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Metrization of function spaces with the Fell topology

HANBIAO YANG

Abstract. For a Tychonoff space $X$, let $\downarrow C_F(X)$ be the family of hypographs of all continuous maps from $X$ to $[0,1]$ endowed with the Fell topology. It is proved that $X$ has a dense separable metrizable locally compact open subset if $\downarrow C_F(X)$ is metrizable. Moreover, for a first-countable space $X$, $\downarrow C_F(X)$ is metrizable if and only if $X$ itself is a locally compact separable metrizable space. There exists a Tychonoff space $X$ such that $\downarrow C_F(X)$ is metrizable but $X$ is not first-countable.

Keywords: space of continuous maps, Fell topology, hyperspace, metrizable, hypograph, separable, first-countable

Classification: 54C35, 54E45, 54B20

1. Introduction and main results

For a topological space $X$, let $C(X)$ denote the set of all continuous maps from $X$ to the unit closed interval $I = [0,1]$ with the usual topology. Then we can endow $C(X)$ with various topologies. For example, the topology of uniform convergence, the topology of pointwise convergence and the compact-open topology are well known. In [4]–[10], $C(X)$ is endowed with other natural topologies inherited from the spaces $\text{Cld}(X \times I)$ of nonempty closed sets in $X \times I$.

For a space $Y$, let $\text{Cld}(Y)$ be the set of all nonempty closed sets in $Y$. For an open set $U$ in $Y$, let

$$U^- = \{ A \in \text{Cld}(Y) : A \cap U \neq \emptyset \} \quad \text{and} \quad U^+ = \{ A \in \text{Cld}(Y) : A \subset U \}.$$ 

The most well-known topology of $\text{Cld}(Y)$, called the Vietoris topology, is generated by

$$\{U^-, U^+ : U \text{ is open in } Y\}.$$ 

In this paper, we consider the Fell topology of $\text{Cld}(Y)$, which is generated by

$$\{U^-, (Y \setminus K)^+ : U \text{ is open and } K \text{ is compact in } Y\}.$$ 

The hyperspaces $\text{Cld}(Y)$ with the above two topologies are denoted by $\text{Cld}_V(Y)$ and $\text{Cld}_F(Y)$, respectively. It is well-known that $\text{Cld}_V(Y)$ (resp. $\text{Cld}_F(Y)$) is metrizable if and only if $Y$ is a compact (resp. locally compact and separable) metrizable space. Obviously, when $Y$ is compact, the Fell topology of $\text{Cld}(Y)$ is equal to the Vietoris topology.
For every \( f \in C(X) \), let
\[
\downarrow f = \{ (x, s) \in X \times I : s \leq f(x) \} \in \text{Cld}(X \times I),
\]
which is called the hypograph of \( f \). By identifying each \( f \in C(X) \) with \( \downarrow f \in \text{Cld}(X \times I) \), we can regard \( C(X) \) as the subset
\[
\downarrow C(X) = \{ \downarrow f : f \in C(X) \} \subset \text{Cld}(X \times I).
\]

Let \( \downarrow CV(X) \) and \( \downarrow CF(X) \) be the spaces with the topologies inherited from \( \text{Cld}(X \times I) \) and \( \text{Cld}(X \times I) \), respectively. These topologies are different from the three topologies mentioned previously (see [4, Corollary 1]). In [9, Theorem 1], it was proved that, for a Tychonoff space \( X \), \( \downarrow CV(X) \) is metrizable if and only if \( \downarrow CV(X) \) is second-countable if and only if \( X \) is compact and metrizable. The following theorem is our main result.

**Theorem 1.** For a Tychonoff space \( X \), the following conditions are equivalent:

(a) \( \downarrow CF(X) \) is separable metrizable;
(b) \( \downarrow CF(X) \) is metrizable.

In case \( X \) is first-countable, the above two conditions are equivalent to

(c) \( X \) is a locally compact and separable metrizable space.

We also prove the following theorem.

**Theorem 2.** Let \( \bigoplus_{s \in S} Y_s \) be the topological sum of Tychonoff spaces \( Y_s \), \( s \in S \), and \( a_s \in Y_s \) a non-isolated point for every \( s \in S \). Let, further, \( Y \) be the quotient space of \( \bigoplus_{s \in S} Y_s \) with the set \( \{ a_s : s \in S \} \) identified to a point. Then \( \downarrow CF(Y) \) is homeomorphic to a subspace of the product space \( \prod_{s \in S} \downarrow CF(Y_s) \).

Applying this theorem, we show the following.

**Corollary 1.** There exists a Tychonoff space \( X \) such that \( \downarrow CF(X) \) is separable metrizable but \( X \) is not first-countable.

The above corollary shows that the first-countability of \( X \) is essential for the equivalence between (a) and (c) in Theorem 1. The following Theorem 3 tells us that, the non-compact case is very different from the compact one.

**Theorem 3.** There exists a countable Tychonoff space \( X \) such that \( \downarrow CF(X) \) is Hausdorff and second-countable but not regular.

In [1, 5.1.2 Proposition], it was proved that, for a Tychonoff space \( X \), the following conditions are equivalent: (a) \( \text{Cld}(X) \) is Hausdorff, (b) \( \text{Cld}(X) \) is regular, (c) \( \text{Cld}(X) \) is Tychonoff, and (d) \( X \) is locally compact. Theorem 3 shows that we cannot replace \( \text{Cld}(X) \) by \( \downarrow CF(X) \) in [1, 5.1.2 Proposition].

The following Theorem 4 states that, even for a compact space \( X \), the regularity and the first-countability of \( \downarrow CF(X) \) do not imply the metrizability of it.

**Theorem 4.** There exists a compact space \( X \) such that \( \downarrow CF(X) \) is Tychonoff, separable and first-countable but not metrizable.
Finally, we will give a necessary condition for the metrizability of $\downarrow C_F(X)$.

**Theorem 5.** For a Tychonoff space $X$, if $\downarrow C_F(X)$ is metrizable, then there exists a dense, locally compact, open and separable metrizable subspace of $X$. But the converse is not true.

2. Preparatory results

In the following, we always assume that $X$ is a Tychonoff space and $p : X \times I \to X$ is the projection. For $s \in I$, we use $s$ to denote the constant function from $X$ to $I$ which maps all elements to $s$. By $\mathbb{R}$ and $\mathbb{Q}$, we denote the sets of all real numbers and of all rational numbers, respectively. Let $\text{cl}_Y$ and $\text{int}_Y$ be the closure-operator and the interior-operator in a space $Y$. If $Y = X$, the subscript in the above operators will be omitted. And, for a closed set $F$ in $Y$, let $F^* = (\text{cl}_Y \setminus F)^+ = \{ A \in \text{cl}(Y) : A \cap F = \emptyset \}$.

By the definition, the topology of $\downarrow C_F(X)$ is generated, as a base, by the following sets:

$$\bigcap_{i=1}^n G_i^- \cap K^* \cap \downarrow C(X),$$

where $G_1, G_2, \cdots, G_n$ are open sets in $X \times (0, 1]$ and $K$ is a compact set in $X \times (0, 1]$. In particular,

$$\left\{ \bigcap_{i=1}^n G_i^- \cap \downarrow C(X) : G_1, \cdots, G_n \text{ are nonempty open in } X \times (0, 1] \right\}$$

and

$$\{ K^* \cap \downarrow C(X) : K \text{ is compact in } X \times (0, 1] \}$$

are neighborhood bases at $\downarrow 1$ and $\downarrow 0$ in $\downarrow C_F(X)$, respectively.

To prove our theorems, we need some lemmas. At first, we show the following lemma.

**Lemma 1.** For a space $X$, the following hold:

1. $\downarrow C_F(X)$ is $T_1$;
2. $\downarrow C_F(X)$ is Hausdorff if and only if there exists a dense open subset $U$ of $X$ which is locally compact.

**Proof:**

(1): Let $f \neq g \in C(X)$. We may assume that $f(x_0) < g(x_0)$ for some $x_0 \in X$. Then $x_0$ has an open neighborhood $W$ such that $f(x) < a < g(x)$ for every $x \in W$, where $a = \frac{f(x_0) + g(x_0)}{2}$. Thus $\downarrow f \in (\{ x_0 \} \times [a, 1])^* \not\equiv \downarrow g$ and $\downarrow g \in (W \times (a, 1])^- \not\equiv \downarrow f$.

(2): The “if” part: Take $f, g \in C(X)$, $x_0 \in W$ and $a \in I$ as the same as in (1). Since $f$ and $g$ are continuous, we assume that $x_0 \in U$. Because $U$ is locally compact, we have an open set $V$ in $X$ such that $x_0 \in V \subset \text{cl}V \subset U \cap W$ and $\text{cl}V$ is compact. Since $f(x) < a < g(x)$ for $x \in \text{cl}V$, $(\text{cl}V \times [a, 1])^* \cap \downarrow C(X)$ and $(V \times (a, 1])^- \cap \downarrow C(X)$ are disjoint neighborhoods of $\downarrow f$ and $\downarrow g$, respectively.
The “only if” part: We define an open set
\[ U = \bigcup \{ \text{int } K : K \text{ is compact in } X \} \subset X. \]

Then \( U \) is locally compact. We show that \( U \) is dense in \( X \). Assume that \( U \) is not dense in \( X \). Then there exists a nonempty open set \( V \) in \( X \) such that the interior of every compact subset of \( V \) is empty. Because \( X \) is Tychonoff, we can choose \( f \in C(X) \) such that \( f(X \setminus V) \subset \{1\} \) and \( f(x_0) = 0 \) for some \( x_0 \in V \). Since \( \downarrow C_F(X) \) is Hausdorff, there exist disjoint open sets \( U \) and \( V \) in \( \downarrow C_F(X) \) such that \( \downarrow U \in U \) and \( \downarrow f \in V \). Then we can find nonempty open sets \( G_1, G_2, \ldots, G_n, \ldots, G_m \subset X \times (0,1] \) and a compact set \( K \subset X \times (0,1] \) such that

\[ \downarrow U \in G_1 \cap G_2 \cap \cdots \cap G_n \cap \downarrow C(X) \subset U \quad \text{and} \quad \downarrow f \in G_{n+1} \cap \cdots \cap G_m \cap K \cap \downarrow C(X) \subset V. \]

Since \( f(X \setminus V) \subset \{1\} \), it follows that \( p(K) \subset V \), which implies that \( \text{int } p(K) = \emptyset \).

For every \( i \leq m \), \( p(G_i) \cap p(K) \neq \emptyset \) since \( p(G_i) \) is a nonempty open set in \( X \). Take \( x_i \in p(G_i) \cap p(K) \). Because \( X \) is Tychonoff, we have \( g \in C(X) \) satisfying

\[ g(x_i) = 1 \quad \text{for } i \leq m \quad \text{and} \quad g(p(K)) = \{0\}. \]

Then \( \downarrow g \in U \cap V \), which contradicts \( U \cap V = \emptyset \).

**Lemma 2.** If \( \downarrow C_F(X) \) is first-countable, then there exist compact sets \( C_1 \subset C_2 \subset \cdots \) in \( X \) such that every compact set in \( X \) is contained in some \( C_n \). In particular, \( X = \bigcup_{n=1}^{\infty} C_n \).

**Proof:** Because \( \downarrow C_F(X) \) is first-countable, we can find compact sets \( K_1 \subset K_2 \subset \cdots \) in \( X \times (0,1] \) such that \( \{ K_n^* \cap \downarrow C(X) : n = 1, 2, \ldots \} \) is a neighborhood base of \( \downarrow 0 \) in \( \downarrow C_F(X) \). Then \( C_n = p(K_n) \), \( n = 1, 2, \ldots \), are the desired compact sets in \( X \). We verify that every compact set \( C \) in \( X \) is contained in some \( C_n \). Otherwise, for every \( n \), we can choose \( x_n \in C \setminus C_n \) and define \( f_n \in C(X) \) such that \( f_n(x_n) = 1 \) and \( f_n(C_n) = \{0\} \). Then \( \downarrow f_n \in K_n^* \) for every \( n \) and hence \( \downarrow f_n \to \downarrow 0 \) in \( \downarrow C_F(X) \). But every \( \downarrow f_n \) is not contained in the neighborhood \( (C \times \{1\})^* \) of \( \downarrow 0 \), which is a contradiction.

**Lemma 3.** If \( X \) and \( \downarrow C_F(X) \) are first-countable, then \( X \) is locally compact.

**Proof:** Suppose there exists \( x_0 \in X \), which has no compact neighborhood. Because \( X \) is first-countable, \( x_0 \) has a countable open neighborhood base \( \{ U_n : n = 1, 2, \ldots \} \), where \( U_n \supset U_{n+1} \) for every \( n \). Since \( \downarrow C_F(X) \) is also first-countable, we can find compact sets \( K_1 \subset K_2 \subset \cdots \) in \( X \times (0,1] \) such that \( \{ K_n^* \cap \downarrow C(X) : n = 1, 2, \ldots \} \) is a neighborhood base at \( \downarrow 0 \) in \( \downarrow C(X) \). By the assumption, \( p(K_n) \nsubseteq U_n \) for every \( n = 1, 2, \ldots \), hence we can take \( x_n \in U_n \setminus p(K_n) \). Then \( x_n \to x_0 \) in \( X \). Since \( X \) is Tychonoff, we have \( f_n \in C(X) \) such that

\[ f_n(x_n) = 1 \quad \text{and} \quad f_n(p(K_n) \cup (X \setminus U_n)) = \{0\}. \]
Then \( \downarrow f_n \in K^*_n \) and hence \( \downarrow f_n \to \downarrow 0 \). On the contrary,

\[
\left( \{ x_n : n = 0, 1, 2, \cdots \} \times \{ 1 \} \right)^* \cap \downarrow C(X)
\]

is a neighborhood of \( \downarrow 0 \) in \( \downarrow C_F(X) \) which does not contain any \( \downarrow f_n \). \( \square \)

When \( X \) is locally compact and non-compact, let \( \alpha X = X \cup \{ \infty \} \) be the one-point compactification of \( X \). Using Lemmas 2 and 3, we may prove the following

**Proposition 1.** If \( X \) and \( \downarrow C_F(X) \) are first-countable, then

1. \( X \) is locally compact and \( \alpha X \) is also first-countable;
2. \( \downarrow C_F(\alpha X) \) is first-countable;
3. \( \downarrow C_F(\alpha X) \) is second-countable if \( \downarrow C_F(X) \) is second-countable.

**Proof:** The assertion (1) directly follows from Lemmas 2 and 3. To show (2) and (3), we only consider the case that \( X \) is not compact. Let \( \{ U_n : n = 1, 2, \ldots \} \) be a countable open neighborhood base at \( \infty \) in \( \alpha X \), and let \( \phi : C(\alpha X) \to C(X) \) be the restriction, that is,

\[
\phi(f) = f|X \quad \text{for every} \quad f \in C(\alpha X).
\]

Then it is not hard to verify that \( \downarrow \phi : \downarrow C_F(\alpha X) \to \downarrow C_F(X) \) is a continuous injection. Unfortunately, it is not an embedding. However, the following \( S \) is a subbase of \( \downarrow C_F(\alpha X) \):

\[
S = \{(\downarrow \phi)^{-1}(G) : G \in \mathcal{G}\} \cup \{(\text{cl}_{\alpha X} U_n \times [r,1])^* \cap \downarrow C(\alpha X) : r \in \mathbb{Q} \cap (0,1], n = 1, 2, \ldots \},
\]

where \( \mathcal{G} \) is an open base for \( \downarrow C_F(X) \). Obviously, \( S \) is a subfamily of the topology of \( \downarrow C_F(\alpha X) \). For every open set \( V \) in \( \alpha X \times I \), \( V \cap (X \times I) \) is open in \( X \times I \) and

\[
V^{-} \cap \downarrow C(\alpha X) = (\downarrow \phi)^{-1}((V \cap (X \times I))^{-} \cap \downarrow C(\alpha X)).
\]

For every compact set \( K \) in \( \alpha X \times (0,1] \), if \( K \cap (\{ \infty \} \times I) = \emptyset \), then \( K \) is also compact in \( X \times I \) and

\[
K^* \cap \downarrow C(\alpha X) = (\downarrow \phi)^{-1}(K^* \cap \downarrow C(X)).
\]

If \( K \cap (\{ \infty \} \times I) \neq \emptyset \), then for every \( \downarrow f \in K^* \cap \downarrow C(\alpha X) \), using the Wallace’s Theorem, there exist \( n \) and a rational number \( r \in (0,1] \) such that

\[
(\text{cl}_{\alpha X} U_n \times [r,1]) \cap \downarrow f = \emptyset \quad \text{and} \quad K \cap (\text{cl}_{\alpha X} U_n \times I) \subset \text{cl}_{\alpha X} U_n \times [r,1].
\]

Let

\[
K_1 = (K \cap ((\alpha X \setminus U_n) \times I)) \cup (\text{cl}_{\alpha X} U_n \times [r,1]).
\]
Then $K_1$ is compact in $\alpha X \times (0, 1]$, $K_1 \supset K$ and $K_1 \cap \downarrow f = \emptyset$. Thus, $\downarrow f \in K_1^* \subset K^*$. Note that

$$K_1^* \cap \downarrow C_F(\alpha X) = (\downarrow \phi)^{-1}((K \cap ((\alpha X \setminus U_n) \times I))^*)$$

$$\cap (\text{cl}(U_n) \times [r, 1])^* \cap \downarrow C_F(\alpha X),$$

that is, $K_1^* \cap \downarrow C_F(\alpha X)$ is an intersection of two elements of $S$.

As a conclusion, $S$ is a subbase for $\downarrow C_F(\alpha X)$. Therefore, $\downarrow C_F(\alpha X)$ is first-countable. Moreover, $\downarrow C_F(\alpha X)$ is second-countable if $\downarrow C_F(X)$ is second-countable. Hence (2) and (3) hold.

Lemma 4. We consider the following statements.

(a) $\downarrow C_F(X)$ is first-countable.

(b) $\downarrow C_F(X)$ has a countable neighborhood base at $\downarrow 1$.

(c) There exists a countable family $\mathcal{U}$ of nonempty open sets in $X$ such that every nonempty open set in $X$ includes an element of $\mathcal{U}$, that is, $\mathcal{U}$ is a countable $\pi$-base for $X$.

(d) $\downarrow C_F(X)$ is separable.

Then the implications (a)$\Rightarrow$(b)$\Rightarrow$(c)$\Rightarrow$(d) hold.

Furthermore, when $X$ is compact, the implication (c)$\Rightarrow$(a) holds and hence (a), (b) and (c) are equivalent.

Proof: The implication (a)$\Rightarrow$(b) is trivial.

(b)$\Rightarrow$(c): We may assume that

$$\{G^n_i)^- \cap (G^n_2)^- \cap \cdots \cap (G^n_{k(n)})^- \cap \downarrow C(X) : n = 1, 2, \ldots \}$$

is a countable neighborhood base at $\downarrow 1$ in $\downarrow C_F(X)$. Let

$$\mathcal{U} = \{p(G^n_i) : i = 1, 2, \ldots, k(n), n = 1, 2, \ldots \}. $$

Then $\mathcal{U}$ is a countable family of nonempty open sets in $X$. We show that every nonempty open set $U$ in $X$ includes an element of $\mathcal{U}$. Take $f \in C(X)$ such that $f(\alpha X \setminus U) \subset \{1\}$ and $f(x_0) = 0$ for some point $x_0 \in U$. Because $\downarrow C_F(X)$ is $T_1$ by Lemma 1(1), $\downarrow f \not\subseteq \bigcap_{i=1}^{k(n)} (G^n_i)^-$ for some $n$, hence $\downarrow f \not\subseteq (G^n_i)^-$ for some $i \leq k(n)$. Then $\downarrow f \cap G^n_i = \emptyset$. Since $f(\alpha X \setminus U) \subset \{1\}$, we have $U \supset p(G^n_i)$, as required.

(c)$\Rightarrow$(d): Let $\mathcal{U}$ be a countable $\pi$-base for $X$. For every $U \in \mathcal{U}$ and $r \in \mathbb{Q} \cap (0, 1]$, we can take a continuous map $f_{(U, r)} : X \to [0, r]$ such that $f_{(U, r)}(\alpha X \setminus U) \subset \{0\}$ and $f_{(U, r)}(x) = r$ for some $x \in U$. Let

$$D = \{\max\{f_{(U, r)} : U \in \mathcal{F}, r \in F\} : \mathcal{F} \text{ and } F \text{ are finite subsets of } \mathcal{U} \text{ and } \mathbb{Q} \cap (0, 1], \text{ resp.}\}.$$ 

Then $\downarrow D = \{\downarrow f : f \in D\}$ is a countable subset of $\downarrow C(X)$. It remains to verify that $\downarrow D$ is dense in $\downarrow C_F(X)$. Let $f \in C(X)$, $K$ be compact in $X \times (0, 1]$ and $G_i$,
$i \leq k$, open in $X \times (0,1]$, such that
\[
\downarrow f \in G_1^- \cap G_2^- \cap \cdots \cap G_k^- \cap K^* \cap \downarrow C(X).
\]

We have $x_1, \ldots, x_k \in X$ such that $\{x_i\} \times [0,f(x_i)] \cap G_i \neq \emptyset$ for each $i \leq k$. Because $\{x_i\} \times [0,f(x_i)] \cap K = \emptyset$, we have an open neighborhood $W_i$ of $x_i$ in $X$ and $s_i < t_i$ such that $W_i \times (s_i,t_i) \subset G_i$ and $W_i \times [0,t_i] \cap K = \emptyset$. Thus, by (c), choose $r_i \in \mathbb{Q} \cap (s_i,t_i)$ and $U_i \in \mathcal{U}$ such that $U_i \subset W_i$. Then $\downarrow f(U_i,r_i) \in G_i^- \cap K^*$ and hence
\[
\downarrow \max\{f(U_i,r_i) : i \leq k\} \in \downarrow D \cap G_1^- \cap G_2^- \cap \cdots \cap G_k^- \cap K^*.
\]

Now, we show (c)$\Rightarrow$(a) under the assumption that $X$ is compact. Let $\mathcal{U}$ be a countable $\pi$-base of $X$. Then, $X \times I$ has the following countable $\pi$-base:
\[
\mathcal{G} = \{U \times (s,t) : U \in \mathcal{U}, s < t \in \mathbb{Q} \cap (0,1)\}.
\]

For every $f \in C(X)$ and $n = 1,2,\ldots$, let
\[
\mathcal{G}(f) = \{G \in \mathcal{G} : f \in G^-\}, \quad K_n(f) = \{(x,t) \in X \times I : t \geq f(x) + n^{-1}\}.
\]

For every open set $H$ in $X \times (0,1]$ with $H^- \ni \downarrow f$, there exists $x_0 \in X$ such that $\{x_0\} \times [0,f(x_0)] \cap H \neq \emptyset$. Since $f(x_0) > 0$, we can find an open neighborhood $V$ of $x_0$ in $X$ and $s < t \in \mathbb{Q} \times (0,1)$ such that $s < f(x_0)$, $V \times (s,t) \subset H$ and $s < f(x)$ for every $x \in V$. Since $\mathcal{U}$ is a $\pi$-base for $X$, $V$ contains some $U \in \mathcal{U}$. Then we have $G = U \times (s,t) \in \mathcal{G}$ and $\downarrow f \in G^- \subset H^-$. Moreover, for every compact set $K$ in $X \times I$ with $K^* \ni \downarrow f$, by the compactness of $X$, there exists $n$ such that $K_n(f) \supset K$ and hence $\downarrow f \in K_n(f)^* \subset K^*$. Therefore,
\[
\{G_1^- \cap \cdots \cap G_k^- \cap K_n(f)^* \cap \downarrow C(X) : G_i \in \mathcal{G}(f) \text{ for } i \leq k, k, n = 1,2,\ldots\}
\]
is a countable neighborhood base at $\downarrow f$ in $\downarrow C_{F}(X)$.

As a consequence of Lemma 4, we have the equivalence between (a) and (b) in Theorem 1, that is,

**Proposition 2.** The space $\downarrow C_{F}(X)$ is metrizable if and only if it is separable metrizable.

We need the following two lemmas which were proved in [8], [9], respectively.

**Lemma 5.** If $V$ is open in $X$ such that $\text{cl} V$ is compact, then the restriction $\phi : \downarrow C_{F}(X) \to \downarrow C_{F}(\text{cl} V)$ defined by $\phi(\downarrow f) = \downarrow f|\text{cl} V$ is a continuous open surjection.

**Lemma 6.** If $X$ is compact and $\downarrow C_{F}(X) = \downarrow C_{V}(X)$ is second-countable, then $X$ is metrizable.
3. Proofs of main results

In this section, we show our main results.

**Proof of Theorem 1:** The equivalence between (a) and (b) is Proposition 2. If $X$ is first-countable, then $X$ is locally compact by Proposition 1(1). Using Proposition 1(3), the condition (b) implies that $\downarrow C(\alpha X)$ is second-countable. It follows from Lemma 6 that $\alpha X$ is metrizable. Hence the condition (c) holds. That is, the implication (b) $\Rightarrow$ (c) holds under the assumption that $X$ is first-countable. The condition (c) implies that $\text{Cld}_F(X \times I)$ is metrizable ([1, 5.1.5 Theorem]), hence so is $\downarrow C_F(X)$, i.e., (b) holds. Therefore, the implication (c) $\Rightarrow$ (b) holds. □

**Proof of Theorem 2:** We may think that every $Y_s$ is a subspace of $Y$. Define $\phi : C(Y) \rightarrow \prod_{s \in S} C(Y_s)$ by

$$\phi(f) = (f|Y_s)_{s \in S} \text{ for each } f \in C(Y).$$

Evidently, $\phi$ is an injection and its image is

$$\phi(C(Y)) = \left\{ g \in \prod_{s \in S} C(Y_s) : g(s)(a_s) = g(s')(a_{s'}) \text{ for } s, s' \in S \right\}.$$ 

Now we show that $\downarrow \phi : \downarrow C_F(Y) \rightarrow \prod_{s \in S} \downarrow C_F(Y_s)$ is an embedding. Let $p_s : \prod_{s \in S} \downarrow C_F(Y_s) \rightarrow \downarrow C_F(Y_s)$ be the projection.

To show the continuity of $\downarrow \phi$, it is sufficient to verify that $p_s \circ \downarrow \phi$ is continuous for every $s \in S$. For every open set $G$ in $Y_s \times (0, 1]$, $G \setminus \{(a_s) \times I\}$ is open in $Y \times (0, 1]$. Since $a_s$ is a non-isolated point in $Y_s$,

$$(p_s \circ \downarrow \phi)^{-1}(G \setminus \downarrow C(Y_s)) = (G \setminus \{(a_s) \times I\}) \setminus \downarrow C(Y).$$

For each compact set $K$ in $Y_s \times (0, 1]$, 

$$(p_s \circ \downarrow \phi)^{-1}(K^* \cap \downarrow C(Y_s)) = K^* \cap \downarrow C(Y).$$

Hence, $p_s \circ \downarrow \phi : \downarrow C_F(Y) \rightarrow \downarrow C_F(Y_s)$ is continuous for every $s \in S$.

Moreover, for every open set $H$ in $Y \times (0, 1]$, if $\downarrow f \in H^* \setminus \downarrow C_F(Y)$, then there exists $s \in S$ such that $\downarrow f|Y_s \in (H \setminus (Y_s \times I))^*$. Hence

$$\downarrow \phi(H^* \cap \downarrow C_F(Y)) = \bigcup_{s \in S} \left( (H \cap (Y_s \times I))^* \times \prod_{t \in S \setminus \{s\}} \downarrow C(Y_t) \right) \cap \downarrow \phi(\downarrow C(Y)).$$

It shows that $\downarrow \phi(H^* \cap \downarrow C_F(Y))$ is open in $\downarrow \phi(\downarrow C(F(Y)))$. For every compact set $K$ in $Y \times (0, 1]$, there exists a finite subset $S_0$ of $S$ such that $K \subset \bigcup_{s \in S_0} Y_s \times (0, 1]$. Then $K \cap Y_s \times (0, 1]$ is compact for every $s \in S_0$ and

$$\downarrow \phi(K^* \cap \downarrow C(Y)) = \left( \prod_{s \in S_0} (K \cap Y_s \times (0, 1])^* \times \prod_{s \in S \setminus S_0} \downarrow C(Y_s) \right) \cap \downarrow \phi(\downarrow C(Y)).$$
It follows that $\downarrow \phi(K^* \cap \downarrow C(Y))$ is open in $\downarrow \phi(\downarrow(C_F(Y)))$. Since $\phi$ is one-to-one, we have that $\downarrow \phi$ maps every open set in $\downarrow C_F(Y)$ to an open set in $\downarrow \phi(\downarrow(C_F(Y)))$.

Therefore, $\downarrow \phi : \downarrow C_F(Y) \to \prod_{s \in S} \downarrow C_F(Y_s)$ is an embedding. \qed

Remark 1. Even for a set $S$ of two points, if $a_s$ is an isolated point in $Y_s$ for some $s$, the map $\downarrow \phi$ defined in the above proof needs not be continuous. For example, let $Y_1 = \{1\} \times (\{0\} \cup [1,2])$, $Y_2 = \{2\} \times I$ as subspaces of $\mathbb{R}^2$. If we think that $a_1 = (1,0), a_2 = (2,0)$, then $p_1 \circ \downarrow \phi : \downarrow C(Y) \to \downarrow C(Y_1)$ is not continuous. In fact, choose $f_n \in C(Y)$ such that $f_n(2,0) = f_n(1,0) = 0$ and $f_n(x) = 1$ for every $x \in Y \setminus (\{2\} \times [0, n^{-1}])$. Then $\downarrow f_n \to \downarrow 1$ but $(p_1 \circ \downarrow \phi)(\downarrow f_n) \not\to (p_1 \circ \downarrow \phi)(\downarrow 1)$.

Proof of Corollary 1: Let $\{Y_n : n = 1, 2, \ldots\}$ be a family of pairwise disjoint locally compact separable metrizable spaces $Y_n$ with a non-isolated point $a_n$. Then, by Theorems 1 and 2, the space $Y$ defined in Theorem 2 is as required. \qed

Proof of Theorem 3: Let $\beta \omega$ be the Čech-Stone compactification of the discrete space $\omega$ of non-negative integers and $q \in \beta \omega \setminus \omega$. Then the subspace $X = \omega \cup \{q\}$ of $\beta \omega$ satisfies the conditions in Theorem 3. By Lemma 1(2), $\downarrow C_F(X)$ is Hausdorff.

Before showing that $\downarrow C_F(X)$ is second-countable but not regular, we verify that every compact subset of $X$ is finite. In fact, let $C$ be an infinite compact subset of $X$. Then $q \in C$. Write $C = A \cup B \cup \{q\}$ such that $A$ and $B$ are disjoint infinite subsets of $\omega$. Define a continuous map $f : \omega \to \{0, 1\}$ as $f^{-1}(0) = A$. Then there exists a continuous extension $\overline{f} : X \to \{0, 1\}$ since $X$ is a subspace of $\beta \omega$. If $\overline{f}(q) = 0$, then $B$ is closed in $X$ and hence is compact. But it is impossible since $B$ is infinite discrete. If $\overline{f}(q) = 1$, then $A$ is closed in $X$ and hence is compact. It is also impossible since $A$ is also infinite discrete.

Now, we define a product space $Y = \prod_{x \in X} I_x$, where $I_x$ is a copy of the unit interval $[0,1]$ with the usual topology for $x \in \omega$ and $I_q$ is $[0,1]$ with the topology generated by $\{(0, r) : r \in [0,1] \cap \mathbb{Q}\} \cup \{\{0,1\}\}$. Then $Y$ is second-countable. We may regard $\downarrow C(X) \subset Y$ by identifying $\downarrow f$ with $(f(x))_{x \in X}$ for every $f \in C(X)$. To show that $\downarrow C_F(X)$ is second-countable, it suffices to verify that $\downarrow C_F(X)$ is the subspace of the space $Y$. It is easy to see that for each $x \in X$, the map $p_x : \downarrow C_F(Y) \to I_x$ defined by $p_x(\downarrow f) = f(x)$ is continuous. Hence the subspace topology is coarser than the Fell topology on $\downarrow C(X)$. Conversely, take a compact set $K \subset X \times (0,1]$ and $f \in C(X)$. Then $p(K)$ is compact in $X$. Then $p(K)$ is a finite set in $X$ and $\downarrow f \cap K = \emptyset$ if and only if $f(x) < m(x) = \min\{s : (x,s) \in K\}$ for every $x \in p(X)$. Hence we can identify

$$K^* \cap \downarrow C(X) = \left( \prod_{x \in p(K)} [0, m_x) \times \prod_{x \in X \setminus p(K)} I_x \right) \cap \downarrow C(X)$$

is open in the subspace topology of $Y$. For every open set $G$ in $X \times (0,1]$ and
\[ f \in C(X), \downarrow f \cap G \neq \emptyset \text{ if and only if } \downarrow f \cap G \setminus \{(q) \times I\} \neq \emptyset \text{ if and only if } f(n) > s_n \text{ for some } n \in p(G) \cap \omega, \text{ where } s_n = \inf\{s : (n, s) \in G\}. \]

Hence

\[ G^- \cap \downarrow C(X) = \left( \bigcup_{n \in p(G) \cap \omega} p_n^{-1}(s_n, 1) \right) \cap \downarrow C(X), \]

where \( p_n : Y \to I_n \) is the projection, is open in the subspace topology of \( Y \). Therefore, \( \downarrow C_F(X) \) is the subspace of \( Y \).

To show that \( \downarrow C_F(X) \) is not regular, we consider an open neighborhood \( U = (\{q\} \times \left[\frac{1}{2}, 1\right]) \cap \downarrow C(X) \) of \( \downarrow 0 \). For every compact set \( K \) in \( X \times (0, 1) \), \( p(K) \) is finite. Define \( f \in C(X) \) such that \( f^{-1}(0) = p(K) \cap \omega \) and \( f^{-1}(1) = X \setminus (p(K) \cap \omega) \). Then \( \downarrow f \in \text{cl}_{\downarrow C_F(X)}(K^* \cap \downarrow C_F(X)) \setminus U \). In fact, every neighborhood of \( \downarrow f \) in \( \downarrow C_F(Y) \) contains the following neighborhood of \( \downarrow f \):

\[ \mathcal{G} = G_1^- \cap \cdots \cap G_k^- \cap G^- \cap C^* \cap \downarrow C_F(X), \]

where \( G_i = \{n_i\} \times (s_i, t_i) \) for \( 1 \leq i \leq k \) and \( G = (A \cup \{q\}) \times (s, t) \) are open and \( C \) is compact in \( X \times (0, 1) \). Then \( A \) is an infinite subset of \( \omega \) and hence we may choose \( n_0 \in A \setminus p(K \cup C) \). Now, define \( g \in C(X) \) as

\[
g(x) = \begin{cases} 
0 & \text{if } x \in A \cup \{q\} \setminus \{n_i : 0 \leq i \leq k\}; \\
1 & \text{if } x = n_0; \\
f(x) & \text{otherwise.}
\end{cases}
\]

Then it is easy to verify that \( \downarrow g \in \mathcal{G} \cap K^* \). This shows that \( \downarrow f \in \text{cl}_{\downarrow C_F(X)}(K^* \cap \downarrow C_F(X)) \). Because \( f(q) = 1 \), we have \( \downarrow f \notin U \). Hence, \( \text{cl}_{\downarrow C_F(X)}(K^* \cap \downarrow C_F(X)) \notin U \) for any compact \( K \) in \( X \times (0, 1) \). Note that the family of all of such \( K^* \cap \downarrow C_F(X) \) is a neighborhood base at \( \downarrow \emptyset \) in \( \downarrow C_F(X) \). Therefore, \( \downarrow C_F(X) \) is not regular.

**Proof of Theorem 4:** Choose a compact Hausdorff non-metrizable space \( X \) satisfying (c) in Lemma 4, for example, \( \beta \omega \) or Helly space (see [2, Problem 5.M]). Then, by Lemma 4, \( \downarrow C_F(X) \) is separable and first-countable. By [3] (cf. [1, 5.1.2 Proposition]), \( \text{Cl}_{F}(X \times I) = \text{Cl}_{V}(X \times I) \) is Tychonoff and hence so is \( \downarrow C_F(X) \). Since \( X \) is compact and non-metrizable, \( \downarrow C_F(X) \) is not second-countable because of Lemma 6. According to Proposition 2, if \( \downarrow C_F(X) \) is metrizable, then \( \downarrow C_F(X) \) is separable metrizable, hence second-countable. Therefore, \( \downarrow C_F(X) \) is not metrizable.

**Proof of Theorem 5:** Assume that \( \downarrow C_F(X) \) is metrizable, which means that \( \downarrow C_F(X) \) is separable metrizable by Proposition 2. Then \( \downarrow C_F(X) \) is second-countable. By Lemma 1(2), there exists a dense open set \( U \) in \( X \) such that \( U \) is locally compact. To complete the proof, it remains to verify that \( U \) is separable metrizable. By Lemma 2, there exists a countable family \( C = \{C_1, C_2, \cdots \} \) of compact sets in \( X \) such that every compact set in \( X \) is contained in some \( C_n \). For each \( n \), let \( U_n = \text{int}(U \cap C_n) \). Then, \( \text{cl} U_n \) is compact because \( \text{cl} U_n \subset C_n \). By
Lemma 5, there exists a continuous open surjection from $\downarrow C_F(X)$ onto $\downarrow C_F(\text{cl} U_n)$. Therefore, $\downarrow C_F(\text{cl} U_n)$ is second-countable, hence $\text{cl} U_n$ is compact and metrizable by Lemma 6. Thus every $U_n$ is also separable metrizable, hence it is second-countable. Moreover, for every $x \in U$, there exists an open set $V$ such that $x \in V$, $\text{cl} V$ is compact and $\text{cl} V \subset U$. Hence there exists $n$ such that $\text{cl} V \subset C_n$. Then, $x \in V \subset \text{int}(U \cap C_n) = U_n$. It follows that $U = \bigcup_{n=1}^{\infty} U_n$. Therefore, $U$ is second-countable, hence it is separable metrizable.

As mentioned in proof of Theorem 4, $\beta \omega$ is a compact space and $\downarrow C_F(\beta \omega)$ is not metrizable but $\omega$ is a dense, locally compact, open and separable metrizable subspace of $\beta \omega$. Namely, the converse is not true.

\begin{remark}
The referee pointed out that McCoy and Ntantu [11] obtained analogous results in 1992. For example, Theorem 4.12 in [11] is similar to our Theorem 1. Our Theorem 3 for $\downarrow C_F(X, I)$ is true for $\uparrow C_F(X, \mathbb{R})$ using Theorems 3.5, 3.7, 4.11 and Example 3.3 in [11], where $\uparrow C_F(X, \mathbb{R})$ is the subspace of $\text{Cld}_F(X \times \mathbb{R})$ consisting of the epigraphs

$$
\uparrow f = \{(x, s) \in X \times \mathbb{R} : f(x) \leq s\} \in \text{Cld}(X \times \mathbb{R}),
$$

of all $f \in C(X, \mathbb{R})$. However our arguments are quite different from their arguments in [11].

\end{remark}

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