

Veronika Chrastinová; Václav Tryhuk
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PARALLELISMS BETWEEN DIFFERENTIAL
AND DIFFERENCE EQUATIONS

VERONIKA CHRASTINOVÁ, VÁCLAV TRYHUK, Brno

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Abstract. The paper deals with the higher-order ordinary differential equations and the analogous higher-order difference equations and compares the corresponding fundamental concepts. Important dissimilarities appear for the moving frame method.

Keywords: ordinary differential equation, ordinary difference equation, first integral, symmetry, infinitesimal symmetry, moving frame

MSC 2010: 34A34, 39A05

INTRODUCTION

The parallelism between differential and difference equations was discussed in the inspirational article [1]. Retaining the notation [1], we propose an alternative approach where the interrelations become more transparent. Especially the fundamental concepts clarify: compare (e.g.) first integrals, shift operator, symmetries and infinitesimal transformations as stated in [1] with our conceptions introduced below. On the other hand, the common method of moving frames [2] not mentioned in [1] needs essential change for the case of the difference equations.

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1. DIFFERENTIAL EQUATION

We consider the equation

$$(1.1) \quad u^{(N)} = F(x, u, u', \dots, u^{(N-1)}) \quad \left(' = \frac{d}{dx}, u = u(x) \right)$$

in the real domain, where F is a smooth function. Employing the jet space \mathbf{M} with coordinates

$$(1.2) \quad \mathbf{M}: x, u^0, u^1, \dots, u^{N-1},$$

equation (1.1) is expressed by the first order system

$$(1.3) \quad \frac{du^k}{dx} = u^{k+1} \quad (k = 0, \dots, N-2), \quad \frac{du^{N-1}}{dx} = F(x, u^0, \dots, u^{N-1})$$

and its *solution* is represented by the curve (the left-hand figure)

$$(1.4) \quad \mathbf{P}(x) = (x, u^0(x), \dots, u^{N-1}(x)) = (x, u(x), \dots, u^{(N-1)}(x)) \in \mathbf{M}.$$

Such a solution is uniquely determined by any of its points

$$(1.5) \quad \mathbf{P}(\bar{x}) = (\bar{x}, u^0(\bar{x}), \dots, u^{N-1}(\bar{x})) \in \mathbf{M} \quad (\text{fixed } \bar{x} \in \mathbb{R}).$$

A function

$$(1.6) \quad \Phi = \Phi(x, u^0, u^1, \dots, u^{N-1})$$

on \mathbf{M} is called the *first integral* of equation (1.1) if $\Phi(\mathbf{P}(x)) = \text{const.}$ for every solution (1.4). *It follows that the restriction of the function Φ to $x = \bar{x}$ can be prescribed.*

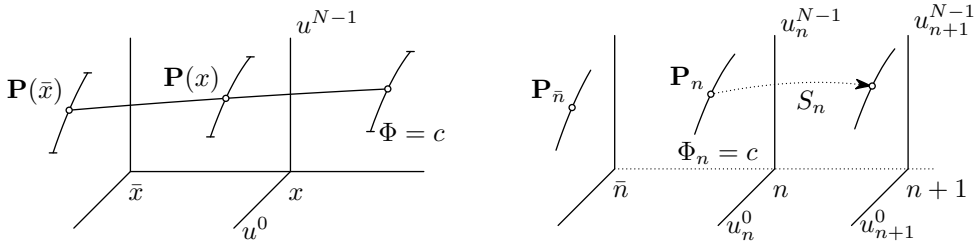


Fig. 1

2. DIFFERENCE EQUATION

We consider the equation

$$(2.1) \quad u_{n+N} = F_n(u_n, \dots, u_{n+N-1}) \quad (u_n \in \mathbb{R}, n \in \mathbb{Z})$$

where F_n are smooth functions. We suppose that it is exactly of the order N , that is, (2.1) can be equivalently expressed as $u_n = G_{n+1}(u_{n+1}, \dots, u_{n+N})$. After a formal adjustment of indices, we obtain the equation

$$(2.2) \quad u_{n-1} = G_n(u_n, \dots, u_{n+N-1}) \quad (u_n \in \mathbb{R}, n \in \mathbb{Z})$$

equivalent to (2.1).

In order to obtain a parallel theory, we introduce discrete counterparts u_n^k to the derivatives $u^{(k)}(x)$. In full detail, we introduce spaces \mathbf{M}_n with coordinates

$$(2.3) \quad \mathbf{M}_n: u_n^0, u_n^1, \dots, u_n^{N-1} \quad (n \in \mathbb{Z}).$$

Roughly speaking, the coordinate u_n^k corresponds to the variable u_{n+k} appearing in (2.1). It follows that equation (2.1) turns into the first order system

$$(2.4) \quad u_{n+1}^k = u_n^{k+1} \quad (k = 0, \dots, N-2), \quad u_{n+1}^{N-1} = F_n(u_n^0, \dots, u_n^{N-1}) \quad (n \in \mathbb{Z})$$

which may be regarded as a shift transformation $S_n: \mathbf{M}_n \rightarrow \mathbf{M}_{n+1}$. (More precisely, S_n is given by the formulae

$$S_n^* u_{n+1}^k = u_n^{k+1} \quad (k = 0, \dots, N-2), \quad S_n^* u_{n+1}^{N-1} = F_n(u_n^0, \dots, u_n^{N-1}) \quad (n \in \mathbb{Z})$$

which are a mere transcription of (2.4). Alternatively, equation (2.2) is expressed by

$$(2.5) \quad u_{n-1}^0 = G_n(u_n^0, \dots, u_n^{N-1}), \quad u_{n-1}^1 = u_n^0, \dots, u_{n-1}^{N-1} = u_n^{N-2}$$

and may be regarded as a transformation $S_n^{-1}: \mathbf{M}_n \rightarrow \mathbf{M}_{n-1}$.) Then a *solution* is represented by a sequence of points

$$(2.6) \quad \mathbf{P}_n = (u_n^0, \dots, u_n^{N-1}) \in \mathbf{M}_n \quad (u_n^k \in \mathbb{R}, n \in \mathbb{Z})$$

satisfying (2.4) (hence $S_n \mathbf{P}_n = \mathbf{P}_{n+1}$, see the above right-hand figure). It follows that it is determined by any of its points

$$(2.7) \quad \mathbf{P}_{\bar{n}} = (u_{\bar{n}}^0, \dots, u_{\bar{n}}^{N-1}) \in \mathbf{M}_{\bar{n}} \quad (u_{\bar{n}}^k \in \mathbb{R}, \text{ fixed } \bar{n} \in \mathbb{Z}).$$

A sequence of functions

$$(2.8) \quad \Phi_n = \Phi_n(u_n^0, \dots, u_n^{N-1}) \quad (n \in \mathbb{Z})$$

is called the *first integral* of equation (2.1) if the values $\Phi_n(\mathbf{P}_n)$ are independent of n for every solution (2.6). *It follows that the term $\Phi_{\bar{n}}$ (fixed $\bar{n} \in \mathbb{Z}$) can be prescribed.*

3. SYMMETRIES OF DIFFERENTIAL EQUATION

We are interested in mappings Γ that preserve differential equation (1.1), however, reasonable counterparts for the difference equation (2.1) represent only the x -preserving mappings. In terms of coordinates (1.2), such *x -preserving symmetry* $\Gamma: \mathbf{M} \rightarrow \mathbf{M}$ is given by certain formulae

$$(3.1) \quad \Gamma^* x = x, \quad \Gamma^* u^k = \hat{u}^k(x, u^0, \dots, u^{N-1}) \quad (k = 0, \dots, N-1)$$

and equations (1.3) are preserved if and only if

$$(3.2) \quad D\hat{u}^k = \hat{u}^{k+1} \quad (k = 0, \dots, N-2), \quad D\hat{u}^{N-1} = F(x, \hat{u}^0, \dots, \hat{u}^{N-1}),$$

$$(D = \partial/\partial x + u^1 \partial/\partial u^0 + \dots + u^{N-1} \partial/\partial u^{N-2} + F \partial/\partial u^{N-1}).$$

We also recall the *infinitesimal symmetry*

$$(3.3) \quad \Gamma^* x = x, \quad \Gamma^* u^k = u^k + \varepsilon Q^k(x, u^0, \dots, u^{N-1}) + \dots$$

represented by the vector field

$$(3.4) \quad X = \sum Q^k(x, u^0, \dots, u^{N-1}) \frac{\partial}{\partial u^k}$$

on the space \mathbf{M} . The coefficients Q^k satisfy the identities

$$DQ^k = Q^{k+1} \quad (k = 0, \dots, N-2), \quad DQ^{N-1} = \sum Q^k \frac{\partial F}{\partial u^k}.$$

Since symmetries are mappings permuting the solutions, it follows that *the restriction of symmetries to the fiber $x = \bar{x}$ (fixed $\bar{x} \in \mathbb{R}$) can be arbitrarily prescribed.*

There is a huge literature on the symmetry theory of differential equations. In particular, we refer to [2] for the moving frame method. The particular *pointwise subcase* when $\Gamma^* u^0 = \hat{u}^0(x, u^0)$ is supposed is not easier and nontrivial pointwise symmetries of a given equation (1.1) *need not exist.*

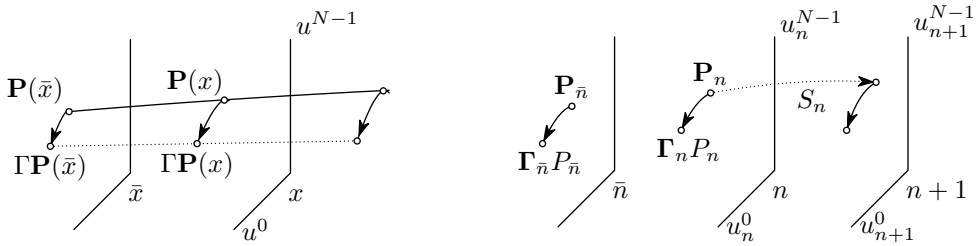


Fig. 2

4. SYMMETRIES OF DIFFERENCE EQUATIONS

We introduce mappings $\Gamma_n: \mathbf{M}_n \rightarrow \mathbf{M}_n$ ($n \in \mathbb{Z}$) given by certain formulae

$$(4.1) \quad \Gamma_n^* u_n^k = \hat{u}_n^k(u_n^0, \dots, u_n^{N-1}) \quad (k = 0, \dots, N-1).$$

Then equations (2.4) are preserved if and only if the requirements

$$(4.2) \quad \hat{u}_{n+1}^k = \hat{u}_n^{k+1} \quad (k = 0, \dots, N-2), \quad \hat{u}_{n+1}^{N-1} = F_n(\hat{u}_n^0, \dots, \hat{u}_n^{N-1})$$

are satisfied by virtue of (2.4). In full detail, we have the requirements

$$\hat{u}_{n+1}^k(u_n^1, \dots, u_n^{N-1}, F_n) = \begin{cases} \hat{u}_n^{k+1} & (\text{if } k = 0, \dots, N-2), \\ \Gamma_n^* F_n & (\text{if } k = N-1). \end{cases}$$

The sequence Γ_n ($n \in \mathbb{Z}$) represents a *symmetry* of difference equation (2.1). The relevant *infinitesimal symmetry*

$$(4.3) \quad \Gamma_n^* u_n^k = u_n^k + \varepsilon Q_n^k(u_n^0, \dots, u_n^{N-1}) + \dots$$

is analogously represented by the sequence of vector fields

$$(4.4) \quad X_n = \sum Q_n^k(u_n^0, \dots, u_n^{N-1}) \frac{\partial}{\partial u_n^k} \quad (n \in \mathbb{Z})$$

on the spaces \mathbf{M}_n . The coefficients Q_n^k satisfy the identities

$$Q_{n+1}^k(u_n^1, \dots, u_n^{N-1}, F_n) = \begin{cases} Q_n^{k+1} & (\text{if } k = 0, \dots, N-2), \\ \sum Q_n^l \partial F_n / \partial u^l & (\text{if } k = N-1). \end{cases}$$

Since symmetries are just the mappings that permute the solutions, it follows that *the data* $\Gamma_{\bar{n}}$ and $Q_{\bar{n}}^k$ (fixed $\bar{n} \in \mathbb{Z}$) *can be arbitrarily prescribed.*

We will mention in more detail the particular *pointwise subcase* when the transformation formula

$$(4.5) \quad \Gamma_n^* u_n^0 = g_n(u_n^0) (= g_n(u_n)) \quad (n \in \mathbb{Z}, \text{ abbreviation } g_n = \hat{u}_n^0)$$

is supposed. Then the remarkable identities

$$(4.6) \quad \begin{aligned} \Gamma_n^* u_n^k &= \Gamma_n^* u_{n+k}^0 = g_{n+k}(u_{n+k}^0) = g_{n+k}(u_n^k) \quad (k = 0, \dots, N-1), \\ \Gamma_n^* F_n &= \hat{u}_{n+1}^{N-1} = g_{n+N}(u_{n+1}^{N-1}) = g_{n+N}(F_n) \end{aligned}$$

follow from (4.2) and (2.4).

We refer to [1] for the infinitesimal pointwise symmetries. Instead we will mention the alternative method of moving frames. It is rather dissimilar from the common theory of symmetries of differential equations [2].

5. MOVING FRAMES FOR POINTWISE SYMMETRIES

Functions g_{n+k}, g_{n+N} in identities (4.6) are unknown. Therefore identities (4.6) can be expressed by saying that Γ_n transforms every level set $u_n^k = c_n^k, F_n = c_n$ (c_n^k, c_n are constants) again into such a level set. Alternatively: every equation $du_n^k = 0, dF_n = 0$ is preserved. Still otherwise: *let us consider spaces \mathbf{N}_n with coordinates and differential forms*

$$(5.1) \quad \begin{aligned} \mathbf{N}_n: & u_n^0, \dots, u_n^{N-1}, a_n^0, \dots, a_n^{N-1}, a_n \quad (n \in \mathbb{Z}), \\ \alpha_n^k &= a_n^k du_n^k \quad (k = 0, \dots, N-1), \quad \alpha_n = a_n dF_n. \end{aligned}$$

Then

$$(5.2) \quad \Gamma_n^* \alpha_n^k = \alpha_n^k \quad (k = 0, \dots, N-1), \quad \Gamma_n^* \alpha_n = \alpha_n$$

if the transformation formulae (4.5), (4.6) are appropriately completed by the additional coordinates. One can directly verify that the additional transformation formulae (in fact needless in the sequel) ensuring the invariance (5.2) are

$$(5.3) \quad \Gamma_n^* a_n^k = \frac{a_n^k}{g'_{n+k}(u_n^k)} \quad (k = 0, \dots, N-1), \quad \Gamma_n^* a_n = \frac{a_n}{g'_{n+N}(F_n)}.$$

6. THE ALGORITHM

Identities (5.2) express the invariance of the differential forms α_n^k, α_n and we will search for more invariants in order to determine the mapping $\Gamma_n: \mathbf{N}_n \rightarrow \mathbf{N}_n$. The indices $n \in \mathbb{Z}$ may be kept fixed in this section.

First of all, we have

$$\alpha_n = \sum A_n^k \alpha_n^k \quad \left(A_n^k = \frac{a_n}{a_n^k} \partial F_n / \partial u_n^k \right)$$

and it follows that $A_n^k = \Gamma_n^* A_n^k$ are invariants. Assume $\partial F_n / \partial u_n^k \neq 0$ for a certain k . Since the level set $A_n^k = 1$ is preserved, we may restrict our calculations to this level set. Alternatively, the coordinate a_n^k in (5.1) may be omitted and the coefficient a_n^k in (5.1) is replaced by $a_n \partial F_n / \partial u_n^k$. (This is the common procedure in the moving frame method.) Repeatedly applying this reduction, we have the invariant forms

$$(6.1) \quad \alpha_n^k = \begin{cases} a_n \partial F_n / \partial u_n^k \cdot du_n^k & (\text{if } \partial F_n / \partial u_n^k \neq 0) \\ a_n^k du_n^k & (\text{if } \partial F_n / \partial u_n^k = 0) \end{cases}$$

on the “reduced” space \mathbf{N}_n (some coordinates a_n^k are not occurring here).

Secondly, clearly

$$(6.2) \quad d\alpha_n = da_n \wedge dF_n = \beta_n \wedge \alpha_n \quad \left(\beta_n = \frac{da_n}{a_n} + b_n dF_n \right)$$

where b_n is regarded as a new variable. Since this β_n is the most general differential form satisfying (6.2), it follows that $\beta_n = \Gamma_n^* \beta_n$ is invariant. The invariance is ensured if

$$\Gamma_n^* b_n = \frac{1}{g'_{n+N}(F_n)} \left(b_n + a_n \frac{g''_{n+N}(F_n)}{g'_{n+N}(F_n)} \right)$$

by using (4.6) and (5.3). (The formula will be in fact needless.)

Thirdly, assuming $\partial F_n / \partial u_n^k \neq 0$ and $\partial F_n / \partial u_n^l \neq 0$, (6.1) gives

$$(6.3) \quad d\alpha_n^k = \left(\beta_n + \frac{1}{a_n} \sum (A_n^{kl} - b_n) \alpha_n^l \right) \wedge \alpha_n^k \quad \left(A_n^{kl} = \frac{\partial^2 F_n / \partial u_n^k \partial u_n^l}{\partial F_n / \partial u_n^k \cdot \partial F_n / \partial u_n^l} \right)$$

with help of (6.2). The coefficients are invariant. Since every level set $A_n^{kl} - b_n = 0$ with $k \neq l$ is preserved, we may restrict the calculations to the level set and put $b_n = A_n^{kl}$ for a certain appropriate k, l (where $k \neq l$).

Fourthly, after this restriction, we have the invariant form

$$(6.4) \quad d\beta_n = dA_n^{kl} \wedge dF_n = \frac{1}{(a_n)^2} \sum \mathcal{A}_n^{rs} \alpha^r \wedge \alpha^s$$

where

$$(6.5) \quad \frac{1}{(a_n)^2} \mathcal{A}_n^{rs} = \frac{1}{(a_n)^2} \left(\frac{\partial A_n^{kl}}{\partial u_n^r} \frac{\partial u_n^r}{\partial u_n^s} - \frac{\partial A_n^{kl}}{\partial u_n^s} \frac{\partial u_n^r}{\partial u_n^r} \right)$$

are invariant coefficients. If $\mathcal{A}_n^{rs} \neq 0$, the level set $\mathcal{A}_n^{rs}/(a_n)^2 = 1$ may be employed in order to determine a_n which clarifies the form β_n . Then $\beta_n = \sum I_n^k \alpha_n^k$, with the true invariant functions I_n^k depending only on the primary variables u_n^0, \dots, u_n^{N-1} .

In particular, we have proved the following assertion needful below.

Theorem 6.1. *Assume $\partial F_n/\partial u_n^k \neq 0$ ($n \in \mathbb{Z}$; $k = 0, \dots, N-1$). Then the pointwise symmetry Γ_n ($n \in \mathbb{Z}$) given by (4.5) preserves all differential forms*

$$\alpha_n^k = a_n \frac{\partial F_n}{\partial u_n^k} du_n^k, \quad \beta_n = \frac{da_n}{a_n} + \frac{\partial^2 F_n / \partial u_n^k \partial u_n^l}{\partial F_n / \partial u_n^k \cdot \partial F_n / \partial u_n^l} \quad (n \in \mathbb{Z}; k, l = 0, \dots, N-1).$$

Here the additional parameters a_n are subject to formulae (5.32).

7. EXAMPLES

(i) *The linear equation.* Let us consider an equation

$$u_{n+N} = C_n^0 u_n + \dots + C_n^{N-1} u_{n+N-1} \quad (C_n^k \in \mathbb{R}, C_n^0 \neq 0),$$

therefore $F_n = \sum C_n^k u_n^k$ in terms of variables (2.3). We have invariant forms

$$\alpha_n^k = a_n C_n^k du_n^k \quad (\text{if } C_n^k \neq 0), \quad \beta_n = \frac{da_n}{a_n};$$

see (6.1), (6.2) where $b_n = A_n^{kl} = 0$ identically. The invariance equations (5.2) with $\Gamma_n^* u_n^k = g_{n+k}(u_n^k)$ together with the additional invariance $\Gamma_n^* \beta_n = \beta_n$ read $\Gamma_n^* a_n \cdot C_n^k dg_{n+k} = a_n C_n^k du_n^k$, $d\Gamma_n^* \ln a_n = d \ln a_n$ whence

$$\Gamma_n^* a_n = E_n a_n, \quad g_{n+k}(u_n^k) = \frac{1}{E_n} (u_n^k + D_n^k) \quad (E_n, D_n^k \in \mathbb{R})$$

and it follows that $E_n = E$ is independent of n and $D_n^k = D_{n+k}$. In terms of original variables, we have the substitution $u_{n+k} \mapsto g_{n+k}(u_{n+k}) = (u_{n+k} + D_{n+k})/E$ which can be simplified as

$$u_n \mapsto g_n(u_n) = \frac{1}{E} (u_n + D_n) \quad (n \in \mathbb{Z}).$$

This is indeed a symmetry if the recurrence $D_{n+N} = \sum C_n^k D_{n+k}$ ($n \in \mathbb{Z}$) is satisfied. It follows that *the point symmetries depend on the choice of N constants $E \neq 0, D_0, \dots, D_{N-1}$.*

(ii) *Nontrivial invariants.* Let us consider the equation

$$u_{n+N} = u_n + \ln G_n \quad (G_n = u_n + \dots + u_{n+N-1}).$$

In terms of variables (2.3), we have invariant forms

$$\alpha_n^0 = a_n \left(1 + \frac{1}{G_n}\right) du_n^0, \quad \alpha_n^k = a_n \frac{1}{G_n} u_n^k \quad (k = 1, \dots, N-1),$$

and β_n which will be needless. One can then find that all coefficients to appear (e.g., $A_n^{kl}, b_n, \mathcal{A}_n^{kl}, I_n^k$) depend only on functions G_n and we may introduce the *invariance requirement* $\Gamma_n^* G_n = G_n$. This implies $\Gamma_n^* du_n^k = du_n^k$ whence

$$g_{n+k}(u_n^k) = \Gamma_n^* u_n^k = u_n^k + E_{n+k}.$$

In terms of original variables, we have the substitution $u_{n+k} \mapsto g_{n+k}(u_{n+k}) = u_{n+k} + E_{n+k}$ which can be simplified to

$$u_n \mapsto g_n(u_n) = u_n + E_n.$$

This is indeed a symmetry if the *compatible requirements*

$$E_{n+N} = E_n, E_n + \dots + E_{n+N-1} = 0$$

equivalent to the *single recurrence*

$$E_{n+1} + \dots + E_{n+N} = 0 \quad (n \in \mathbb{Z})$$

are satisfied. It follows that *the point symmetries depend on the choice of $N-1$ constants.* One can however see that a slight change of data may provide a difference equation without any point symmetries, see also [4] for a quite general discussion if $N=2$ is supposed.

(iii) *The zero curvature examples.* Let us introduce a large class of equations

$$u_{n+N} = \varphi_n(p_n^0(u_n) + \dots + p_n^{N-1}(u_{n+N-1})) \quad (\varphi_n' \neq 0, p_n^{k'} \neq 0)$$

where φ_n, p_n^k are smooth functions. In terms of variables (2.3), we have

$$F_n = \varphi_n \left(\sum p_n^k(u_n^k) \right), \quad A_n^{kl} = \frac{\varphi_n''}{(\varphi_n')^2} = b_n, \quad \mathcal{A}_n^{kl} = 0.$$

We recall the invariant forms

$$\alpha_n^k = a_n \varphi_n' p_n^k \, du_n^k = a_n \varphi_n' dp_n^k, \quad \beta_n = \frac{da_n}{a_n} + \frac{\varphi_n''}{(\varphi_n')^2} d\varphi_n = d(\ln a_n \varphi_n').$$

The invariance of the forms β_n implies

$$\Gamma_n^*(a_n \varphi_n') = E_n a_n \varphi_n'$$

for appropriate nonvanishing constants $E_n \in \mathbb{R}$. Then the invariance of the forms α_n^k reads

$$E_n a_n \varphi_n' \Gamma_n^* dp_n^k = a_n \varphi_n' dp_n^k$$

whence

$$E_n \Gamma_n^* p_n^k = p_n^k + D_n^k \quad (k = 0, \dots, N-1)$$

for appropriate constants $D_n^k \in \mathbb{R}$. However,

$$\Gamma_n^* p_n^k = p_n^k (\Gamma_n^* u_n^k) = p_n^k (g_{n+k})$$

and therefore $E_n = E$ is independent of n and $D_n^k = D_{n+k}$. Altogether we have the substitution

$$(7.1) \quad u_{n+k} \mapsto g_{n+k}(u_{n+k}) = (p_n^k)^{-1} \left(\frac{1}{E} (p_n^k(u_{n+k}) + D_{n+k}) \right) \quad (k = 0, \dots, N-1)$$

(with the inverse function $(p_n^k)^{-1}$) in terms of original variables. In particular,

$$(7.2) \quad u_n \mapsto g_n(u_n) = (p_n^0)^{-1} \left(\frac{1}{E} (p_n^0(u_n) + D_n) \right)$$

and this result is quite reasonable in applications. We have an overdetermined system of requirements. *If requirements (7.1), (7.2) are compatible, they determine all possible point symmetries.*

(iv) *Continuation with $N = 2$.* Let us mention two intentionally simple examples

$$u_{n+2} = \frac{u_n u_{n+1}}{u_n + u_{n+1}}, \quad u_{n+2} = \frac{1 - u_n u_{n+1}}{u_n + u_{n+1}}.$$

They can be transcribed as

$$u_{n+2} = \frac{1}{1/u_n + 1/u_{n+1}}, \quad u_{n+2} = \frac{1}{\tan(\arctan u_n + \arctan u_{n+1})},$$

respectively. The choice of functions $\varphi_n = \varphi$, $p_n^k = p$ (independent of n, k) is obvious here and formula (7.2) may be applied. We obtain substitutions

$$u_n \mapsto \frac{1}{E^{-1}(1/u_n + D_n)}, \quad u_n \mapsto \frac{1}{\tan E^{-1}(\arctan u_n + D_n)} \quad (n \in \mathbb{Z})$$

and they are true symmetries if and only if the recurrence $D_{n+2} = D_n + D_{n+1}$ ($n \in \mathbb{Z}$) is satisfied (direct verification). The same formulae were obtained in [1] with the use of infinitesimal transformations and rather lengthy calculations.

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Authors' addresses: *V. Chrastinová, V. Tryhuk*: Brno University of Technology, Faculty of Civil Engineering, Department of Mathematics, Veveří 331/95, 602 00 Brno, Czech Republic, e-mail: chrastinova.v@fce.vutbr.cz, tryhuk.v@fce.vutbr.cz.