{i\neq j} \int_0^T \int_0^\infty |w_i(t,x)| \cdot |w_j(t,x)|\,\mathrm{d}x\,\mathrm{d}t, \tag{4.25}$$

$$Q_{UW}(T) = \sum_{j=1}^n \sum_{i\neq j} \int_0^T \int_0^\infty |u_i(t,x)| \cdot |w_j(t,x)|\,\mathrm{d}x\,\mathrm{d}t, \tag{4.26}$$

and

$$J_1 \stackrel{\text{def}}{=} \{1,\ldots,m\}, \quad J_2 \stackrel{\text{def}}{=} \{m+1,\ldots,n\}. \tag{4.27}$$

To begin with, we estimate $Q_U(T)$.

(i) For $i, j \in J_1$ and $i \neq j$, by (2.33) and Lemma 3.2 we have

$$\begin{aligned}(4.28)\qquad &\int_0^T\int_0^\infty |u_i(t,x)|\cdot|u_j(t,x)|\,\mathrm{d}x\,\mathrm{d}t \\ &\leqslant C\Big(\|\varphi_0\|_{L^1(\mathbb{R}^+)} + \int_0^T \|F(t,\cdot)\|_{L^1(\mathbb{R}^+)}\,\mathrm{d}t\Big)^2,\end{aligned}$$

where $F = (F_1, \ldots, F_n)^\top$ with

$$(4.29)\qquad F_i = \sum_{j,k=1}^n F_{ijk}(u)u_j w_k + \sum_{j,k=1}^n P_{ijk}(u)u_j u_k.$$

By Hadamard's formula (see [19]), we get

$$(4.30)\qquad |F_{ijj}(u)| \leqslant C\sum_{m\neq j}|u_m|, \quad |P_{ijj}(u)| \leqslant C\sum_{m\neq j}|u_m|, \quad \forall\, i, j \in \{1, \ldots, n\};$$

as a consequence, it follows that

$$(4.31)\qquad \int_0^T \|F(t,\cdot)\|_{L^1(\mathbb{R}^+)}\,\mathrm{d}t \leqslant C(Q_U(T) + Q_{UW}(T)).$$

Thus, for $i, j \in J_1$ and $i \neq j$, we conclude that

$$(4.32)\qquad \int_0^T\int_0^\infty |u_i(t,x)|\cdot|u_j(t,x)|\,\mathrm{d}x\,\mathrm{d}t \leqslant C(\|\varphi_0\|_{L^1(\mathbb{R}^+)} + Q_U(T) + Q_{UW}(T))^2.$$

(ii) For $i \in J_1$, $j \in J_2$, it also follows from (2.33) and Lemma 3.2 that

$$\begin{aligned}(4.33)\qquad &\int_0^T\int_0^\infty |u_i(t,x)|\cdot|u_j(t,x)|\,\mathrm{d}x\,\mathrm{d}t \\ &\leqslant C\Big(\|\varphi_0\|_{L^1(\mathbb{R}^+)} + \int_0^T \|F_i(t,\cdot)\|_{L^1(\mathbb{R}^+)}\,\mathrm{d}t\Big) \\ &\quad\times\Big(\|\varphi_0\|_{L^1(\mathbb{R}^+)} + \int_0^T \|F_j(t,\cdot)\|_{L^1(\mathbb{R}^+)}\,\mathrm{d}t + \int_0^T \lambda_j(u(t,0))|u_j(t,0)|\,\mathrm{d}t\Big).\end{aligned}$$

By (4.8)–(4.12), taking into account (4.2) and Lemma 3.1, we can obtain

$$
\begin{aligned}
(4.34)\quad & \int_0^T \lambda_j(u(t,0))|u_j(t,0)|\,\mathrm{d}t \\
&= \int_0^T \lambda_j(u(t,0))|\overline{g}_j(\overline{\alpha}(t),u_1,\ldots,u_m)+\bar{h}_j(t)|\,\mathrm{d}t \\
&\leqslant \sum_{r=1}^{m}\int_0^T \lambda_j(u(t,0))\Big|\frac{\partial \overline{g}_j}{\partial u_r}\Big|\cdot|u_r(t,0)|\,\mathrm{d}t + \int_0^T \lambda_j(u(t,0))|\bar{h}_j(t)|\,\mathrm{d}t \\
&\leqslant C(\|h\|_{L^1(\mathbb{R}^+)}+U_b(T) \\
&\leqslant \Big(\|h\|_{L^1(\mathbb{R}^+)}+\|\varphi_0\|_{L^1(\mathbb{R}^+)}+\int_0^T \|F(t,\cdot)\|_{L^1(\mathbb{R}^+)}\,\mathrm{d}t.
\end{aligned}
$$

Due to (4.31), we conclude from (4.33)–(4.34) that

$$
\begin{aligned}
(4.35)\qquad & \int_0^T\int_0^\infty |u_i(t,x)|\cdot|u_j(t,x)|\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad \leqslant C(\|h\|_{L^1(\mathbb{R}^+)}+\|\varphi_0\|_{L^1(\mathbb{R}^+)}+Q_U(T)+Q_{UW}(T))^2.
\end{aligned}
$$

(iii) For $i\in J_2$, $j\in J_1$ or $i,j\in J_2$ and $i\neq j$, a similar procedure yields

$$
\begin{aligned}
(4.36)\qquad & \int_0^T\int_0^\infty |u_i(t,x)|\cdot|u_j(t,x)|\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad \leqslant C(\|h\|_{L^1(\mathbb{R}^+)}+\|\varphi_0\|_{L^1(\mathbb{R}^+)}+Q_U(T)+Q_{UW}(T))^2.
\end{aligned}
$$

Combining the three cases above gives

$$
(4.37)\qquad Q_U(T)\leqslant C(\|h\|_{L^1(\mathbb{R}^+)}+\|\varphi_0\|_{L^1(\mathbb{R}^+)}+Q_U(T)+Q_{UW}(T))^2.
$$

Similarly to $Q_U(T)$, due to (2.35) and (4.6), it follows that

$$
(4.38)\quad Q_W(T)\leqslant C\Big(\|h'\|_{L^1(\mathbb{R}^+)}+\|\alpha'\|_{L^1(\mathbb{R}^+)}+W_1(0)+\int_0^T\|G(t,\cdot)\|_{L^1(\mathbb{R}^+)}\,\mathrm{d}t\Big)^2
$$

where $G=(G_1,\ldots,G_n)^\top$ with

$$
(4.39)\qquad G_i=\sum_{j,k=1}^{n}\Gamma_{ijk}(u)w_jw_k+\sum_{j,k=1}^{n}Q_{ijk}(u)u_kw_j.
$$

By (2.31) and Hadamard's formula, we have

$$
(4.40)\qquad |Q_{ijj}(u)\leqslant C\sum_{m\neq j}|u_m|,\quad \forall\, i,j\in\{1,\ldots,n\},
$$

and then

$$\int_0^T \|G(t,\cdot)\|_{L^1(\mathbb{R}^+)}\,\mathrm{d}t \leqslant C(Q_W(T)+(1+W_\infty(T))Q_{UW}(T)). \tag{4.41}$$

Thus, it follows that

$$\begin{aligned} Q_W(T) \leqslant C\big(&\|h'\|_{L^1(\mathbb{R}^+)}+\|\alpha'\|_{L^1(\mathbb{R}^+)}+W_1(0)+Q_W(T)\\ &+(1+W_\infty(T))Q_{UW}(T)\big)^2. \end{aligned} \tag{4.42}$$

By virtue of (2.33) and (2.35), we similarly have

$$\begin{aligned} Q_{UW}(T) \leqslant C\big(&\|h\|_{L^1(\mathbb{R}^+)}+\|\varphi_0\|_{L^1(\mathbb{R}^+)}+Q_U(T)+Q_{UW}(T))\\ &\times(\|h'\|_{L^1(\mathbb{R}^+)}+\|\alpha'\|_{L^1(\mathbb{R}^+)}+W_1(0)+Q_W(T)\\ &+(1+W_\infty(T))Q_{UW}(T)\big). \end{aligned} \tag{4.43}$$

We now estimate $W_1(T)$.

(i) For $i\in J_1$, by (2.35) and Lemma 3.1 we have

$$\begin{aligned} \int_0^\infty |w_i(t,x)|\,\mathrm{d}x &\leqslant W_1(0)+\int_0^T \|G(t,\cdot)\|_{L^1(\mathbb{R}^+)}\,\mathrm{d}t\\ &\leqslant W_1(0)+C\big(Q_W(t)+(1+W_\infty(T))Q_{UW}(T)\big). \end{aligned} \tag{4.44}$$

(ii) For $i\in J_2$, by (2.35) and 3.1 we also have

$$\begin{aligned} &\int_0^\infty |w_i(t,x)|\,\mathrm{d}x\\ &\qquad\leqslant W_1(0)+\int_0^T \|G(t,\cdot)\|_{L^1(\mathbb{R}^+)}\,\mathrm{d}t+\int_0^T \lambda_i(u(t,0))|w_i(t,0)|\,\mathrm{d}t. \end{aligned} \tag{4.45}$$

Using (4.6), we obtain

$$\begin{aligned} &\int_0^T \lambda_i(u(t,0))|w_i(t,0)|\,\mathrm{d}t\\ &\quad=\int_0^T \lambda_i(u(t,0))\Big|\sum_{r=1}^m f_{ir}(\alpha(t),u(t,0))w_r(t,0)\\ &\qquad+\sum_{l=1}^k \overline{f}_{il}(\alpha(t),u(t,0))\alpha_l'(t)+\sum_{\bar{s}=m+1}^n \tilde{f}_{i\bar{s}}(\alpha(t),u(t,0))h_{\bar{s}}'(t)\Big|\,\mathrm{d}t \end{aligned} \tag{4.46}$$

$$
\begin{aligned}
&\leqslant \sum_{r=1}^{m}\int_0^T \frac{\lambda_i(u(t,0))}{-\lambda_r(u(t,0))}|f_{ir}(\alpha(t),u(t,0))|(-\lambda_r(u(t,0))|w_r(t,0)|)\,\mathrm{d}t\\
&\quad+\int_0^T \lambda_i(u(t,0))\Big(\sum_{l=1}^{k}|\overline{f}_{il}(\alpha(t),u(t,0))|\cdot|\alpha_l'(t)|\\
&\qquad\qquad\qquad\qquad+\sum_{\bar{s}=m+1}^{n}|\tilde{f}_{i\bar{s}}(\alpha(t),u(t,0))|\cdot|h_{\bar{s}}'(t)|\Big)\,\mathrm{d}t\\
&\leqslant C(\|h'\|_{L^1(\mathbb{R}^+)}+\|\alpha'\|_{L^1(\mathbb{R}^+)}\\
&\quad+W_1(0)+Q_W(T)+(1+W_\infty(T))Q_{UW}(T));
\end{aligned}
$$

consequently, for $i\in J_2$ it follows that

$$
\int_0^\infty |w_i(t,x)|\,\mathrm{d}x \leqslant C\big(\|h'\|_{L^1(\mathbb{R}^+)}+\|\alpha'\|_{L^1(\mathbb{R}^+)}+W_1(0)+Q_W(T) + (1+W_\infty(T))Q_{UW}(T)\big). \tag{4.47}
$$

Thus, combining (4.44) and (4.47) yields

$$
W_1(T)\leqslant \big(\|h'\|_{L^1(\mathbb{R}^+)}+\|\alpha'\|_{L^1(\mathbb{R}^+)}+W_1(0)+Q_W(T) + (1+W_\infty(T))Q_{UW}(T)\big). \tag{4.48}
$$

We next estimate $\widetilde{W}_1(T)$.

For this purpose, we need to estimate

$$
\int_{C_j}|w_i|\,\mathrm{d}t\quad (i\neq j).
$$

For any given point $(t,x)\in D(T)$, denoted by A, we have:

(i) For $i\in J_1$, there are only two possibilities:

(a) The jth characteristic passing through the point A intersects the x-axis at a point B $(0,\alpha_j)$ and the ith characteristic passing through the point A intersects the x-axis at a point P' $(0,\alpha_i)$. One can rewrite (2.35) as

$$
\mathrm{d}(|w_i(t,x)|(\mathrm{d}x-\lambda_i(u)\,\mathrm{d}t))=\operatorname{sgn}(w_i)G_i\,\mathrm{d}t\wedge\mathrm{d}x. \tag{4.49}
$$

Integrating(4.49) on the domain ABP, utilizing the Stokes formula and using (4.41), we have

$$
\begin{aligned}
&\Big|\int_{C_j}|w_i(t,x)|(\lambda_j(u)-\lambda_i(u))\,\mathrm{d}t\Big|\\
&\qquad\leqslant \Big|\int_{\alpha_i}^{\alpha_j}|w_i(0,x)|\,\mathrm{d}x\Big|+\iint_{ABP}|G_i|\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad\leqslant W_1(0)+C(Q_W(T)+(1+W_\infty(T))Q_{UW}(T)).
\end{aligned} \tag{4.50}
$$

By the definition of $\widetilde{W_1}$, we have $j \neq i$ and then

$$|\lambda_j(u) - \lambda_i(u)| \geqslant \delta_0; \tag{4.51}$$

consequently, it follows that

$$\int_{C_j} |w_i(t,x)| \, \mathrm{d}t \leqslant W_1(0) + C(Q_W(T) + (1 + W_\infty(T))Q_{UW}(T)). \tag{4.52}$$

(b) The jth characteristic passing through the point A intersects the t-axis at a point $B\,(\beta_j, 0)$ and the ith characteristic passing through the point A intersects the x-axis at a point $P\,(0, \alpha_i)$. Denote the origin $(0,0)$ by O. We integrate (4.49) on the domain $ABOP$, utilize the Stokes formula and apply Lemma 3.1 to obtain

$$\begin{aligned} &\left| \int_{C_j} |w_i(t,x)|(\lambda_j(u) - \lambda_i(u)) \, \mathrm{d}t \right| \\ &\leqslant \int_0^{\alpha_i} |w_i(0,x)| \, \mathrm{d}x + \iint_{ABOP} |G_i| \, \mathrm{d}x \, \mathrm{d}t + \int_0^{\beta_j} (-\lambda_i(u(t,0))|w_i(t,0)|) \, \mathrm{d}t \\ &\leqslant C\big(W_1(0) + Q_W(T) + (1 + W_\infty(T))Q_{UW}(T)\big). \end{aligned} \tag{4.53}$$

Hence, by (4.51)–(4.53), for $i \in J_1$ it follows that

$$\int_{C_j} |w_i(t,x)| \, \mathrm{d}t \leqslant C\big(W_1(0) + Q_W(T) + (1 + W_\infty(T))Q_{UW}(T)\big). \tag{4.54}$$

(ii) For $i \in J_2$, there are four possibilities as follows:

(a$'$) The jth characteristic passing through the point A intersects the x-axis at a point $B\ (0, \alpha_j)$ and the ith characteristic passing through the point A intersects the x-axis at a point $P\,(0, \alpha_i)$. Similarly to the case (a) in (i), we integrate (4.49) on the domain ABP and use the Stokes formula to obtain

$$\int_{C_j} |w_i(t,x)| \, \mathrm{d}t \leqslant C\big(W_1(0) + Q_W(T) + (1 + W_\infty(T))Q_{UW}(T)\big). \tag{4.55}$$

(b$'$) The jth characteristic passing through the point A intersects the x-axis at a point $B\,(0, \alpha_j)$ and the ith characteristic passing through the point A intersects the t-axis at a point $P\ (\beta_i, 0)$. Integrating (4.49) on the domain $ABOP$, using the Stokes formula and taking into account boundary condition (4.6), (1.22), and Lemma 3.1, we have

$$
\begin{aligned}
(4.56)\quad & \left|\int_{C_j} |w_i(t,x)|(\lambda_j(u)-\lambda_i(u))\,\mathrm{d}t\right| \\
& \leqslant \int_0^{\alpha_j} |w_i(0,x)|\,\mathrm{d}x + \iint_{ABOP} |G_i|\,\mathrm{d}x\,\mathrm{d}t + \int_0^{\beta_i} (\lambda_i(u(t,0))|w_i(t,0)|)\,\mathrm{d}t \\
& \leqslant C(W_1(0)+Q_W(T)+Q_{UW}(T)) \\
& \quad + \sum_{r=1}^{m} \int_0^T \lambda_i(u(t,0))|f_{ir}(\alpha(t),u(t,0))|\cdot|w_r(t,0)|\,\mathrm{d}t \\
& \quad + \int_0^T \lambda_i(u(t,0))\Big(\sum_{l=1}^{k} |\overline{f}_{il}(\alpha(t),u(t,0))|\cdot|\alpha_l'(t)| \\
& \qquad\qquad + \sum_{\bar{s}=m+1}^{n} |\tilde{f}_{i\bar{s}}(\alpha(t),u(t,0))|\cdot|h_{\bar{s}}'(t)|\Big)\,\mathrm{d}t \\
& \leqslant C\big(\|h'\|_{L^1(\mathbb{R}^+)} + \|\alpha'\|_{L^1(\mathbb{R}^+)} + W_1(0) + Q_W(T) + (1+W_\infty(T))Q_{UW}(T)\big).
\end{aligned}
$$

Thus, by (4.51), it follows that

$$
\begin{aligned}
(4.57)\qquad \int_{C_j} |w_i(t,x)|\,\mathrm{d}t \leqslant C\big(&\|h'\|_{L^1(\mathbb{R}^+)} + \|\alpha'\|_{L^1(\mathbb{R}^+)} + W_1(0) \\
&+ Q_W(T) + (1+W_\infty(T))Q_{UW}(T)\big).
\end{aligned}
$$

(c′) The jth characteristic passing through the point A intersects the t-axis at a point $B\,(\beta_j,0)$ and the ith characteristic passing through the point A intersects the x-axis at a point $P\,(0,\alpha_i)$. Similarly to case (b′), we integrate (4.49) on the domain $ABOP$, utilize the Stokes formula and due to the boundary condition (4.6), (1.22), and (4.51) we obtain

$$
\begin{aligned}
(4.58)\quad & \int_{C_j} |w_i(t,x)|\,\mathrm{d}t \\
& \leqslant C\big(\|h'\|_{L^1(\mathbb{R}^+)} + \|\alpha'\|_{L^1(\mathbb{R}^+)} + W_1(0) + Q_W(T) + (1+W_\infty(T))Q_{UW}(T)\big),
\end{aligned}
$$

where we have used Lemma 3.1.

(d′) The jth characteristic passing through the point A intersects the t-axis at a point $B\,(\beta_j,0)$ and the ith characteristic passing through the point A intersects the t-axis at a point $P\,(\beta_i,0)$. Similarly, integrating (4.49) on the domain ABP, utilizing the Stokes formula and the boundary conditions, we have

$$
\begin{aligned}
(4.59)\quad & \left|\int_{C_j} |w_i(t,x)|(\lambda_j(u)-\lambda_i(u))\,\mathrm{d}t\right| \\
& \leqslant \iint_{ABP} |G_i|\,\mathrm{d}x\,\mathrm{d}t + \left|\int_{\beta_i}^{\beta_j} (\lambda_i(u(t,0))|w_i(t,0)|)\,\mathrm{d}t\right| \\
& \leqslant C\big(\|h'\|_{L^1(\mathbb{R}^+)} + \|\alpha'\|_{L^1(\mathbb{R}^+)} + W_1(0) + Q_W(T) + (1+W_\infty(T))Q_{UW}(T)\big).
\end{aligned}
$$

Hence, for any $i,j\in\{1,\dots,n\}$ and $i\neq j$, it follows that

$$
\begin{aligned}
(4.60)\qquad & \int_{C_j} |w_i|\,\mathrm{d}t \\
& \qquad \leqslant C\big(\|h'\|_{L^1(\mathbb{R}^+)} + \|\alpha'\|_{L^1(\mathbb{R}^+)} + W_1(0) + Q_W(T) \\
& \qquad\quad + (1+W_\infty(T))Q_{UW}(T)\big),
\end{aligned}
$$

so that

$$
\begin{aligned}
(4.61)\qquad \widetilde{W}_1(T) \leqslant C\big(& \|h'\|_{L^1(\mathbb{R}^+)} + \|\alpha'\|_{L^1(\mathbb{R}^+)} + W_1(0) + Q_W(T) \\
& + (1+W_\infty(T))Q_{UW}(T)\big).
\end{aligned}
$$

Similarly,

$$
(4.62)\qquad \tilde{U}_1(T) \leqslant C\big(\|h\|_{L^1(\mathbb{R}^+)} + \|\varphi_0\|_{L^1(\mathbb{R}^+)} + Q_U(T) + Q_{UW}(T)\big).
$$

We finally estimate $W_\infty(T)$.

(i) For $i\in J_1$ and any given point $(t,x)\in D(T)$, the ith characteristic passing through the point (t,x) must intersects the x-axis at a point $(0,\alpha_i)$. By (2.34), we have

$$
(4.63)\qquad |w_i(t,x)| \leqslant W_\infty(0) + \left|\int_{C_i}\Big(\sum_{j,\,k=1}^{n}\gamma_{ijk}(u)w_jw_k + \sum_{j,k=1}^{n} Q_{ijk}(u)u_kw_j\Big)\right|.
$$

By (2.21) and (2.22), it follows that

$$
\begin{aligned}
(4.64)\quad & \left|\int_{C_i}\Big(\sum_{j,\,k=1}^{n}\gamma_{ijk}(u)w_jw_k + \sum_{j,k=1}^{n} Q_{ijk}(u)u_kw_j\Big)\right| \\
& \leqslant \int_{C_i} |\gamma_{iii}(u)w_i^2| + CW_\infty(T)\widetilde{W}_1(T) + \int_{C_i} |Q_{iii}(u)u_iw_i| + C\delta\widetilde{W}_1(T) \\
& \leqslant CW_\infty^2(T)\int_{C_i} |\gamma_{iii}(u)| + CW_\infty(T)\widetilde{W}_1(T) \\
& \qquad + \delta W_\infty(T)\int_{C_i} |Q_{iii}(u)| + C\delta\widetilde{W}_1(T).
\end{aligned}
$$

By (2.22), Hadamard's formula yields

$$|\gamma_{iii}(u)| = |\gamma_{iii}(u) - \gamma_{iii}(u_i e_i)| \leqslant C \sum_{j \neq i} |u_j|. \tag{4.65}$$

Then

$$\int_{C_i} |\gamma_{iii}(u)| \leqslant C\tilde{U}_1(T). \tag{4.66}$$

Due to (2.31), we also have

$$\int_{C_i} |Q_{iii}(u)| \leqslant C\tilde{U}_1(T). \tag{4.67}$$

Hence, for $i \in J_1$, it follows that

$$\begin{aligned} \|w_i(t,\cdot)\|_{C^0} \leqslant{} & W_\infty(0) + CW_\infty^2(T)\tilde{U}_1(T) + CW_\infty(T)\widetilde{W}_1(T) \\ & + C\delta W_\infty(T)\tilde{U}_1(T) + C\delta\widetilde{W}_1(T). \end{aligned} \tag{4.68}$$

(ii) For $i \in J_2$ and any given point $(t,x) \in D(T)$, there are only two possibilities:

(a$''$) The ith characteristic passing through the point (t,x) intersects the x-axis at a point $(0,\alpha_i)$, and similarly to (i) above we have

$$\begin{aligned} \|w_i(t,\cdot)\|_{C^0} \leqslant{} & W_\infty(0) + CW_\infty^2(T)\tilde{U}_1(T) + CW_\infty(T)\widetilde{W}_1(T) \\ & + C\delta W_\infty(T)\tilde{U}_1(T) + C\delta\widetilde{W}_1(T). \end{aligned} \tag{4.69}$$

(b$''$) The ith characteristic passing through the point (t,x) intersects the t-axis at a point $(\beta_i, 0)$. By (4.6), (4.68), and (1.20), it follows that

$$\begin{aligned} |w_i(t,x)| \leqslant{} & |w_i(\beta_i,0)| + \left| \int_{C_i} \Big(\sum_{j,k=1}^n \gamma_{ijk}(u) w_j w_k + \sum_{j,k=1}^n Q_{ijk}(u) u_k w_j \Big) \right| \\ \leqslant{} & \Bigg| \sum_{r=1}^m f_{ir}(\alpha(\beta_i), u(\beta_i,0)) w_r(\beta_i,0) + \sum_{l=1}^k \overline{f}_{il}(\alpha(\beta_i), u(\beta_i,0)) \alpha_l'(\beta_i) \\ & + \sum_{\bar{s}=m+1}^n \tilde{f}_{i\bar{s}}(\alpha(\beta_i), u(\beta_i,0)) h_{\bar{i}}'(\beta_i) \Bigg| + CW_\infty^2(T)\tilde{U}_1(T) \\ & + CW_\infty(T)\widetilde{W}_1(T) + C\delta W_\infty(T)\tilde{U}_1(T) + C\delta\widetilde{W}_1(T) \\ \leqslant{} & C\big(M + W_\infty(0) + W_\infty^2(T)\tilde{U}_1(T) + W_\infty(T)\widetilde{W}_1(T) \\ & + \delta W_\infty(T)\tilde{U}_1(T) + \delta\widetilde{W}_1(T)\big). \end{aligned} \tag{4.70}$$

By virtue of (4.68) and (4.70), it is easy to see that

$$W_\infty(T) \leqslant C\big(M + W_\infty(0) + W_\infty^2(T)\tilde{U}_1(T) + W_\infty(T)\widetilde{W}_1(T) + \delta W_\infty(T)\tilde{U}_1(T) + \delta\widetilde{W}_1(T)\big). \tag{4.71}$$

Combining (4.37), (4.42)–(4.43), (4.48), (4.61), (4.62), and (4.71), we can prove that

$$Q_U(T) \leqslant C\frac{\varepsilon^2}{(M+1)^2}, \tag{4.72}$$

$$Q_W(T) \leqslant C\varepsilon^2, \tag{4.73}$$

$$Q_{UW(}T) \leqslant C\frac{\varepsilon^2}{M+1} \tag{4.74}$$

as well as the conclusion of Lemma 4.2. This completes the proof of Lemma 4.2. □

By (4.20), it follows that

$$U_\infty(T) \leqslant CW_1(T) \leqslant C\varepsilon. \tag{4.75}$$

Taking ε sufficiently small, we get

$$U_\infty(T) \leqslant \frac{1}{2}\,\delta, \tag{4.76}$$

so the hypothesis (4.2) is reasonable.

Theorem 1.1 is a direct consequence of Lemma 4.2.

5. Proof of Theorem 1.2

P r o o f. To estimate

$$\int_0^\infty |u^{(0)}(t,x)|\,\mathrm{d}x,$$

it suffices to estimate

$$\int_0^\infty |\xi_i^{(1)}(t,x)|\,\mathrm{d}x, \quad \forall\, i \in \{1,\ldots,n\} \tag{5.1}$$

on a fixed time interval $[0,T]$.

By the boundary condition (1.11), we have

$$x = 0\colon\ v_s^{(1)} = f_s(\alpha^{(1)}(t), v_1^{(1)}, \ldots, v_m^{(1)}) + h_s^{(1)}(t) \quad (s = m+1,\ldots,n) \tag{5.2}$$

and

$$x=0\colon\ v_s^{(2)}=f_s(\alpha^{(2)}(t),v_1^{(2)},\dots,v_m^{(2)})+h_s^{(2)}(t)\quad (s=m+1,\dots,n),\tag{5.3}$$

where $v_i^{(l)}=l_i(u^{(l)})u^{(l)}$ $(i=1,\dots,n;l=1,2)$. By (2.38) and (2.40), it follows that

$$\begin{aligned}v_i^{(1)}-v_i^{(2)}&=l_i(u^{(1)})u^{(1)}-l_i(u^{(2)})u^{(2)}\\&=\xi_i^{(1)}+\sum_{k=1}^n\xi_k^{(1)}r_k^\top(u^{(1)})\frac{\partial l_i}{\partial u}\,u^{(2)}\quad (i=1,\dots,n).\end{aligned}\tag{5.4}$$

Then, subtracting (5.2) from (5.3) yields, at $x=0$, that

$$\begin{aligned}&\xi_s^{(1)}+\sum_{k=1}^n\xi_k^{(1)}r_k^\top(u^{(1)})\frac{\partial l_s}{\partial u}\,u^{(2)}\\&\quad=\Big(\frac{\partial f_s}{\partial\alpha}\Big)(\alpha^{(0)}(t))^\top+\sum_{r=1}^m\frac{\partial f_s}{\partial v_r}\Big[\xi_r^{(1)}+\sum_{k=1}^n\xi_k^{(1)}r_k^\top(u^{(1)})\frac{\partial l_r}{\partial u}\,u^{(2)}\Big]+h_s^{(0)}(t)\\&\qquad\qquad(s=m+1,\dots,n),\end{aligned}\tag{5.5}$$

where $\alpha^{(0)}(t)=\alpha^{(1)}(t)-\alpha^{(2)}(t)$, $h_s^{(0)}(t)=h_s^{(1)}(t)-h_s^{(2)}(t)$. When ε is small enough, we conclude from Theorem 1.1 that

$$\begin{aligned}x=0\colon\ \xi_s^{(1)}=\sum_{r=1}^m g_{sr}^{(1)}\xi_r^{(1)}+\sum_{i=1}^k\overline{g}_{si}^{(1)}\alpha_i^{(0)}(t)+\sum_{\bar s}^n\tilde g_{s\bar s}^{(1)}h_{\bar s}^{(0)}(t)\\(s=m+1,\dots,n),\end{aligned}\tag{5.6}$$

where $g_{sr}^{(1)}$, $\overline{g}_{si}^{(1)}$, and $\tilde g_{s\bar s}^{(1)}$ are continuous functions with respect to $(\alpha^{(1)}(t)$, $\alpha^{(2)}(t)$, $u^{(1)}$, $u^{(2)})$.

Similarly,

$$x=0\colon\ \xi_s^{(2)}=\sum_{r=1}^m g_{sr}^{(2)}\xi_r^{(2)}+\sum_{i=1}^k\overline{g}_{si}^{(2)}\alpha_i^{(0)}(t)+\sum_{\bar s}^n\tilde g_{s\bar s}^{(2)}h_{\bar s}^{(0)}(t)\quad(s=m+1,\dots,n),\tag{5.7}$$

where $g_{sr}^{(2)}$, $\overline{g}_{si}^{(2)}$, and $\tilde g_{s\bar s}^{(2)}$ are continuous functions with respect to $(\alpha^{(1)}(t)$, $\alpha^{(2)}(t)$, $u^{(1)}$, $u^{(2)})$.

(i) For $i\in J_1$, due (2.41), it follows from Lemma 3.1 that

$$\begin{aligned}&\int_0^\infty|\xi_i^{(1)}(t,x)|\,\mathrm{d}x+\int_0^T(-\lambda_i(u^{(1)}(t,0))|\xi_i^{(1)}(t,0)|)\,\mathrm{d}t\\&\quad\leqslant\int_0^\infty|\xi_i^{(1)}(0,x)|\,\mathrm{d}x+\int_0^T\|F_i^{(1)}(t,\cdot)\|_{L^1(\mathbb{R}^+)}\,\mathrm{d}t,\end{aligned}\tag{5.8}$$

where

$$
\begin{aligned}
(5.9)\quad F_i^{(1)} = &\sum_{j,k=1}^{n} \Theta_{ijk}(u^{(1)})\xi_j^{(1)} w_k^{(1)} - (\lambda_i(u^{(1)}) - \lambda_i(u^{(2)}))l_i(u^{(1)})r_i(u^{(2)})w_i^{(2)} \\
&+ \sum_{j\neq i}(\lambda_i(u^{(1)}) - \lambda_j(u^{(2)}))l_i(u^{(1)})(r_j(u^{(1)}) - r_j(u^{(2)}))w_j^{(2)} \\
&- \sum_{j=1}^{n} l_i(u^{(1)})\nabla r_j(u^{(1)})\xi_j^{(1)} + l_i(u^{(1)})(B(u^{(1)}) - B(u^{(2)})).
\end{aligned}
$$

By Hadamard's formula, some estimates are obtained in [19], [20] as follows:

$$
\sum_{j\neq i} |u_j^{(0)}| \leqslant C \sum_{j\neq i} |\xi_j^{(2)} + C \sum_{j\neq i} |u_j^{(2)}| \cdot |\xi_i^{(2)}, \tag{5.10}
$$

$$
|\lambda_i(u^{(1)}) - \lambda_i(u^{(2)})| \leqslant C \sum_{j\neq i} |\xi_j^{(2)}| + C \sum_{j\neq i} |u_j^{(2)}| \cdot |\xi_i^{(2)}|, \tag{5.11}
$$

$$
|r_j(u^{(1)}) - r_j(u^{(2)})| \leqslant C \sum_{k\neq j} |\xi_k^{(2)}| + C \sum_{k\neq j} |u_k^{(2)}| \cdot |\xi_j^{(2)}|, \tag{5.12}
$$

$$
|B(u^{(1)})| \leqslant C \sum_{k\neq j} |u_k^{(1)}|, \tag{5.13}
$$

$$
|\Theta_{ijj}(u^{(1)})| \leqslant C \sum_{k\neq j} |u_k^{(1)}|, \quad \forall\, i,j \in \{1,\dots,n\}, \tag{5.14}
$$

and

$$
|B(u^{(1)}) - B(u^{(2)})| \leqslant C \sum_{j=1}^{n} \sum_{k\neq j} |u_k^{(2)}| \cdot |\xi_j^{(2)}| + C \sum_{j=1}^{n} \sum_{k\neq j} |u_k^{(1)}| \cdot |\xi_j^{(1)}|. \tag{5.15}
$$

Combining the above facts, we obtain

$$
\begin{aligned}
(5.16)\quad &\int_0^T \|F_i^{(1)}(t,\cdot)\|_{L^1(\mathbb{R}^+)}\,\mathrm{d}t \\
&\leqslant C\big(Q_{\xi^{(1)}}(T) + (M+1)D_{\xi^{(1)}}(T) + Q_{\xi^{(2)}}(T) + (M+1)D_{\xi^{(2)}}(T)\big),
\end{aligned}
$$

where

$$
Q_{\xi^{(1)}}(T) = \sum_{j=1}^{n} \sum_{i\neq j} \int_0^T \int_0^\infty |\xi_i^{(1)}(t,x)| \cdot |w_j^{(1)}(t,x)|\,\mathrm{d}x\,\mathrm{d}t, \tag{5.17}
$$

$$
Q_{\xi^{(2)}}(T) = \sum_{j=1}^{n} \sum_{i\neq j} \int_0^T \int_0^\infty |\xi_i^{(2)}(t,x)| \cdot |w_j^{(2)}(t,x)|\,\mathrm{d}x\,\mathrm{d}t, \tag{5.18}
$$

$$D_{\xi^{(1)}}(T) = \sum_{j=1}^{n} \sum_{h \neq j} \int_0^T \int_0^\infty |u_h^{(1)}(t,x)| \cdot |\xi_j^{(1)}(t,x)| \,\mathrm{d}x \,\mathrm{d}t, \tag{5.19}$$

$$D_{\xi^{(2)}}(T) = \sum_{j=1}^{n} \sum_{h \neq j} \int_0^T \int_0^\infty |u_h^{(2)}(t,x)| \cdot |\xi_j^{(2)}(t,x)| \,\mathrm{d}x \,\mathrm{d}t. \tag{5.20}$$

We first estimate $Q_{\xi^{(1)}}(T)$.

(a) For $i, j \in J_1$ and $i \neq j$, by (2.35) and (2.41), it follows from Lemma 3.2 and Lemma 4.2 that

$$\begin{aligned}
&\int_0^T \int_0^\infty |\xi_i^{(1)}(t,x)| \cdot |w_j^{(1)}(t,x)| \,\mathrm{d}x \,\mathrm{d}t \\
&\quad \leqslant C\bigg(\|\varphi^{(0)}\|_{L^1(\mathbb{R}^+)} + \sum_{i=1}^{n} \int_0^T \|F_i^{(1)}(t,\cdot)\|_{L^1(\mathbb{R}^+)} \,\mathrm{d}t \bigg) \\
&\qquad \times \bigg(W_1^{(1)}(0) + \sum_{i,j,k=1}^{n} \int_0^T \int_0^\infty |\Gamma_{ijk}(u^{(1)}) w_j^{(1)} w_k^{(1)} \\
&\qquad\qquad + Q_{ijk}(u^{(1)}) u_k^{(1)} w_j^{(1)}| \,\mathrm{d}x \,\mathrm{d}t \bigg) \\
&\quad \leqslant C\varepsilon \bigg(\|\varphi^{(0)}\|_{L^1(\mathbb{R}^+)} + \sum_{i=1}^{n} \int_0^T \|F_i^{(1)}(t,\cdot)\|_{L^1(\mathbb{R}^+)} \,\mathrm{d}t \bigg),
\end{aligned} \tag{5.21}$$

where $\varphi^{(0)} = \varphi^{(1)} - \varphi^{(2)}$.

(b) For $i \in J_1$ and $j \in J_2$, by virtue of (4.46) and Lemma 4.2, we conclude from Lemma 3.2 that

$$\begin{aligned}
&\int_0^T \int_0^\infty |\xi_i^{(1)}(t,x)| \cdot |w_j^{(1)}(t,x)| \,\mathrm{d}x \,\mathrm{d}t \\
&\leqslant C\bigg(\|\varphi^{(0)}\|_{L^1(\mathbb{R}^+)} + \sum_{i=1}^{n} \|F_i^{(1)}(t,\cdot)\|_{L^1(\mathbb{R}^+)} \,\mathrm{d}t \bigg) \\
&\quad \times \bigg(W_1^{(1)}(0) + \int_0^T \lambda_j(u^{(1)}(t,0)) |w_j^{(1)}(t,0)| \,\mathrm{d}t \\
&\qquad + \sum_{i,j,k=1}^{n} \int_0^T \int_0^\infty |\Gamma_{ijk}(u^{(1)}) w_j^{(1)} w_k^{(1)} + Q_{ijk}(u^{(1)}) u_k^{(1)} w_j^{(1)}| \,\mathrm{d}x \,\mathrm{d}t \bigg) \\
&\leqslant C\varepsilon \bigg(\|\varphi^{(0)}\|_{L^1(\mathbb{R}^+)} + \sum_{i=1}^{n} \int_0^T \|F_i^{(1)}(t,\cdot)\|_{L^1(\mathbb{R}^+)} \,\mathrm{d}t \bigg).
\end{aligned} \tag{5.22}$$

(c) For $i \in J_2$ and $j \in J_1$, due to the boundary condition (5.6), it follows from Lemma 3.2 that

$$
\begin{aligned}
(5.23)\qquad & \int_0^T \int_0^\infty |\xi_i^{(1)}(t,x)| \cdot |w_j^{(1)}(t,x)| \,\mathrm{d}x \,\mathrm{d}t \\
&\leqslant C\Big(\|\varphi^{(0)}\|_{L^1(\mathbb{R}^+)} + \sum_{i=1}^n \int_0^T \|F_i^{(1)}(t,\cdot)\|_{L^1(\mathbb{R}^+)} \,\mathrm{d}t \\
&\qquad + \int_0^T \lambda_i(u^{(1)}(t,0)) |\xi_i^{(1)}(t,0)| \,\mathrm{d}t \Big) \\
&\quad \times \Big(W_1^{(1)}(0) + \sum_{i,j,k=1}^n \int_0^T \int_0^\infty |\Gamma_{ijk}(u^{(1)}) w_j^{(1)} w_k^{(1)} \\
&\qquad + Q_{ijk}(u^{(1)}) u_k^{(1)} w_j^{(1)}| \,\mathrm{d}x \,\mathrm{d}t \Big) \\
&\leqslant C\varepsilon\Big(\|\varphi^{(0)}\|_{L^1(\mathbb{R}^+)} + \|\alpha^{(0)}\|_{L^1(\mathbb{R}^+)} + \|h^{(0)}\|_{L^1(\mathbb{R}^+)} \\
&\qquad + \sum_{i=1}^n \int_0^T \|F_i^{(1)}(t,\cdot)\|_{L^1(\mathbb{R}^+)} \,\mathrm{d}t \Big).
\end{aligned}
$$

(d) For $i, j \in J_2$ and $i \neq j$, the similar procedure gives

$$
\begin{aligned}
(5.24)\qquad & \int_0^T \int_0^\infty |\xi_i^{(1)}(t,x)| \cdot |w_j^{(1)}(t,x)| \,\mathrm{d}x \,\mathrm{d}t \\
&\leqslant C\varepsilon\Big(\|\varphi^{(0)}\|_{L^1(\mathbb{R}^+)} + \|\alpha^{(0)}\|_{L^1(\mathbb{R}^+)} + \|h^{(0)}\|_{L^1(\mathbb{R}^+)} \\
&\qquad + \sum_{i=1}^n \int_0^T \|F_i^{(1)}(t,\cdot)\|_{L^1(\mathbb{R}^+)} \,\mathrm{d}t \Big).
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
(5.25)\qquad Q_{\xi^{(1)}}(T) \leqslant C\varepsilon\Big(& \|\varphi^{(0)}\|_{L^1(\mathbb{R}^+)} + \|\alpha^{(0)}\|_{L^1(\mathbb{R}^+)} + \|h^{(0)}\|_{L^1(\mathbb{R}^+)} \\
& + \sum_{i=1}^n \int_0^T \|F_i^{(1)}(t,\cdot)\|_{L^1(\mathbb{R}^+)} \,\mathrm{d}t \Big).
\end{aligned}
$$

Because of (2.23) and (2.41), we conclude from Lemma 3.2 that

$$
\begin{aligned}
(5.26)\qquad D_{\xi^{(1)}}(T) \leqslant C\frac{\varepsilon}{M+1}\Big(& \|\varphi^{(0)}\|_{L^1(\mathbb{R}^+)} + \|\alpha^{(0)}\|_{L^1(\mathbb{R}^+)} + \|h^{(0)}\|_{L^1(\mathbb{R}^+)} \\
& + \sum_{i=1}^n \int_0^T \|F_i^{(1)}(t,\cdot)\|_{L^1(\mathbb{R}^+)} \,\mathrm{d}t \Big).
\end{aligned}
$$

Similar estimates hold true for $Q_{\xi^{(2)}}(T)$ and $D_{\xi^{(2)}}(T)$.

Consequently, it follows from (5.16) that

$$\sum_{i=1}^{n}\int_0^T \|F_i^{(1)}(t,\cdot)\|_{L^1(\mathbb{R}^+)}\,\mathrm{d}t \leqslant C\varepsilon(\|\varphi^{(0)}\|_{L^1(\mathbb{R}^+)}+\|\alpha^{(0)}\|_{L^1(\mathbb{R}^+)}+\|h^{(0)}\|_{L^1(\mathbb{R}^+)}). \tag{5.27}$$

Hence, for $i\in J_1$ we have

$$\int_0^\infty |\xi_i^{(1)}(t,x)|\,\mathrm{d}x \leqslant C(\|\varphi^{(0)}\|_{L^1(\mathbb{R}^+)}+\|\alpha^{(0)}\|_{L^1(\mathbb{R}^+)}+\|h^{(0)}\|_{L^1(\mathbb{R}^+)}). \tag{5.28}$$

(ii) For $i\in J_2$, due (5.6), it follows from Lemma 3.1 that

$$\begin{aligned}\int_0^\infty |\xi_i^{(1)}(t,x)|\,\mathrm{d}x &\leqslant \int_0^\infty |\xi_i^{(1)}(0,x)|\,\mathrm{d}x+\int_0^T \|F_i^{(1)}(t,\cdot)\|_{L^1(\mathbb{R}^+)}\,\mathrm{d}t\\ &\quad+\int_0^T \big(\lambda_i(u^{(1)}(t,0))|\xi_i^{(1)}(t,0)|\big)\,\mathrm{d}t\\ &\leqslant C\bigg(\|\varphi^{(0)}\|_{L^1(\mathbb{R}^+)}+\sum_{i=1}^{n}\int_0^T \|F_i^{(1)}(t,\cdot)\|_{L^1(\mathbb{R}^+)}\,\mathrm{d}t\\ &\qquad+\|\alpha^{(0)}\|_{L^1(\mathbb{R}^+)}+\|h^{(0)}\|_{L^1(\mathbb{R}^+)}\bigg).\end{aligned} \tag{5.29}$$

Thus, from (5.27) and (5.29), for $i\in J_2$, we conclude that

$$\int_0^\infty |\xi_i^{(1)}(t,x)|\,\mathrm{d}x \leqslant C\big(\|\varphi^{(0)}\|_{L^1(\mathbb{R}^+)}+\|\alpha^{(0)}\|_{L^1(\mathbb{R}^+)}+\|h^{(0)}\|_{L^1(\mathbb{R}^+)}\big). \tag{5.30}$$

Therefore, by (5.28) and (5.30), we prove (5.1), which completes the proof of Theorem 1.2. □

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