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Abstract. We introduce a general notion of covering property, of which many classical definitions are particular instances. Notions of closure under various sorts of convergence, or, more generally, under taking kinds of accumulation points, are shown to be equivalent to a covering property in the sense considered here (Corollary 3.10). Conversely, every covering property is equivalent to the existence of appropriate kinds of accumulation points for arbitrary sequences on some fixed index set (Corollary 3.5).

We discuss corresponding notions related to sequential compactness, and to pseudocompactness, or, more generally, properties connected with the existence of limit points of sequences of subsets. In spite of the great generality of our treatment, many results here appear to be new even in very special cases, such as D-compactness and D-pseudocompactness, for D an ultrafilter, and weak (quasi) M-(pseudo)-compactness, for M a set of ultrafilters, as well as for $[\beta, \alpha]$-compactness, with β and α ordinals.

Keywords: covering property, subcover, compactness, accumulation point, convergence, pseudocompactness, limit point

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1. Introduction

"Covering property" in the title refers to a property of the form "every open cover has a subcover by a tractable class of elements". The most general and easiest form of establishing what "tractable" is to be intended simply amounts to enumerate those sets which are to be considered *tractable*. We are thus led to the following definition, where $\mathcal{P}(A)$ denotes the set of all subsets of the set A.

Definition 1.1 ([19, Definition 7.7]). If A is a set, and $B \subseteq \mathcal{P}(A)$, we say that a topological space X is $[B, A]$-*compact* if and only if, whenever $(O_a)_{a \in A}$ is a sequence of open sets of X such that $\bigcup_{a \in A} O_a = X$, then there is $H \in B$ such that $\bigcup_{a \in H} O_a = X$.

Of course, (full) *compactness* is the particular case when A is infinite and arbitrarily large, and B is the set of all finite subsets of A. If in the above sentence we replace finite by countable, we get *Lindelöfness*. On the other hand, if we instead restrict only to countable A, we get *countable compactness*. More general examples of cardinal (and ordinal) notions reducible to $[B, A]$-compactness will be presented below.

We first show how to produce counterexamples to $[B, A]$-compactness in a standard way.

Example 1.2. Suppose that A is a set, $B \subseteq \mathcal{P}(A)$, B is nonempty, and $A \notin B$ (the assumption $A \notin B$ is necessary by Fact 2.2(1) below).

(a) As a typical counterexample to $[B, A]$-compactness, we can exhibit B itself, with the topology a subbase of which consists of the sets $a^{\not<} = \{H \in B \mid a \notin H\}$, a varying in A.

With the above topology, B is not $[B, A]$-compact, as the $a^{\not<}$s themselves witness. Indeed, the $a^{\not<}$s are a cover of B, since $A \notin B$. However, for every $H \in B$, $(a^{\not<})_{a\in H}$ is not a cover of B, since H belongs to no $a^{\not<}$, for $a \in H$.

We believe that, in a sense still to be made precise, B with the above topology is the typical example of a non $[B, A]$-compact topological space. This is suggested by particular cases concerning ordinal compactness, see [19, Theorems 5.4 and 5.7].

Notice that B, with the above topology, is T_0, but, in general, not even T_1. However, the example can be turned into a Tychonoff topological space by introducing a finer topology as in (b) below.

Observe that $\mathcal{P}(A)$ is in a bijective correspondence, via characteristic functions, with ${}^A\{0, 1\}$, the set of all functions from A to $\{0, 1\}$, hence with the product of A-many copies of $\{0, 1\}$. Via the above identifications, if we give to $\{0, 1\}$ the topology in which $\{0\}$ is open, but $\{1\}$ is not open, then the topology described above is the subspace topology induced on B by the (Tychonoff) product topology on ${}^A\{0, 1\}$.

(b) If we instead give to $\{0, 1\}$ the discrete topology, then the subset topology induced on B by the topology on ${}^A\{0, 1\}$ makes B a Tychonoff topological space, which is still a counterexample to $[B, A]$-compactness. This latter topology, too, admits an explicit description: it is the topology a subbase of which consists of the sets which have either the form $a^{\not<} = \{H \in B \mid a \notin H\}$, or $a^{<} = \{H \in B \mid a \in H\}$, for some $a \in A$.

If B is closed under symmetrical difference, then, with this topology, B inherits from ${}^A\{0, 1\}$ the structure of a topological group. If B is closed both under finite unions and finite intersections, then B inherits from ${}^A\{0, 1\}$ the structure of a topological lattice.

We now consider some more specific instances of Definition 1.1.

The most general form of a covering notion involving cardinality as a measure of "tractability" is $[\mu, \lambda]$-*compactness*, where μ and λ are cardinals. It is the particular case of Definition 1.1 when $A = \lambda$ and $B = \mathcal{P}_\mu(\lambda)$ is the set of all subsets of λ of cardinality $< \mu$. The notion of $[\mu, \lambda]$-compactness originated in the 20's in the past century [1], and thus has a very long history. See, e.g., [8], [14], [22], [23], [25], [26], [27], [28] for results and references.

In [19] we generalized cardinal compactness to ordinals, that is, we considered the particular case of Definition 1.1 in which A is an ordinal (or, anyway, a well-ordered set), and the "tractability" of some subset H of A is measured by considering the order type of H. In more detail, for α and β ordinals, $[\beta, \alpha]$-*compactness* is obtained from Definition 1.1 by letting $A = \alpha$, and letting B equal to the set of all subsets of α having order type $< \beta$. The notion is interesting, since one can prove many non trivial results of the form "every $[\beta, \alpha]$-compact space is $[\beta', \alpha']$-compact", for various ordinals, while only trivial results of this kind hold, when restricted to cardinals. Moreover, there are examples of spaces satisfying exactly the same $[\mu, \lambda]$-compactness (cardinal) properties, but which behave in a very different way as far as (ordinal) $[\beta, \alpha]$-compactness is concerned. Just to present the simplest possible example, if κ is a regular uncountable cardinal, then κ, with the order topology, is $[\kappa + \kappa, \kappa + \kappa]$-compact, but the disjoint union of two copies of κ is not $[\kappa + \kappa, \kappa + \kappa]$-compact (here $+$ denotes ordinal sum). Furthermore, there are many rather deep connections among $[\beta, \alpha]$-compactness, cardinalities and separation properties of spaces. In [19] we also introduced an ordinal version of the Lindelöf number of a topological space, and showed that this ordinal version gives much more information about the space than the cardinal version.

So far, we have not yet provided really strong motivations in favor of the great generality of Definition 1.1. Indeed, at first sight, the ordinal version of compactness, that is, $[\beta, \alpha]$-compactness, appears to be a quite very sensitive and fine notion, well suited for exactly measuring the covering properties enjoyed by some topological space. However, other interesting properties naturally insert themselves into the general framework given by Definition 1.1. In fact, besides considering $[\beta, \alpha]$-compactness, we reached the notion of $[B, A]$-compactness after a careful look at the proposition below, which characterizes D-compactness.

Recall that if D is an ultrafilter, say over some set I, then a topological space X is said to be D-*compact* if and only if every sequence $(x_i)_{i \in I}$ of elements of X D-converges to some point of x, where a sequence $(x_i)_{i \in I}$ is said to D-*converge* to some point $x \in X$ if and only if $\{i \in I \mid x_i \in U\} \in D$, for every neighborhood U of x in X.

In [18, Corollary 34] we proved the following proposition, which is also a particular case of Theorem 3.9 below (see Remark 3.12).

Proposition 1.3. *Let D be an ultrafilter over I. A topological space X is D-compact if and only if, for every open cover $(O_Z)_{Z \in D}$ of X, there is some $i \in I$ such that $(O_Z)_{i \in Z}$ is a cover of X.*

Thus also D-compactness is equivalent to a covering property, namely, the particular case of Definition 1.1 in which A is D itself, and $B = \{i^< \mid i \in I\}$, where, for $i \in I$, we put $i^< = \{Z \in D \mid i \in Z\}$. In words, B is the set of all the intersections of D with some principal ultrafilter. Hence, in the sense of D-compactness, being "tractable" means (having indices) lying in the intersection of D with some principal ultrafilter.

Reflecting on the above example, we soon realized that many other conditions asking closure under appropriate types of convergence are equivalent to covering properties. Furthermore, this is the case also for the existence of kinds of accumulation points, as we shall show in Section 3. Historically, the interplay between covering properties and accumulation properties has been a central theme in topology, starting from [1], if not earlier. In this respect, see also the discussion in Remark 2.5.

Also a generalization of D-compactness, weak M-compactness, involving a set M of ultrafilters, is equivalent to a covering property, as will be shown in Corollary 3.14. See Corollary 5.15 for a characterization of a further related notion: quasi M-compactness.

If in Definition 1.1 we take $B = \mathcal{P}(A) \setminus \{A\}$, then a counterexample to $[B, A]$-compactness is what is usually called an *irreducible* (or *minimal*) cover. Irreducible covers, as well as spaces in which every cover can be refined to a (possibly finite) irreducible cover have been the object of some study. See [2], [15] and further references there. In a sense, an infinite irreducible cover produces a maximal form of incompactness. Indeed, e.g. by Fact 2.2(2)(6) below in contrapositive form, if some topological space X has an irreducible cover of cardinality λ, then X is not $[B, A]$-compact, for every set A such that $|A| \leq \lambda$, and every $B \subseteq \mathcal{P}(A)$ such that $A \notin B$.

If X is a T_1 topological space which is not countably compact, then any open cover witnessing countable incompactness can be refined to an irreducible countably infinite open cover. This follows, for example, from the proof of [19, Lemma 6.4]. Compare also with [2, Theorem 2.1]. Thus we get the following proposition.

Proposition 1.4. *For a T_1 topological space X, the following conditions are equivalent.*

1. *X is not countably compact.*
2. *X is not $[\mathcal{P}(A) \setminus \{A\}, A]$-compact, for every countable nonempty set A.*
3. *X is not $[B, A]$-compact, for some countably infinite set A, and some $B \subseteq \mathcal{P}(A)$ such that B contains all finite subsets of A.*

The above equivalences do not generalize to uncountable cardinals. The space κ, with the order topology, is not $[\kappa, \kappa]$-compact, but it is $[\kappa + \omega, \kappa + \omega]$-compact [19, Example 3.2(3)] (here $+$ denotes ordinal sum). Moreover the hypothesis that X is T_1 is necessary, by [19, Example 3.2(2)].

Though simple, Definition 1.1 unifies many disparate situations, and allows for the possibility of proving some interesting and non trivial results, which sometimes are new and useful even in very particular cases.

When restricted to (cardinal) $[\mu, \lambda]$-compactness, some of the results presented in this note might be seen as a revisitation of known results. They are new in the case of (ordinal) $[\beta, \alpha]$-compactness. Actually, the study of properties of $[\beta, \alpha]$-compactness has been the leading motivation for the present research. Restricted to this particular case, this note may be seen as a continuation of [19]. As soon as we realized that the results naturally fit into a more general setting, with no essential further technical complication, we decided to present them in their more general form.

As far as D-compactness, and other notions of convergence are concerned, the results presented here can improve shedding new light into the subject. In particular, they hopefully provide a new point of view about the relationship between convergence, accumulation and covering properties.

It might be of some interest the fact that there is also a version for notions related to pseudocompactness. As well known, for Tychonoff spaces, there is an equivalent formulation of pseudocompactness which involves open covers: a Tychonoff space is *pseudocompact* if and only if every countable open cover has a subset with dense union. Here the premise is the same as in countable compactness, with a weakened conclusion. Definition 1.1, too, can be modified in the same way (Definition 4.1), and essentially all the results we prove for $[B, A]$-compactness have a version for this pseudocompact-like notion. The notion of convergence (or accumulation) of a sequence of *points* will be replaced with notions of limit points of a sequence of *subsets*.

Furthermore, in Section 5, we present variations which include covering properties equivalent to sequential compactness, sequential pseudocompactness, quasi M-compactness, the Menger property and the Rothberger property. Many other notions can be obtained as particular cases of Definition 5.7. Definition 5.7 probably deserves further study.

We assume no separation axiom, unless otherwise specified.

2. Equivalents of a covering property

In this section we show that, for every B and A as in Definition 1.1, there are many equivalent formulations of $[B, A]$-compactness. In particular, it can be characterized by a sort of accumulation property, in a sense which will be explicitly described in the next section. Parts of the results presented in this section are known for $[\mu, \lambda]$-compactness, hence, in particular, for countable compactness, Lindelöfness etc. They are new for (ordinal) $[\beta, \alpha]$-compactness, and for other general notions of compactness.

We begin with a trivial but useful fact.

Fact 2.1. *A topological space is $[B, A]$-compact if and only if, for every sequence $(C_a)_{a\in A}$ of closed sets, if $\bigcap_{a\in H} C_a \neq \emptyset$, for every $H \in B$, then $\bigcap_{a\in A} C_a \neq \emptyset$.*

Proof: Immediate from the definition of $[B, A]$-compactness, in contrapositive form, and by taking complements. □

We now state some other easy facts about $[B, A]$-compactness.

Fact 2.2. *Suppose that X is a topological space, A is a set, and $B, B' \subseteq \mathcal{P}(A)$.*

(1) *If $A \in B$, then every topological space is $[B, A]$-compact. In particular, every topological space is $[\{A\}, A]$-compact.*
(2) *If $B \subseteq B'$, and X is $[B, A]$-compact, then X is $[B', A]$-compact.*

(3) *More generally, if, for every* $H \in B$, *there is* $H' \in B'$ *such that* $H \subseteq H'$, *then every* $[B, A]$*-compact topological space is* $[B', A]$*-compact.*

(4) *If* X *is* $[B, A]$*-compact, and* $A' \subseteq A$, *then* X *is* $[B|_{A'}, A']$*-compact, where* $B|_{A'} = \{H \cap A' \mid H \in B\}$.

(5) *Suppose that, for every* $H \in B$, $D_H \subseteq \mathcal{P}(H)$, *and let* $D = \bigcup_{H \in B} D_H$. *If* X *is* $[B, A]$*-compact, and* $[D_H, H]$*-compact, for every* $H \in B$, *then* X *is* $[D, A]$*-compact.*

(6) *If* C *is a set,* $f : C \to A$ *is a function, and* $D = \{f^{-1}(H) \mid H \in B\}$, *then every* $[B, A]$*-compact topological space is* $[D, C]$*-compact.*

Fact (5) above is a broad generalization of standard arguments, e.g., the argument showing that Lindelöfness and countable compactness imply compactness.

Fact (6) follows immediately from the fact that a union of open sets is still open. Indeed, if $(O_c)_{c \in C}$ is an open cover of X, then $(Q_a)_{a \in A}$ is an open cover of X, where $Q_a = \bigcup_{f(c)=a} O_c$, for $a \in A$.

Remark 2.3. Let us say that $B \subseteq \mathcal{P}(A)$ is *closed under subsets* if and only if, whenever $H' \in B$ and $H \subseteq H'$, then $H \in B$. Notice that, by (2) and (3) above, if $B' \subseteq \mathcal{P}(A)$ and B is the smallest subset of $\mathcal{P}(A)$ which contains B' and is closed under subsets, then a topological space X is $[B, A]$-compact if and only if it is $[B', A]$-compact. Thus, in the definition of $[B, A]$-compactness, it is no loss of generality to consider only those B which are closed under subsets.

If X is a topological space, and $P \subseteq X$, we denote by $\overline{P}$ the closure of P in X, and by P° its interior. The topological space in which we are taking closure and interior will always be clear from the context.

If $B \subseteq \mathcal{P}(A)$ and $a \in A$, we let $a^{\lesssim}_B = \{H \in B \mid a \in H\}$.

Theorem 2.4. *Suppose that* A *is a set,* $B \subseteq \mathcal{P}(A)$, *and* X *is a topological space. Then the following conditions are equivalent.*

(1) X *is* $[B, A]$*-compact.*

(2) *For every sequence* $(P_a)_{a \in A}$ *of subsets of* X, *if, for every* $H \in B$, $\bigcap_{a \in H} P_a \neq \emptyset$, *then* $\bigcap_{a \in A} \overline{P}_a \neq \emptyset$.

(3) *Same as (2), with the further assumption that* $|P_a| \leq |a^{\lesssim}_B|$, *for every* $a \in A$.

(4) *For every sequence* $\{x_H \mid H \in B\}$ *of elements of* X, *it happens that* $\bigcap_{a \in A} \overline{\{x_H \mid H \in a^{\lesssim}_B\}} \neq \emptyset$.

(5) *For every sequence* $\{x_H \mid H \in B\}$ *of elements of* X, *there is* $x \in X$ *such that, for every neighborhood* U *of* x *in* X, *and for every* $a \in A$, *there is* $H \in B$ *such that* $a \in H$ *and* $x_H \in U$.

(6) *For every sequence* $\{Y_H \mid H \in B\}$ *of nonempty subsets of* X, *it happens that* $\bigcap_{a \in A} \overline{\bigcup \{Y_H \mid H \in a^{\lesssim}_B\}} \neq \emptyset$.

(7) *For every sequence* $\{D_H \mid H \in B\}$ *of nonempty closed subsets of* X, $\bigcap_{a \in A} \overline{\bigcup \{D_H \mid H \in a^{\lesssim}_B\}} \neq \emptyset$.

(8) *For every sequence* $\{O_H \mid H \in B\}$ *of open proper subsets of* X*, if, for every* $a \in A$*, we put* $Q_a = (\bigcap\{O_H \mid H \in a^{\lesssim}_B\})^\circ$*, then* $(Q_a)_{a\in A}$ *is not a cover of* X*.*

Proof: (1) ⇒ (2) Just take $C_a = \overline{P}_a$, for $a \in A$, and use the equivalent formulation of $[B, A]$-compactness in terms of closed sets, as given in Fact 2.1.

(2) ⇒ (3) is trivial.

(3) ⇒ (4) For $a \in A$, put $P_a = \{x_H \mid H \in a^{\lesssim}_B\}$. Thus $|P_a| \leq |a^{\lesssim}_B|$. Moreover, if $a \in H \in B$, then $x_H \in P_a$, hence $x_H \in \bigcap_{a\in H} P_a$, thus $\bigcap_{a\in H} P_a \neq \emptyset$. By applying (3), $\bigcap_{a\in A} \overline{P_a} = \bigcap_{a\in A} \overline{\{x_H \mid H \in a^{\lesssim}_B\}} \neq \emptyset$.

(5) is clearly a reformulation of (4), hence they are equivalent.

(4) trivially implies (6), since if, for every $H \in B$, we choose $x_H \in Y_H \neq \emptyset$, then $\bigcap_{a\in A} \overline{\bigcup\{Y_H \mid H \in a^{\lesssim}_B\}} \supseteq \bigcap_{a\in A} \overline{\{x_H \mid H \in a^{\lesssim}_B\}}$, and this latter set is nonempty by (4).

(6) ⇒ (7) is trivial, since (7) is a particular case of (6).

(7) ⇒ (1) We shall use the equivalent formulation of $[B, A]$-compactness given by Fact 2.1. Suppose that $(C_a)_{a\in A}$ are closed subsets of X such that $\bigcap_{a\in H} C_a \neq \emptyset$, for every $H \in B$. For each $H \in B$, put $D_H = \bigcap_{a\in H} C_a$, thus $C_a \supseteq D_H$, whenever $a \in H$, hence, for every $a \in A$, $C_a \supseteq \overline{\bigcup\{D_H \mid H \in a^{\lesssim}_B\}}$. By (7), $\bigcap_{a\in A} C_a \supseteq \bigcap_{a\in A} \overline{\bigcup\{D_H \mid H \in a^{\lesssim}_B\}} \neq \emptyset$.

(8) ⇔ (7) is immediate by taking complements. □

Notice that Conditions (6) and (7) can be reformulated in a way similar to the reformulation (5) of (4). As we shall explain in detail in Section 3, Condition (5) can be seen as a statement that asserts the existence of some kind of accumulation point for the sequence $\{x_H \mid H \in B\}$.

Remark 2.5. Some particular cases of Theorem 2.4 are known, sometimes being classical results.

As we mentioned in the introduction, countable compactness is the particular case of Definition 1.1 when A is countable (without loss of generality we can take $A = \omega$), and B is the set of all finite subsets of ω. It is easy to see that we can equivalently take $B = \{[0, n) \mid n \in \omega\}$; this follows, for example, from Remark 2.3. In a different context, a similar argument has been exploited in [18]; see in particular Remark 24 there. Remark 2.3 (and Fact 2.2(2)(3)) have further interesting applications which will be presented elsewhere.

Recall that, for an infinite cardinal λ, a topological space X satisfies CAP_λ if and only if every subset $Y \subseteq X$ with $|Y| = \lambda$ has a complete accumulation point x, that is, a point x such that $|U \cap Y| = \lambda$, for every neighborhood U of x.

For the above choice of $A = \omega$ and $B = \{[0, n) \mid n \in \omega\}$, the equivalence of (1) and (5) in Theorem 2.4 shows that countable compactness is equivalent to CAP_ω. This is because a sequence $(x_H)_{H\in B}$ can be thought as a sequence $(x_n)_{n\in\omega}$, via the obvious correspondence between B and ω. The astute reader will notice that the above argument (and Theorem 2.4, in general) deals with sequences, while the definition of CAP_ω deals with subsets; that is, in the former case, repetitions are allowed, while in the latter case they are not allowed. However, it is easy to see that, in the particular case at hand, the difference produces no substantial effect. See Remark 3.3 below and [17, Section 3] for further details.

Arguments similar to the above ones can be carried over, with no essential change, for every regular cardinal λ. In this case, we get that $[\lambda, \lambda]$-compactness is equivalent to CAP_λ. These results are very classical, and, indeed, are immediate consequences of [1, Section 9]. For λ singular, the characterization of $[\lambda, \lambda]$-compactness is not that neat. The point is that, for λ regular, a subset of λ cofinal in λ has necessarily cardinality λ; this is false when λ is singular.

We have discussed in some detail the equivalence between CAP_λ and $[\lambda, \lambda]$-compactness, for λ regular, since it might be seen as a prototype of all the results proved in the present paper. In fact, we establish an interplay between notions of compactness, on one hand, and satisfaction of accumulation properties, on the other hand. Such an interplay holds in very general situations, sometimes rather far removed from the above particular and nowadays standard example.

Turning to the more general notion of $[\mu, \lambda]$-compactness, the special case of the equivalence of (1) and (2) in Theorem 2.4 appears in [8, Theorem 1.1]. See [26, Lemma 5(b)]. For $[\mu, \lambda]$-compactness, Conditions (1)–(4) in Theorem 2.4 are the particular case of [18, Proposition 32(1)–(4)], taking $\mathcal{F}$ to be the set of all singletons of X. In the particular case $\mu = \omega$, $[\omega, \lambda]$-compactness is usually called *initial λ-compactness*. In this case there are much more characterizations: see [23, Section 2] and [26]. Some equivalences hold also for $\mu > \omega$, under additional assumptions. See [26, Theorem 2].

The equivalences in Theorem 2.4 have been inspired by results from Caicedo [4, Section 3], who implicitly uses similar methods in order to deal with $[\mu, \lambda]$-compactness. In our opinion, Caicedo [4] has provided an essentially new point of view about $[\mu, \lambda]$-compactness. Apart from [4], it is difficult to track back which parts of Theorem 2.4, in this particular case, have appeared in some form or another in the literature. This is due to the hidden assumption, used by many authors, of the regularity of some of the cardinals involved, or of some forms of the generalized continuum hypothesis. See [26].

Theorem 2.4 is new in the particular case of $[\beta, \alpha]$-compactness, for β and α ordinals. Since it was our leading motivation for working on such matters, we state explicitly the equivalence of (1) and (4) in Theorem 2.4 for this special case. We let $\mathcal{P}_\beta(\alpha)$ denote the set of all subsets of α having order type $< \beta$. Notice that this notation is consistent with the case introduced before when α and β are cardinals.

Corollary 2.6. *Suppose that X is a topological space and α and β are ordinals. Then the following conditions are equivalent.*

(1) *X is $[\beta, \alpha]$-compact.*
(2) *For every sequence $\{x_z \mid z \in \mathcal{P}_\beta(\alpha)\}$ of elements of X, if, for $\gamma \in \alpha$, we put $P_\gamma = \{x_z \mid z \in \mathcal{P}_\beta(\alpha) \text{ and } \gamma \in z\}$, then $\bigcap_{\gamma \in \alpha} \overline{P}_\gamma \neq \emptyset$.*

As we mentioned in the introduction, also D-compactness turns out to be equivalent to a covering property in the sense of Definition 1.1. More generally, many notions of being closed under convergence, or under taking particular kinds of accumulation points are equivalent to a covering property, as we shall show in the next section. Theorem 2.4 applies in each of the above cases.

As a final remark in this section, let us mention that Condition (5) in Theorem 2.4 suggests the following relativized notion of a cluster point of a net.

Definition 2.7. Suppose that $(\Sigma, \leq)$ is a directed set, and $(x_\sigma)_{\sigma \in \Sigma}$ is a net in a topological space X. If $T \subseteq \Sigma$, we say that $x \in X$ is a *cluster point restricted to* T of the net $(x_\sigma)_{\sigma \in \Sigma}$ if and only if for every $\tau \in T$ and every neighborhood U of x, there is $\sigma \in \Sigma$ such that $\sigma \geq \tau$ and $x_\sigma \in U$.

In fact, if $\Sigma = B \subseteq \mathcal{P}(A)$, $\leq$ is inclusion, and we suppose that B contains all singletons of $\mathcal{P}(A)$, then, in the terminology of Definition 2.7, Condition 2.4(5) asserts that every Σ-indexed net $(x_\sigma)_{\sigma \in \Sigma}$ has some cluster point restricted to the set of all singletons of $\mathcal{P}(A)$.

This might explain the difficulties in finding an equivalent formulation of $[\mu, \lambda]$-compactness in terms of cluster points of nets [26]. The condition in Definition 2.7 is generally weaker than the request for a cluster point: the definition of a *cluster point* of a net is obtained from 2.7 in the particular case when $T = \Sigma$ (or, more generally, when T is *cofinal* in Σ, that is, T is such that, for every $\sigma \in \Sigma$, there is $\sigma' \in T$ such that $\sigma \leq \sigma'$).

3. Every notion of accumulation (and more) is a covering property

An uncompromising way of defining a general notion of “accumulation point” is simply to fix some index set I, and to prescribe exactly which subsets of I are allowed to be the (index sets of) elements contained in the neighborhoods of some x — supposed to be an accumulation point of some I-indexed sequence.

Just to present the simplest nontrivial example, if I is infinite, and we allow all subsets of I with cardinality $|I|$, we get the notion of a complete accumulation point (for sequences all whose points are distinct).

To state it precisely, let us give the following definition.

Definition 3.1. Let I be a set, E be a subset of $\mathcal{P}(I)$, and $\mathbf{x} = (x_i)_{i \in I}$ be an I-indexed sequence of elements of some topological space X.

If $U \subseteq X$, let $I_{\mathbf{x},U} = \{i \in I \mid x_i \in U\}$. We say that a point $x \in X$ is an *accumulation point in the sense of* E, or simply an E*-accumulation point*, of the sequence $\mathbf{x}$ if and only if $I_{\mathbf{x},U} \in E$, for every open neighborhood U of x.

We say that X satisfies the E*-accumulation property* if and only if every I-indexed sequence of elements of X has some (not necessarily unique) accumulation point in the sense of E.

Remark 3.2. Trivially, if $E = \mathcal{P}(I)$, then every space satisfies the E-accumulation property. Under certain assumptions, we can get a smaller “minimal” E.

For every I-indexed sequence $\mathbf{x}$ of elements of X, and every $x \in X$, there is a smallest set $E \subseteq \mathcal{P}(I)$ such that x is an E-accumulation point of $\mathbf{x}$: just take $E = E_{\mathbf{x},x} = \{I_{\mathbf{x},U} \mid U$ an open neighborhood of $x\}$. Notice that $E_{\mathbf{x},x}$ is closed under finite intersections and arbitrary unions.

More generally, if Σ is a set of I-indexed sequences of elements of X and, for every $\mathbf{x} \in \Sigma$, $Y_{\mathbf{x}}$ is a subset of X, then $E = \bigcup\{E_{\mathbf{x},x} \mid \mathbf{x} \in \Sigma,\ x \in Y_{\mathbf{x}}\}$ is the smallest set E such that x is an E-accumulation point of $\mathbf{x}$, for every $\mathbf{x} \in \Sigma$ and $x \in Y_{\mathbf{x}}$. In other words, if we fix in advance some abstract relation of being an accumulation point of a sequence, then there is a minimal E which realizes this relation (of course, in general, E will realize many more instances of accumulation).

Remark 3.3. As we hinted before Definition 3.1, if I is infinite, and E is the set of all subsets of I of cardinality $|I|$, then the notion of an E-accumulation point corresponds to that of a complete accumulation point. There is a technical difference that should be mentioned: here we are dealing with sequences, rather than subsets. In order to get the standard definition of a complete accumulation point, we should require that all the elements of the sequence are distinct, otherwise some differences might occur. However, if $|I|$ is a regular cardinal, then a topological space satisfies $\mathrm{CAP}_{|I|}$ if and only if it satisfies the E-accumulation property, for the above E.

The whole matter has been discussed in detail in [17, Section 3], see in particular Remark 3.2 and Proposition 3.3 there, taking $\mathcal{F}$ to be the set of all singletons of X. We believe that, in general, dealing with sequences is the most natural way; for sure, it is the best way for our purposes here.

Remark 3.4. Definition 3.1 has some resemblance with the notion of filter convergence. However, we are not asking E to be necessarily a filter. This is because we want to include notions of accumulation and since, for example, in the case of complete accumulation points the corresponding E is not closed under intersection. Indeed, the intersection of two subsets of I of cardinality $|I|$ may have cardinality strictly smaller than $|I|$.

Of course, given some *fixed* sequence $(x_i)_{i\in I}$ and some *fixed* element $x \in X$, the topological relations between $(x_i)_{i\in I}$ and x are completely determined by the (possibly improper) filter F generated by the sets $\{i \in I \mid x_i \in U\}$, U varying among the neighborhoods of x in X. However, as the example of complete accumulation points shows, if we allow x vary, we get a more general (and useful) notion by considering an arbitrary subset E, rather than just a filter.

In this connection, however, see also Remark 5.4.

Definition 3.1 incorporates essentially all possible notions of “accumulation”. It captures also many notions of convergence. For example, a sequence $(x_n)_{n\in\omega}$ converges to x if and only if, for every neighborhood U of X, the set $\omega \setminus \{n \in \omega \mid x_n \in U\}$ is finite. In this case, $I = \omega$ and E consists of the cofinite subsets of ω. In a similar way, we can deal with convergence of transfinite sequences. Actually, even net convergence is a particular case of Definition 3.1. If $(\Sigma, \leq)$ is the directed set on which the net is built, then the net converges to x if and only if x is an

E-accumulation point in the sense of Definition 3.1 for the following choice of E. Take $I = \Sigma$ and let E be the set of all subsets of I which contain at least one set of the form $\sigma^<$, where, for $\sigma \in \Sigma$, we put $\sigma^< = \{\sigma' \in \Sigma \mid \sigma \leq \sigma'\}$. Of course, this is the usual argument showing that net convergence can be seen as an instance of filter convergence.

Definition 3.1 is more general. If, for a net as above, we take E to be the set of all subsets of Σ which are cofinal in Σ, then an E-accumulation point corresponds to a cluster point of the net. Also the notion of a restricted cluster point, as introduced in Definition 2.7, can be expressed in terms of E-accumulation, for some appropriate E.

If $E = D$ is an ultrafilter over I, then the existence of an E-accumulation point corresponds exactly to D-convergence.

It is rather astonishing that such a bunch of disparate notions turn out to be each equivalent to some covering property in the sense of Definition 1.1, as we shall show in Corollary 3.10 below.

Before embarking in the proof, we notice that also the converse holds, that is, every covering property is equivalent to some accumulation property. This is simply a reformulation, in terms of E-accumulation, of the equivalence (1) $\Leftrightarrow$ (5) in Theorem 2.4.

Corollary 3.5. *Suppose that X is a topological space, A is a set, $B \subseteq \mathcal{P}(A)$, and put $I = B$ and $E = \{Z \subseteq B \mid$ for every $a \in A$ there is $H \in Z$ such that $a \in H\} = \{Z \subseteq B \mid \bigcup Z = A\}$. Then the following conditions are equivalent.*

(1) *X is $[B, A]$-compact.*

(2) *X satisfies the E-accumulation property.*

Example 3.6. As in Remark 2.5, if $A = \lambda$ is a regular infinite cardinal, and $B = \{[0, \alpha) \mid \alpha < \lambda\}$, then the E given by Corollary 3.5 consists of all subsets of B of cardinality λ. In this particular case, Corollary 3.5 amounts exactly to the equivalence of $[\lambda, \lambda]$-compactness and CAP_λ.

Example 3.7. As another simple example, suppose that A is any set, and let $B = \{A \setminus \{a\} \mid a \in A\}$. For this choice of B, a topological space X is $[B, A]$-compact if and only if X has no irreducible cover of cardinality $|A|$. The E given by Corollary 3.5 in this situation is the set of all subsets of B which contain at least two elements from B. In this case, the failure of the E-accumulation property means that there exists an $|A|$-indexed sequence of elements of X such that every element of X has a neighborhood intersecting at most one element from the sequence. If X is T_1, this is equivalent to saying that X has a discrete closed subset of cardinality $|A|$.

In conclusion, in this particular case, Corollary 3.5 shows that a T_1 topological space has an irreducible cover of cardinality λ if and only if it has a discrete closed subset of cardinality λ. This is a classical result, implicit in the proof of [2, Theorem 2.1].

Now we are going to prove the promised converse of Corollary 3.5, namely, that every E-accumulation property in the sense of Definition 3.1 is equivalent to some covering property, under the reasonable hypothesis that E is closed under taking supersets.

Definition 3.8. If I is a set, and $E \subseteq \mathcal{P}(I)$, we let $E^+ = \{a \subseteq I \mid a \cap e \neq \emptyset$, for every $e \in E\}$.

We say that $E \subseteq \mathcal{P}(I)$ is *closed under supersets* if and only if, whenever $e \in E$ and $e \subseteq f \subseteq I$, then $f \in E$ (this is half the definition of a *filter*: if E is also closed under finite intersections, then it is a filter).

Trivially, for every E, we have that E^+ is closed under supersets. Moreover, it is easy to see that $E^{++} = E$ if and only if E is closed under supersets. Notice that if E is a filter, then E is an ultrafilter if and only if $E = E^+$.

If $A \subseteq \mathcal{P}(I)$, then, for every $i \in I$, we put $i_A^{\lessdot} = \{a \in A \mid i \in a\}$.

We can now state the main result of this section.

Theorem 3.9. *Suppose that X is a topological space, I is a set, $A \subseteq \mathcal{P}(I)$, and let $E = A^+$. Then the following conditions are equivalent.*

(1) *X satisfies the E-accumulation property.*
(2) *For every open cover $(O_a)_{a \in A}$ of X, there is $i \in I$ such that $(O_a)_{i \in a \in A}$ is a cover of X.*
(3) *X is $[B, A]$-compact, for $B = \{i_A^{\lessdot} \mid i \in I\}$.*
(4) *For every sequence $(x_i)_{i \in I}$ of elements of X, if, for each $a \in A$, we put $C_a = \overline{\{x_i \mid i \in a\}}$, then $\bigcap_{a \in A} C_a \neq \emptyset$.*

Before proving Theorem 3.9, we state its main corollary, and then we present a stronger local version for the equivalence of Conditions (1) and (4).

Corollary 3.10. *For every $E \subseteq \mathcal{P}(I)$ such that E is closed under supersets, there are $A \subseteq \mathcal{P}(I)$ and $B \subseteq \mathcal{P}(A)$ such that, for every topological space, the E-accumulation property is equivalent to $[B, A]$-compactness.*

Proof: If $E \subseteq \mathcal{P}(I)$ is closed under supersets, then $E = E^{++}$, hence, by taking $A = E^+$, we have $E = E^{++} = A^+$. Thus we get from Theorem 3.9 (1) $\Leftrightarrow$ (3) that, for every E closed under supersets, the E-accumulation property is equivalent to some compactness property in the sense of Definition 1.1. $\square$

Proposition 3.11. *Suppose that X is a topological space, $x \in X$, I is a set, and $(x_i)_{i \in I}$ is a sequence of elements of X. Suppose that $A \subseteq \mathcal{P}(I)$, $E = A^+$, and, for $a \in A$, put $C_a = \overline{\{x_i \mid i \in a\}}$. Then the following conditions are equivalent.*

(1) *x is an E-accumulation point of $(x_i)_{i \in I}$.*
(2) *$x \in \bigcap_{a \in A} C_a$.*

Proof: If (1) holds, then, for every open neighborhood U of x, the set $e_U = \{i \in I \mid x_i \in U\}$ belongs to E. We are going to show that $x \in \bigcap_{a \in A} C_a$.

Hence, suppose that $a \in A$. For every open neighborhood U of x, $a \cap e_U \neq \emptyset$, by the first statement, and the definition of E. This means that there is $i \in I$ such that $i \in a \cap e_U$, that is, $x_i \in C_a \cap U$, hence $C_a \cap U \neq \emptyset$. Since C_a is closed, and the above inequality holds for every open neighborhood U of x, then $x \in C_a$. Since this holds for every $a \in A$, we have $x \in \bigcap_{a \in A} C_a$.

Now assume that (2) holds. Suppose that U is a neighborhood of x, and let $e = \{i \in I \mid x_i \in U\}$. We have to show that $e \in E = A^+$, that is, $e \cap a \neq \emptyset$, for every $a \in A$. Let us fix $a \in A$. By (2), $x \in C_a$ and, by the definition of C_a, there is $i \in a$ such that $x_i \in U$. By the definition of e, $i \in e$, thus $i \in e \cap a \neq \emptyset$. Since this argument works for every neighborhood U of x, we have proved (1). □

The particular case of Proposition 3.11 in which x is a cluster point of some net is Exercise 1.6.A in [7]. Cf. also [5, IV], and Remark 4.7 below.

Proof of Theorem 3.9: (2) ⇔ (3) is immediate from the definitions.

(3) ⇔ (4) is a particular case of Theorem 2.4 (1) ⇔ (4). Indeed, in the situation at hand, members of B have the form $H = i^{\leq}_A$, for $i \in I$. For such an H, we have that $H \in a^{\leq}_B$ if and only if $a \in H = i^{\leq}_A$ if and only if $i \in a$, thus Condition (4) in Theorem 2.4 reads exactly as Condition (4) in Theorem 3.9.

(1) ⇔ (4) is immediate from Proposition 3.11.

Alternatively, the proof of 3.9 can be completed avoiding the use of Proposition 3.11, and using Corollary 3.5 in order to prove (1) ⇔ (3). Indeed, under the respective assumptions, and modulo the obvious correspondence between I and $B = \{i^{\leq}_A \mid i \in I\}$, the E given by the statement of 3.5 corresponds exactly to the E given by the statement of 3.9. To check this, let $I' = B$ and, for $e \subseteq I$, let $e' \subseteq I'$ be defined by $e' = \{i^{\leq}_A \mid i \in e\}$. Applying Corollary 3.5 to I', the resulting E' turns out to be equal to $\{e' \subseteq I' \mid$ for every $a \in A$, there is $i \in I$ such that $i^{\leq}_A \in e'$ and $i \in a\} = \{e' \subseteq I' \mid e \cap a \neq \emptyset$, for every $a \in A\} = \{e' \mid e \in E\}$. Corollary 3.5 thus shows that $[B, A]$-compactness is equivalent to the E'-accumulation property, which, through the above mentioned correspondence, is trivially equivalent to the E-accumulation property. □

Remark 3.12. If D is an ultrafilter over I, then, by taking $A = D$ in Theorem 3.9, the equivalence of (1) and (2) furnishes a proof of Proposition 1.3, since, for D an ultrafilter, we have that $D^+ = D$.

In [18, Proposition 17] we also proved a characterization of D-pseudocompactness analogous to Proposition 1.3. The methods of Sections 2 and 3 do apply also in case of notions related to pseudocompactness. We shall devote the next section to this endeavor. Before proceeding, we show that Theorem 3.9 furnishes a characterization of weak M-compactness.

Definition 3.13. If M is a set of ultrafilters over some set I, a topological space is said to be *weakly M-compact* if and only if, for every sequence $(x_i)_{i \in I}$ of elements of X, there is $x \in X$ such that, for every neighborhood U of x, there is $D \in M$ such that $\{i \in I \mid x_i \in U\}$. See [9] for more information, credits, references and a characterization. In the terminology of Definition 3.1, X is weakly M-compact if and only if it satisfies the E-accumulation property, for $E = \bigcup_{D \in M} D$.

Corollary 3.14. *Suppose that X is a topological space, M is a set of ultrafilters over I, and let $F = \bigcap_{D \in M} D$. Then the following conditions are equivalent.*

(1) *X is weakly M-compact.*

(2) *For every open cover $(O_Z)_{Z \in F}$ of X, there is some $i \in I$ such that $(O_Z)_{i \in Z \in F}$ is a cover of X.*

Proof: By Theorem 3.9, taking $A = F$, and noticing that $E = A^+ = \bigcup_{D \in M} D$. □

4. Pseudocompactness and the like

Definitions 1.1 and 3.1 can be generalized in the setting presented in [17], [18]; in particular, in such a way that incorporates pseudocompact-like notions.

Let us fix a family $\mathcal{F}$ of subsets of a topological space X. The most interesting case will be when $\mathcal{F} = \mathcal{O}$ is the family of all the nonempty open sets of X. At first reading, the reader might want to consider this particular case only.

We relativize Definitions 1.1 and 3.1 to $\mathcal{F}$. The notion of $[B, A]$-compactness is modified by replacing the conclusion with the requirement that the union of the elements of an appropriate subsequence intersects ever member of $\mathcal{F}$. As far as notions of accumulation are concerned, instead of considering accumulation points of elements, we shall now consider limit points of sequences of elements of $\mathcal{F}$.

The two most significant cases are when $\mathcal{F}$ is the family of all singletons of X, in which case we get back the definitions and results of Sections 2 and 3, and, as we mentioned, when $\mathcal{F} = \mathcal{O}$ is the family of all the nonempty open sets of X, in which case we get notions and results related to pseudocompactness or variants of pseudocompactness.

Definition 4.1. If A is a set, $B \subseteq \mathcal{P}(A)$, X is a topological space, and $\mathcal{F}$ is a family of subsets of X, we say that X is $\mathcal{F}$-$[B, A]$-*compact* if and only if one of the following equivalent conditions holds.

(1) For every open cover $(O_a)_{a \in A}$ of X, there is $H \in B$ such that $\bigcup_{a \in H} O_a$ intersects every member of $\mathcal{F}$ (that is, for every $F \in \mathcal{F}$, there is $a \in H$ such that $O_a \cap F \neq \emptyset$).

(2) For every sequence $(C_a)_{a \in A}$ of closed subsets of X, if, for every $H \in B$, there exists $F \in \mathcal{F}$ such that $\bigcap_{a \in H} C_a \supseteq F$, then $\bigcap_{a \in A} C_a \neq \emptyset$.

The equivalence of the above conditions is trivial, by taking complements.

Notice that, in the particular case when $\mathcal{F} = \mathcal{O}$, the conclusion in Definition 4.1(1) asserts that $\bigcup_{a \in A} O_a$ is dense in X.

Definition 4.2. Let I be a set, E be a subset of $\mathcal{P}(I)$, and $(F_i)_{i \in I}$ be an I-indexed sequence of subsets of some topological space X.

We say that a point $x \in X$ is a *limit point in the sense of* E, or simply an E*-limit point*, of the sequence $(F_i)_{i \in I}$ if and only if, for every open neighborhood U of x, the set $\{i \in I \mid F_i \cap U \neq \emptyset\}$ belongs to E.

If $\mathcal{F}$ is a family of subsets of X, we say that X satisfies the $\mathcal{F}$*-*E*-accumulation property* if and only if every I-indexed sequence of elements of $\mathcal{F}$ has some limit point in the sense of E.

In the particular case when $\mathcal{F}$ is the family of all singletons of X Definitions 4.1 and 4.2 reduce to Definitions 1.1 and 3.1, respectively.

As in Remark 3.2, if $E = \mathcal{P}(I)$, then every space satisfies the $\mathcal{F}$-E-accumulation property, for every $\mathcal{F}$.

More generally, for every sequence $(F_i)_{i \in I}$ of subsets of X, and every $x \in X$, there is a smallest set $E \subseteq \mathcal{P}(I)$ such that x is an E-limit point of $(F_i)_{i \in I}$: just take $E = \{I_U \mid U$ an open neighborhood of $x\}$, where $I_U = \{i \in I \mid F_i \cap U \neq \emptyset\}$. In the same way, and exactly as in Remark 3.2, for every family of I-indexed sequences, and respective families of elements of X, there is the smallest E such that each element in the family is a limit point of the corresponding sequence.

Remark 4.3. If $\mathcal{F}$ is a family of subsets of some topological space X, let $\overline{\mathcal{F}}$ denote the set of all closures of elements of $\mathcal{F}$.

If $\mathcal{G}$ is another family of subsets of X, let us write $\mathcal{F} \rhd \mathcal{G}$ to mean that, for every $F \in \mathcal{F}$, there is $G \in \mathcal{G}$ such that $F \supseteq G$. We write $\mathcal{F} \equiv \mathcal{G}$ to mean that both $\mathcal{F} \rhd \mathcal{G}$ and $\mathcal{G} \rhd \mathcal{F}$.

It is trivial to see that, in Definitions 4.1 and 4.2, as well as in the theorems below, we get equivalent conditions if we replace $\mathcal{F}$ either by $\overline{\mathcal{F}}$, or by $\mathcal{G}$, in case $\mathcal{F} \equiv \mathcal{G}$ (in this latter case, as far as Definition 4.2 is concerned, the condition turns out to be equivalent provided we assume that E is closed under supersets).

In particular, when $\mathcal{F} = \mathcal{O}$, we get equivalent definitions and results if we replace $\mathcal{O}$ by either

(1) the set $\mathcal{B}$ of the nonempty elements of some fixed base of X, or
(2) the set $\overline{\mathcal{O}}$ of all nonempty regular closed subsets of X, or
(3) the set $\overline{\mathcal{B}}$ of the closures of the nonempty elements of some base of X, or
(4) the set $\mathcal{R}$ of all nonempty regular open subsets of X (since $\overline{\mathcal{R}} = \overline{\mathcal{O}}$).

The connection of Definitions 4.1 and 4.2 with pseudocompactness goes as follows. A Tychonoff space X is pseudocompact if and only if every countable open cover of X has a finite subcollection whose union is dense in X. This is Condition (C_5) in [24], and corresponds to the particular case $A = \omega$, $B = \mathcal{P}_\omega(\omega)$ of $\mathcal{O}$-$[B, A]$-compactness, in the sense of Definition 4.1.

As another characterization of pseudocompactness, Glicksberg [13] proved that a Tychonoff space X is pseudocompact if and only if the following condition holds:

(*) for every sequence of nonempty open sets of X, there is some point $x \in X$ such that each neighborhood of x intersects infinitely many elements of the sequence.

This corresponds to the particular case of Definition 4.2 in which $\mathcal{F} = \mathcal{O}$, $I = \omega$ and E equals the set of all infinite subsets of ω. Actually, as a very particular case of Theorem 4.4 (1) $\Leftrightarrow$ (5) below, and arguing as in Remark 2.5, we get another proof of Glicksberg result, in the sense that we get a proof that (*) and (C_5) above are equivalent, for every topological space (no separation axiom assumed).

The situation is entirely parallel to the characterization of countable compactness, which is equivalent to CAP_ω, as discussed in detail in Remark 2.5. Indeed, conditions analogous to (*) and (C_5) above are still equivalent when ω is replaced by any infinite regular cardinal; see [17, Theorem 4.4] for exact statements. This kind of analogies, together with many generalizations, had been the main theme of [17], [18]. In the present paper we show that such analogies can be carried over much further.

The connections between covering properties and general accumulation properties, as described in Section 3, do hold even in the extended setting we are now considering. In other words, the relationships between the properties introduced in Definitions 1.1 and 3.1 are exactly the same as the relationships between the properties introduced in Definitions 4.1 and 4.2. This will be stated in Theorem 4.5.

We first state the result analogous to Theorem 2.4 (and Corollary 3.5).

Theorem 4.4. *Suppose that A is a set, $B \subseteq \mathcal{P}(A)$, X is a topological space, and $\mathcal{F}$ is a family of subsets of X. Then the following conditions are equivalent.*

(1) *X is $\mathcal{F}$-$[B, A]$-compact.*
(2) *For every sequence $(P_a)_{a \in A}$ of subsets of X, if, for every $H \in B$, there exists $F \in \mathcal{F}$ such that $\bigcap_{a \in H} P_a \supseteq F$, then $\bigcap_{a \in A} \overline{P_a} \neq \emptyset$.*
(3) *Same as (2), with the further assumption that, for every $a \in A$, P_a is the union of $\leq \kappa_a$-many elements of $\mathcal{F}$, where $\kappa_a = |a^{\lesssim}_B|$.*
(4) *For every sequence $\{F_H \mid H \in B\}$ of elements of $\mathcal{F}$, it happens that $\bigcap_{a \in A} \overline{\bigcup \{F_H \mid H \in a^{\lesssim}_B\}} \neq \emptyset$.*
(5) *For every sequence $\{F_H \mid H \in B\}$ of elements of $\mathcal{F}$, there is $x \in X$ such that, for every neighborhood U of x in X, and for every $a \in A$, there is $H \in B$ such that $a \in H$ and $F_H \cap U \neq \emptyset$.*
(6) *For every sequence $\{Y_H \mid H \in B\}$ of subsets of X such that each Y_H contains some $F_H \in \mathcal{F}$, $\bigcap_{a \in A} \overline{\bigcup \{Y_H \mid H \in a^{\lesssim}_B\}} \neq \emptyset$.*
(7) *For every sequence $\{D_H \mid H \in B\}$ of closed subsets of X such that each D_H contains some $F_H \in \mathcal{F}$, it happens that $\bigcap_{a \in A} \overline{\bigcup \{D_H \mid H \in a^{\lesssim}_B\}} \neq \emptyset$.*
(8) *For every sequence $\{O_H \mid H \in B\}$ of open subsets of X such that, for each $H \in B$, there is $F_H \in \mathcal{F}$ disjoint from O_H, if, for every $a \in A$, we put $Q_a = (\bigcap \{O_H \mid H \in a^{\lesssim}_B\})^\circ$, then $(Q_a)_{a \in A}$ is not a cover of X.*
(9) *X satisfies the $\mathcal{F}$-E-accumulation property, for $I = B$ and $E = \{Z \subseteq B \mid$ for every $a \in A$ there is $H \in Z$ such that $a \in H\}$.*

In each case, we get equivalent conditions by replacing $\mathcal{F}$ with either $\overline{\mathcal{F}}$, or $\mathcal{G}$, in case $\mathcal{F} \equiv \mathcal{G}$.

Proof: The proof is similar to the proof of Theorem 2.4. Cf. also parts of the proof of [18, Proposition 6].

It is not obvious that we get equivalent statements for all conditions, when $\mathcal{F}$ is replaced by $\overline{\mathcal{F}}$, or by $\mathcal{G}$, when $\mathcal{F} \equiv \mathcal{G}$. However, this is true for, say, Condition (1), and the proof of the equivalences of (1)–(9) works for an arbitrary family. □

As a simple example of the equivalence of (1) and (9), and arguing as in Example 3.7, a topological space X has an open cover of cardinality λ with no proper dense subfamily if and only if X contains a discrete family of λ open sets.

We now state the results corresponding to those in Section 3. There is no essential difference in proofs.

Theorem 4.5. *Suppose that X is a topological space, $\mathcal{F}$ is a family of subsets of X, I is a set, $A \subseteq \mathcal{P}(I)$ and $E = A^+$. Then the following conditions are equivalent.*

(1) *X satisfies the $\mathcal{F}$-E-accumulation property.*
(2) *For every sequence $(C_a)_{a\in A}$ of closed subsets of X, if, for every $i \in I$, there exists $F \in \mathcal{F}$ such that $\bigcap_{i\in a\in A} C_a \supseteq F$, then $\bigcap_{a\in A} C_a \neq \emptyset$.*
(3) *X is $\mathcal{F}$-$[B, A]$-compact, where $B = \{i^{<}_{A} \mid i \in I\}$.*
(4) *For every sequence $(F_i)_{i\in I}$ of elements in $\mathcal{F}$, if, for each $a \in A$, we put $C_a = \overline{\bigcup_{i\in a} F_i}$, then $\bigcap_{a\in A} C_a \neq \emptyset$.*

In each case, we get equivalent conditions by replacing $\mathcal{F}$ with either $\overline{\mathcal{F}}$, or $\mathcal{G}$, in case $\mathcal{F} \equiv \mathcal{G}$.

We state explicitly also the analogue of Proposition 3.11, since it does not follow formally from Theorem 4.5.

Proposition 4.6. *Suppose that X is a topological space, $x \in X$, I is a set, and $(F_i)_{i\in I}$ is a sequence of subsets of X. Suppose that $A \subseteq \mathcal{P}(I)$, $E = A^+$, and, for $a \in A$, put $C_a = \overline{\bigcup_{i\in a} F_i}$. Then the following conditions are equivalent.*

(1) *x is an E-limit point of $(F_i)_{i\in I}$.*
(2) *$x \in \bigcap_{a\in A} C_a$.*

Remark 4.7. A version of Proposition 4.6 appears in [5, IV], using different terminology and notations, and possibly with a misprint. Proposition 4.6 appears to be slightly more general, since E does not necessarily become a filter (cf. Remark 3.4).

As an example, Theorem 4.5 can be applied to notions related to ultrafilter convergence, in particular, to D-pseudocompactness.

Definition 4.8. Let D be an ultrafilter over some set I, X be a topological space, and $\mathcal{F}$ be a family of subsets of X.

We say [17, Definition 2.1] that X is $\mathcal{F}$-D-*compact* if and only if every sequence $(F_i)_{i\in I}$ of members of $\mathcal{F}$ has some D-limit point in X.

In case $\mathcal{F}$ is the set of all singletons of X, we get back the notion of D-compactness. In case $\mathcal{F} = \mathcal{O}$ we get the notion of D-pseudocompactness, as introduced in [12], [11].

Corollary 4.9 ([18, Proposition 33]). *Suppose that X is a topological space, $\mathcal{F}$ is a family of subsets of X, and D is an ultrafilter over some set I. Then the following are equivalent.*

(1) *X is $\mathcal{F}$-D-compact.*
(2) *For every sequence $\{F_i \mid i \in I\}$ of members of $\mathcal{F}$, if, for $Z \in D$, we put $C_Z = \overline{\bigcup_{i\in Z} F_i}$, then we have that $\bigcap_{Z\in D} C_Z \neq \emptyset$.*
(3) *Whenever $(C_Z)_{Z\in D}$ is a sequence of closed sets of X with the property that, for every $i \in I$, there exists some $F \in \mathcal{F}$ such that $\bigcap_{i\in Z} C_Z \supseteq F$, then $\bigcap_{Z\in D} C_Z \neq \emptyset$.*
(4) *For every open cover $(O_Z)_{Z\in D}$ of X, there is some $i \in I$ such that $F \cap \bigcup_{i\in Z} O_Z \neq \emptyset$, for every $F \in \mathcal{F}$.*

In the particular case $\mathcal{F} = \mathcal{O}$, Corollary 4.9 provides a characterization of D-pseudocompactness parallel to the characterization of D-compactness given in Proposition 1.3. This characterization of D-pseudocompactness had been explicitly stated with a direct proof in [18, Proposition 17]. Also Corollary 3.14 can be generalized without difficulty. We leave this to the reader.

Of course, all the results of Sections 2 and 3, in particular, Theorems 2.4 and 3.9, could be obtained as particular cases of the results in the present section, by taking $\mathcal{F}$ to be the set of all singletons of X. In principle, we could have first proved Theorems 4.4 and 4.5, and then obtain Theorems 2.4 and 3.9 as corollaries. We have chosen the other way for easiness of presentation, and since already Sections 2 and 3 appear to be abstract enough. Probably, there are more readers (if any at all!) interested in Theorems 2.4 and 3.9 rather than in Theorems 4.4 and 4.5 in such a generality.

However, the particular case $\mathcal{F} = \mathcal{O}$ in the results of the present section appears to be of interest. We stated the results in the general $\mathcal{F}$-dependent form for three reasons. First, to point out that, even if it is possible that the results are particularly interesting only in the case $\mathcal{F} = \mathcal{O}$, nevertheless almost nowhere we made use of the specific form of the members of $\mathcal{O}$. Second, since it is not always trivial that we can equivalently replace $\mathcal{O}$ with anyone of the families (1)–(4) of Remark 4.3. The general form of our statements thus provides many equivalences at the same time. The third reason for stating the theorems in the $\mathcal{F}$-form is to make clear that there is absolutely no difference, in the proofs and in the arguments, with the case dealt in the preceding sections, that is, when dealing with sequences of points, rather than general subsets. In fact, the statements of Theorems 4.4 and 4.5 unify the two cases. This is similar to what we have done in [17]; indeed, some results of [17] can be obtained as corollaries of results proved here.

Of course, the possibility is left open for interesting applications of Theorems 4.4 and 4.5 in other cases, besides the cases of singletons and of nonempty open sets.

5. Notions related to sequential compactness

Sequential compactness is not a particular case of Definition 3.1. However, Definition 3.1 can be modified in order to include also notions such as sequential compactness. The results in Sections 3 and 4 generalize even to this situation.

Definition 5.1. Suppose that I is a set, $\mathcal{E}$ is a set of subsets of $\mathcal{P}(I)$, and X is a topological space.

(1) If $(x_i)_{i\in I}$ is a sequence of elements of X, we say that $x \in X$ is an *$\mathcal{E}$-accumulation point* of $(x_i)_{i\in I}$ if and only if there is $E \in \mathcal{E}$ such that x is an E-accumulation point of $(x_i)_{i\in I}$ (in the sense of Definition 3.1).

We say that X satisfies the *$\mathcal{E}$-accumulation property* if and only if every I-indexed sequence of elements of X has some $\mathcal{E}$-accumulation point.

(2) If $(F_i)_{i\in I}$ is an I-indexed sequence of subsets of X, we say that a point $x \in X$ is an *$\mathcal{E}$-limit point* of the sequence $(F_i)_{i\in I}$ if and only if, for some $E \in \mathcal{E}$, x is an E-limit point of $(F_i)_{i\in I}$ (cf. Definition 4.2).

If $\mathcal{F}$ is a family of subsets of X, we say that X satisfies the *$\mathcal{F}$-$\mathcal{E}$-accumulation property* if and only if every I-indexed sequence of elements of $\mathcal{F}$ has some $\mathcal{E}$-limit point.

Case (1) in Definition 5.1 is the particular case of (2) when $\mathcal{F}$ is taken to be the set of all singletons of X.

When $\mathcal{E} = \{E\}$ has just one member, Definitions 5.1(1)(2) reduce to Definitions 3.1 and 4.2, respectively.

Remark 5.2. Notice that if in the second statement in Definition 5.1(1) we take $I = \omega$ and we let $\mathcal{E}$ be the set of all nonprincipal ultrafilters over ω, we get still another equivalent formulation of countable compactness. This is the reformulation of a nowadays standard fact (see, e.g., [12]). The equivalence follows also from Remark 5.4 below, and the fact (Remark 2.5) that countable compactness is equivalent to CAP_ω. More generally, if λ is regular, and in Definition 5.1(1) we take $I = \lambda$ and $\mathcal{E}$ the set of all uniform ultrafilters over λ, we get an equivalent formulation of $[\lambda, \lambda]$-compactness, equivalently, of CAP_λ.

We now show how to get the definition of sequential compactness as a particular case of Definition 5.1(1).

Definitions 5.3. As usual, if $W \subseteq \omega$ is infinite, we let $[W]^\omega$ denote the set of all infinite subsets of W. If $Z \in [\omega]^\omega$, we let $F_Z = \{W \subseteq \omega \mid |Z \setminus W| \text{ is finite}\}$, that is, F_Z is the filter on ω generated by the Frechet filter on Z.

We now get *sequential compactness* if in Definition 5.1(1) we take $I = \omega$, and $\mathcal{E} = \{F_Z \mid Z \in [\omega]^\omega\}$.

With the above choice of I and $\mathcal{E}$, and taking $\mathcal{F} = \mathcal{O}$ in 5.1(2) (that is, considering sequences $(O_i)_{i\in I}$ of nonempty open sets of X), we get a notion called *sequential pseudocompactness* in [3], and *sequential feeble compactness* in [6]. Notice that in [3] the O_i's are requested to be pairwise disjoint; however, it can be shown [20] that we get equivalent definitions, whether or not we suppose the O_i's to be disjoint.

Remark 5.4. Suppose that each element of $\mathcal{E}$ is closed under supersets, and let $\mathcal{E}' = \{F \subseteq \mathcal{P}(I) \mid F$ is a filter on I and $F \subseteq E$, for some $E \in \mathcal{E}\}$. Then some point x is an $\mathcal{E}$-accumulation point of some sequence $\mathbf{x} = (x_i)_{i\in I}$ if and only if x is an $\mathcal{E}'$-accumulation point of $\mathbf{x}$. Indeed, $\mathcal{E}'$-accumulation trivially implies $\mathcal{E}$-accumulation. On the other direction, if x is an $\mathcal{E}$-accumulation point of $\mathbf{x}$, then there is $E \in \mathcal{E}$ such that $I_{\mathbf{x},U} = \{i \in I \mid x_i \in U\} \in E$, for every open neighborhood U of x. If F is the filter generated by $G = \{I_{\mathbf{x},U} \mid U$ is an open neighborhood of $x\}$, then $F \subseteq E$, since G is closed under intersection, and E is closed under supersets. Thus $F \in \mathcal{E}'$, and F witnesses that x is an $\mathcal{E}'$-accumulation point of $\mathbf{x}$ (cf. also Remarks 3.2 and 3.4).

In particular, under the above assumptions on $\mathcal{E}$ and $\mathcal{E}'$, a topological space satisfies the $\mathcal{E}$-accumulation property if and only if it satisfies the $\mathcal{E}'$-accumulation property. Thus, in contrast with Remark 3.4, and as far as Definition 5.1 is concerned, it is no loss of generality to assume that all members of $\mathcal{E}$ are filters. Of course, the above observation applies only in case we are not concerned with the cardinality of $\mathcal{E}$, since, in the above situation, the cardinality of $\mathcal{E}'$ is generally strictly larger than the cardinality of $\mathcal{E}$.

Notice that the above argument carries over even when we consider $\mathcal{E}'' = \{F \subseteq \mathcal{P}(I) \mid F$ is a filter on I and, for some $E \in \mathcal{E}$, $F \subseteq E$ and F is maximal among the filters contained in $E\}$ (because every filter $F \subseteq E$ can be extended to a maximal filter with this property, using Zorn's Lemma). Sometimes this turns out to be useful.

We now introduce the generalization of Definitions 1.1 and 4.1 which furnishes the equivalent of Definition 5.1 in terms of properties of open covers.

Definition 5.5. Suppose that A is a set, $B, G \subseteq \mathcal{P}(A)$, and X is a topological space.

(1) We say that X is $[B, G]$-*compact* if and only if one of the following equivalent conditions hold.
   (a) If $(O_a)_{a\in A}$ are open sets of X, and, for every $K \in G$, $(O_a)_{a\in K}$ is a cover of X, then there is $H \in B$ such that $(O_a)_{a\in H}$ is a cover of X,
   (b) If $(C_a)_{a\in A}$ is a sequence of closed subsets of X, and, for every $H \in B$, $\bigcap_{a\in H} C_a \neq \emptyset$, then there is $K \in G$ such that $\bigcap_{a\in K} C_a \neq \emptyset$.

(2) If $\mathcal{F}$ is a family of subsets of X, we say that X is $\mathcal{F}$-$[B, G]$-*compact* if and only if one of the following equivalent conditions hold.

(a) If $(O_a)_{a\in A}$ are open sets of X, and, for every $K \in G$, $(O_a)_{a\in K}$ is a cover of X, then there is $H \in B$ such that, for every $F \in \mathcal{F}$, there is $a \in H$ such that $O_a \cap F \neq \emptyset$.

(b) If $(C_a)_{a\in A}$ are closed sets of X, and, for every $H \in B$, there exists $F \in \mathcal{F}$ such that $\bigcap_{a\in H} C_a \supseteq F$, then there is $K \in G$ such that $\bigcap_{a\in K} C_a \neq \emptyset$.

Case (1) in Definition 5.5 is the particular case of (2) when $\mathcal{F}$ is taken to be the set of all singletons of X.

Definitions 1.1 and 4.1 are the particular cases of the above definition when $G = \{A\}$.

Remark 5.6. Some known notions are particular cases of $[B, G]$-compactness, as introduced in Definition 5.5.

Indeed, in the particular case when G is a partition of A, say into κ classes, the hypothesis in Condition (1)(a) of Definition 5.5 amounts exactly to considering a family of κ open covers of X, each cover having the same cardinality as the corresponding class. In the rest of this remark we shall deal only with the particular case when A is countable and G is a partition of A into ω-many classes, each class having cardinality ω.

If, under the above assumptions, we let B consist of all subsets of A such that B has finite intersection with each element of G, then Condition (1)(a) in Definition 5.5 asserts that, given a countable family of countable covers of X, we can extract a cover of X by selecting a finite number of elements from each one of the original covers. This property turns out to be equivalent to what nowadays is called the *Menger property*, and is denoted by $\mathrm{S}_{\mathrm{fin}}(\mathcal{O}, \mathcal{O})$ in [21, Section 5] (here we are following the notations from [21], and $\mathcal{O}$ denotes the collection of all open covers of X).

On the other hand, if B consists of all subsets of A such that B intersects each element of G in exactly one element, we get the *Rothberger property*, denoted by $\mathrm{S}_1(\mathcal{O}, \mathcal{O})$ in [21, Section 6].

The connections between Definition 5.5 and the notions introduced in [21] probably deserve further analysis. Notice that here we put no restriction on covers, while [21] also deals with special classes of covers, such as *large covers*, *ω-covers* and so on. One probably gets interesting notions modifying Definitions 1.1, 5.5 etc., by putting restrictions on the nature of the starting cover and of the resulting subcover. This suggests the next definition.

Definition 5.7. Suppose that A is a set, $B, G \subseteq \mathcal{P}(A)$, X is a topological space, and $\mathcal{A}$, $\mathcal{B}$ are collections of subsets of X.

X is $[B_{\mathcal{B}}, G_{\mathcal{A}}]$-*compact* (*feebly* $[B_{\mathcal{B}}, G_{\mathcal{A}}]$-*compact*, respectively) if and only if whenever $(O_a)_{a\in A}$ are subsets of X, and, for every $K \in G$, $(O_a)_{a\in K}$ is a cover in $\mathcal{A}$, then there is $H \in B$ such that $(O_a)_{a\in H}$ is a cover in $\mathcal{B}$ ($(O_a)_{a\in H}$ is in $\mathcal{B}$ and its union is dense in X, respectively).

Arguing as in Remark 5.6, the properties $\mathrm{S_{fin}}(\mathcal{A},\mathcal{B})$ and $\mathrm{S_1}(\mathcal{A},\mathcal{B})$ from [21] are particular cases of Definition 5.7.

The particular case of Definition 5.5 in which $A=\lambda$, G is the set of subsets of λ of cardinality λ, and $B=\mathcal{P}_\kappa(\lambda)$ has been briefly hinted on [16, p. 1380] under the name *almost* $[\kappa,\lambda]$*-compactness*.

In the next theorems we give the connections between the notions introduced in Definitions 5.1 and 5.5.

Recall the definition of $a^{\lesssim}_B$ given just before Theorem 2.4.

Theorem 5.8. *Suppose that A is a set, $B,G\subseteq\mathcal{P}(A)$, and X is a topological space. Then the following conditions are equivalent.*

(1) *X is $[B,G]$-compact.*

(2) *For every sequence $(P_a)_{a\in A}$ of subsets of X, if, for every $H\in B$, $\bigcap_{a\in H}P_a\neq\emptyset$, then there is $K\in G$ such that $\bigcap_{a\in K}\overline{P}_a\neq\emptyset$.*

(3) *For every sequence $\{x_H \mid H\in B\}$ of elements of X, there is $K\in G$ such that $\bigcap_{a\in K}\overline{\{x_H \mid H\in a^{\lesssim}_B\}}\neq\emptyset$.*

(4) *For every sequence $\{Y_H \mid H\in B\}$ of nonempty subsets of X, there is $K\in G$ such that $\bigcap_{a\in K}\overline{\bigcup\{Y_H \mid H\in a^{\lesssim}_B\}}\neq\emptyset$.*

(5) *X satisfies the $\mathcal{E}$-accumulation property, for $I=B$ and $\mathcal{E}=\{E_K \mid K\in G\}$ where, for $K\in G$, we put $E_K=\{Z\subseteq B \mid$ for every $a\in K$ there is $H\in Z$ such that $a\in H\}$.*

Theorem 5.9. *Suppose that A is a set, $B,G\subseteq\mathcal{P}(A)$, X is a topological space, and $\mathcal{F}$ is a family of subsets of X. Then the following conditions are equivalent.*

(1) *X is $\mathcal{F}$-$[B,G]$-compact.*

(2) *For every sequence $(P_a)_{a\in A}$ of subsets of X, if, for every $H\in B$, there exists $F\in\mathcal{F}$ such that $\bigcap_{a\in H}P_a\supseteq F$, then there is $K\in G$ such that $\bigcap_{a\in K}\overline{P}_a\neq\emptyset$.*

(3) *For every sequence $\{F_H \mid H\in B\}$ of elements of $\mathcal{F}$, there is $K\in G$ such that $\bigcap_{a\in K}\overline{\bigcup\{F_H \mid H\in a^{\lesssim}_B\}}\neq\emptyset$.*

(4) *For every sequence $\{Y_H \mid H\in B\}$ of subsets of X such that each Y_H contains some $F_H\in\mathcal{F}$, there is $K\in G$ such that $\bigcap_{a\in K}\overline{\bigcup\{Y_H \mid H\in a^{\lesssim}_B\}}\neq\emptyset$.*

(5) *X satisfies the $\mathcal{F}$-$\mathcal{E}$-accumulation property, for I and $\mathcal{E}$ as in Condition 5.8(5) above.*

When $G=\{A\}$, the conditions in Theorems 5.8 and 5.9 turn out to coincide with the corresponding conditions in Theorems 2.4 and 4.4 and Corollary 3.5.

Theorem 5.10. *Suppose that X is a topological space, I is a set, G is a set of subsets of $\mathcal{P}(I)$, and put $\mathcal{E}=\{K^+ \mid K\in G\}$ and $A=\bigcup G$. Then the following conditions are equivalent.*

(1) *X satisfies the $\mathcal{E}$-accumulation property.*

(2) *If $(O_a)_{a\in A}$ are open sets of X, and, for every $K\in G$, $(O_a)_{a\in K}$ is a cover of X, then there is $i\in I$ such that $(O_a)_{i\in a\in A}$ is a cover of X.*

(3) *X is $[B,G]$-compact, where $B = \{i^{<}_{A} \mid i \in I\}$.*
(4) *For every sequence $(x_i)_{i\in I}$ of elements of X, there is $K \in G$ such that if, for each $a \in K$, we put $C_a = \overline{\{x_i \mid i \in a\}}$, then $\bigcap_{a\in K} C_a \neq \emptyset$.*

PROOF: Similar to the proof of Theorem 3.9. Notice that (2) $\Leftrightarrow$ (3) is immediate from the definitions, using Condition (1)(a) in Definition 5.5, and that (1) $\Leftrightarrow$ (4) follows directly from Proposition 3.11. □

Theorem 5.11. *Under the assumptions in Theorem 5.10, and if $\mathcal{F}$ is a family of subsets of X, then the following conditions are equivalent.*

(1) *X satisfies the $\mathcal{F}$-$\mathcal{E}$-accumulation property.*
(2) *X is $\mathcal{F}$-$[B,G]$-compact, where $B = \{i^{<}_{A} \mid i \in I\}$.*
(3) *For every sequence $(F_i)_{i\in I}$ of elements of $\mathcal{F}$, there is $K \in G$ such that if, for each $a \in K$, we put $C_a = \overline{\bigcup_{i\in a} F_i}$, then $\bigcap_{a\in K} C_a \neq \emptyset$.*

Theorem 5.10 is the particular case of Theorem 5.11 when $\mathcal{F}$ is the family of all singletons of X. Theorems 3.9 and 4.5 are the particular cases of, respectively, Theorems 5.10 and 5.11 when $G = \{A\}$ has just one member.

The following characterization of sequential compactness in terms of open covers might be known, but we know no reference for it.

Corollary 5.12. *A topological space X is sequentially compact (sequentially feebly compact, respectively) if and only if, for every open cover $\{O_a \mid a \in [\omega]^\omega\}$ of X such that $\{O_a \mid a \in [Z]^\omega\}$ is still a cover of X, for every $Z \in [\omega]^\omega$, there is $n \in \omega$ such that $\{O_a \mid n \in a \in [\omega]^\omega\}$ is a cover of X (has dense union in X, respectively).*

PROOF: Take $I = \omega$ and $G = \{[Z]^\omega \mid Z \in [\omega]^\omega\}$ in Theorems 5.10 and 5.11. If $K = [Z]^\omega \in G$, then $K^+ = F_Z$, in the notations of Definition 5.3. Thus the corollary is a particular case of the equivalence (1) $\Leftrightarrow$ (2) in Theorems 5.10 and 5.11, respectively,

Of course, also a direct proof of Corollary 5.12 is not difficult. □

As a special case of Theorem 5.8 (1) $\Leftrightarrow$ (3), we get the following characterizations (probably folklore) of the Rothberger and the Menger properties.

Corollary 5.13. *A topological space X satisfies the Rothberger property if and only if, for every sequence $\{x_f \mid f : \omega \to \omega\}$ of elements of X, there is $n \in \omega$ such that $\bigcap_{m\in\omega} \overline{\{x_f \mid f(n) = m\}} \neq \emptyset$.*

A topological space X satisfies the Menger property if and only if, for every sequence $\{x_f \mid f : \omega \to [\omega]^{<\omega}\}$ of elements of X, there is $n \in \omega$ such that $\bigcap_{m\in\omega} \overline{\{x_f \mid m \in f(n)\}} \neq \emptyset$.

The ideas in Section 4 suggest the following definition (known under different terminology).

Definition 5.14. A topological space X is *feebly Rothberger* (*feebly Menger*, respectively) if and only if, for every countable family of countable covers of X, we can select one member (a finite number of members, respectively) from each cover in such a way that the union of the selected members is dense in X.

The above properties can be characterized in a way similar to Corollary 5.13, by means of Theorem 5.9.

If I is a set, and M is a set of ultrafilters over I, then a topological space X is said to be *quasi M-compact* if and only if, for every I-indexed sequence $(x_i)_{i\in I}$ of elements of X, there exists $D \in M$ such that $(x_i)_{i\in I}$ D-converges to some point of X. Of course, if $M = \{D\}$ is a singleton, then quasi M-compactness is the same as D-compactness, and is also equivalent to weak M-compactness (Definition 3.13). See [9] for further references about these notions.

The space X is *quasi M-pseudocompact* if and only if, for every I-indexed sequence $(O_i)_{i\in I}$ of nonempty open sets of X, there exists $D \in M$ such that $(O_i)_{i\in I}$ has some D-limit point in X. Notice that, for $I = \omega$, the above notion is called *M-pseudocompactness* in [10, Definition 2.1]. We have chosen the name quasi M-pseudocompactness in analogy with quasi M-compactness.

Corollary 5.15. *Suppose that M is a set of ultrafilters over some set I, and let $A = \bigcup_{D\in M} D$. Then a topological space X is quasi M-compact (quasi M-pseudocompact, respectively) if and only if, whenever $(O_a)_{a\in A}$ are open sets of X, and, for every $D \in M$, $(O_a)_{a\in D}$ is a cover of X, then there is $i \in I$ such that $(O_a)_{i\in a\in A}$ is a cover of X (has dense union in X, respectively).*

Proof: By Theorems 5.10 and 5.11 (1) $\Leftrightarrow$ (2), with $G = M$, since, as already noticed, if D is an ultrafilter, then $D^+ = D$. □

Remark 5.16. As a final remark, let us mention that not every "covering property" present in the literature has the form given in Definitions 1.1, 4.1, or 5.5, the most notable case being paracompactness. More generally, all covering properties involving some particular properties (local finiteness, point finiteness, etc.) of the original cover, or of the resulting subcover, are not part of the framework given by Definition 1.1, as it stands.

There are even equivalent formulations of countable compactness which, at least formally, are not particular cases of Definition 1.1. Indeed, a space X is countably compact if and only if, for every countable open cover $(O_n)_{n\in\omega}$ such that $O_n \subseteq O_m$, for $n \leq m < \omega$, there is $n \in \omega$ such that $O_n = X$. The above condition cannot be directly expressed as a particular case of Definition 1.1.

In spite of the above remarks, we believe to have demonstrated that Definition 1.1 and its variants are general enough to capture many disparate and seemingly unrelated notions, being at the same time sufficiently concrete and manageable so that interesting results can be proved about them.

Of course, as we did in Definition 5.7, there is the possibility of modifying Definition 1.1 and its variants by considering only particular covers with special properties (cf. also Remark 5.6). We have not yet pursued this promising line of research.

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References

[1] Alexandroff P., Urysohn P., *Mémorie sur les éspaces topologiques compacts*, Verh. Akad. Wetensch. Amsterdam **14** (1929), 1–96.

[2] Arens R., Dugundji J., *Remark on the concept of compactness*, Portugaliae Math. **9** (1950), 141–143.

[3] Artico G., Marconi U., Pelant J., Rotter L., Tkachenko M., *Selections and suborderability*, Fund. Math. **175** (2002), 1–33.

[4] Caicedo X., *The abstract compactness theorem revisited*, in Logic and Foundations of Mathematics (A. Cantini et al. editors), Kluwer Academic Publishers, Dordrecht, 1999, pp. 131–141.

[5] Choquet C., *Sur les notions de filtre et de grille*, C.R. Acad. Sci., Paris **224** (1947), 171–173.

[6] Dow A., Porter J.R., Stephenson R.M., Woods R.G., *Spaces whose pseudocompact subspaces are closed subsets*, Appl. Gen. Topol. **5** (2004), 243–264.

[7] Engelking R., *General Topology*, 2nd edition, Sigma Ser. Pure Math., 6, Heldermann, Berlin, 1989.

[8] Gaal L.S., *On the theory of* (m,n)*-compact spaces*, Pacific J. Math. **8** (1958), 721–734.

[9] García-Ferreira S., *On FU(p)-spaces and p-sequential spaces*, Comment. Math. Univ. Carolin. **32** (1991), 161–171.

[10] García-Ferreira S., *Some generalizations of pseudocompactness*, Papers on General Topology and Applications (Flushing, NY, 1992), Ann. New York Acad. Sci., 728, New York Acad. Sci., New York, 1994, pp. 22–31.

[11] Garcia-Ferreira S., *On two generalizations of pseudocompactness*, Proceedings of the 14th Summer Conference on General Topology and its Applications (Brookville, NY, August 4–8, 1999), Topology Proc. **24** (2001), 149–172.

[12] Ginsburg J., Saks V., *Some applications of ultrafilters in topology*, Pacific J. Math. **57** (1975), 403–418.

[13] Glicksberg I., *Stone-Čech compactifications of products*, Trans. Amer. Math. Soc **90** (1959), 369–382.

[14] Good C., *The Lindelöf property*, in Encyclopedia of General Topology, edited by K.P. Hart, J. Nagata and J.E. Vaughan, Elsevier Science Publishers, Amsterdam, 2004, Chapter d-8, 182–184.

[15] Larson P.B., *Irreducibility of product spaces with finitely many points removed*, Spring Topology and Dynamical Systems Conference, Topology Proc. **30** (2006), 327–333.

[16] Lipparini P., *Compact factors in finally compact products of topological spaces*, Topology Appl. **153** (2006), 1365–1382.

[17] Lipparini P., *Some compactness properties related to pseudocompactness and ultrafilter convergence*, Topology Proc. **40** (2012), 29–51.

[18] Lipparini P., *More generalizations of pseudocompactness*, Topology Appl. **158** (2011), 1655–1666.

[19] Lipparini P., *Ordinal compactness*, submitted, preprint available at arXiv:1012.4737v2 (2011).

[20] Lipparini P., *Products of sequentially pseudocompact spaces*, arXiv:1201.4832.

[21] Scheepers M., *Combinatorics of open covers. I. Ramsey theory*, Topology Appl. **69** (1996), 31–62.

[22] Smirnov Y.M., *On topological spaces compact in a given interval of powers* (Russian), Izvestiya Akad. Nauk SSSR, Ser. Mat. **14** (1950), 155–178.

[23] Stephenson R.M., Jr., *Initially κ-compact and related spaces*, in Handbook of Set-theoretic Topology, edited by K. Kunen and J.E. Vaughan, North-Holland, Amsterdam, 1984, Chapter 13, pp. 603–632.

[24] Stephenson R.M., Jr., *Pseudocompact spaces*, in Encyclopedia of General Topology, edited by K.P. Hart, J. Nagata and J.E. Vaughan, Elsevier Science Publishers, B.V., Amsterdam, 2004, Chapter d-07, pp. 177–181.

[25] Vaughan J.E., *Some recent results in the theory of* $[a, b]$*-compactness*, in TOPO 72–General Topology and its Applications (Proc. Second Pittsburgh Internat. Conf., Pittsburgh, Pa., 1972; dedicated to the memory of Johannes H. de Groot), Lecture Notes in Math., 378, Springer, Berlin, 1974, pp. 534–550.

[26] Vaughan J.E., *Some properties related to* $[a, b]$*-compactness*, Fund. Math. **87** (1975), 251–260.

[27] Vaughan J.E., *Countably compact and sequentially compact spaces*, in Handbook of Set-theoretic Topology, edited by K. Kunen and J.E. Vaughan, North-Holland, Amsterdam, 1984, Chapter 12, pp. 569–602.

[28] Vaughan J.E., *Countable compactness*, in Encyclopedia of General Topology, edited by K.P. Hart, J. Nagata and J.E. Vaughan, Elsevier Science Publishers, Amsterdam, 2004, Chapter d-6, 174–176.

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