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Applications of Mathematics, Vol. 57 (2012), No. 4, 377--390

Persistent URL: `http://dml.cz/dmlcz/142905`

ON THE GROWTH OF SOLUTIONS OF SOME HIGHER ORDER LINEAR DIFFERENTIAL EQUATIONS

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(Received April 12, 2010)

Abstract. In this paper we discuss the growth of solutions of the higher order nonhomogeneous linear differential equation

$$f^{(k)} + A_{k-1}f^{(k-1)} + \ldots + A_2 f'' + (D_1(z) + A_1(z)\mathrm{e}^{az})f' \\ + (D_0(z) + A_0(z)\mathrm{e}^{bz})f = F \quad (k \geqslant 2),$$

where a, b are complex constants that satisfy $ab(a-b) \neq 0$ and $A_j(z)$ $(j = 0, 1, \ldots, k-1)$, $D_j(z)$ $(j = 0, 1)$, $F(z)$ are entire functions with $\max\{\varrho(A_j)$ $(j = 0, 1, \ldots, k-1)$, $\varrho(D_j)$ $(j = 0, 1)\} < 1$. We also investigate the relationship between small functions and the solutions of the above equation.

Keywords: linear differential equations, entire solutions, order of growth, exponent of convergence of zeros, exponent of convergence of distinct zeros

MSC 2010: 34M10, 30D35

1. Introduction and statement of results

Throughout this paper, we assume that the reader is familiar with the fundamental results and the standard notation of the Nevanlinna value distribution theory and the basic notions of Wiman-Valiron as well (see [11], [12], [15]). In addition, we will use $\lambda(f)$ $(\lambda_2(f))$ and $\overline{\lambda}(f)$ $(\overline{\lambda}_2(f))$ to denote respectively the exponents (hyper-exponents) of convergence of the zero-sequence and the sequence of distinct zeros of f, $\varrho(f)$ to denote the order of growth of a meromorphic function f and $\varrho_2(f)$ to denote the hyper-order of f. A meromorphic function $\varphi(z)$ is called a small function with respect to $f(z)$ if $T(r, \varphi) = o(T(r, f))$ as $r \to \infty$ except possibly a set of r of finite linear measure, where $T(r, f)$ is the Nevanlinna characteristic function of f. If

f is of infinite order and φ is of finite order, then clearly $\varphi(z)$ is a small function with respect to $f(z)$. We also define

$$\overline{\lambda}(f-\varphi)=\limsup_{r\to\infty}\frac{\log\overline{N}\Big(r,\dfrac{1}{f-\varphi}\Big)}{\log r}$$

and

$$\overline{\lambda}_2(f-\varphi)=\limsup_{r\to\infty}\frac{\log\log\overline{N}\Big(r,\dfrac{1}{f-\varphi}\Big)}{\log r}$$

for any meromorphic function $\varphi(z)$.

For the second order linear differential equation

$$f''+\mathrm{e}^{-z}f'+B(z)f=0, \tag{1.1}$$

where $B(z)$ is an entire function, it is well known that each solution f of equation (1.1) is an entire function, and that if f_1, f_2 are two linearly independent solutions of (1.1), then by [7], at least one of f_1, f_2 is of infinite order. Hence, "most" solutions of (1.1) will have infinite order. But equation (1.1) with $B(z)=-(1+\mathrm{e}^{-z})$ possesses a solution $f(z)=\mathrm{e}^{z}$ of finite order.

A natural question arises: What conditions on $B(z)$ will guarantee that every solution $f\not\equiv 0$ of (1.1) has infinite order? Many authors, Frei [8], Ozawa [16], Amemiya-Ozawa [1], Gundersen [9], and Langley [13] have studied this problem. They proved that when $B(z)$ is a nonconstant polynomial or $B(z)$ is a transcendental entire function with order $\varrho(B)\neq 1$, then every solution $f\not\equiv 0$ of (1.1) has infinite order.

In 2002, Z. X. Chen [5] considered the question: What conditions on $B(z)$ when $\varrho(B)=1$ will guarantee that every nontrivial solution of (1.1) has infinite order? He proved the following results, which improved results of Frei, Amemiya-Ozawa, Ozawa, Langley, and Gundersen.

Theorem A ([5]). *Let $A_j(z)\,(\not\equiv 0)$ $(j=0,1)$ and $D_j(z)$ $(j=0,1)$ be entire functions with $\max\{\varrho(A_j)\ (j=0,1),\ \varrho(D_j)\ (j=0,1)\}<1$, and let a, b be complex constants that satisfy $ab\neq 0$ and $\arg a\neq\arg b$ or $a=cb$ $(0<c<1)$. Then every solution $f\not\equiv 0$ of the equation*

$$f''+(D_1(z)+A_1(z)\mathrm{e}^{az})f'+(D_0(z)+A_0(z)\mathrm{e}^{bz})f=0 \tag{1.2}$$

is of infinite order.

Setting $D_j\equiv 0$ $(j=0,1)$ in Theorem A, we obtain the following result.

Theorem B. *Let* $A_j(z)\,(\not\equiv 0)$ $(j=0,1)$ *be entire functions with* $\max\{\varrho(A_j)\colon\ j=0,1\}<1$, *and let* a, b *be complex constants that satisfy* $ab\neq 0$ *and* $\arg a\neq\arg b$ *or* $a=cb$ $(0<c<1)$. *Then every solution* $f\not\equiv 0$ *of the equation*

$$f''+A_1(z)\mathrm{e}^{az}f'+A_0(z)\mathrm{e}^{bz}f=0 \tag{1.3}$$

is of infinite order.

Theorem C ([5]). *Let* $A_j(z)\,(\not\equiv 0)$ $(j=0,1)$ *be entire functions with* $\varrho(A_j)<1$ $(j=0,1)$, *and let* a, b *be complex constants that satisfy* $ab\neq 0$ *and* $a=cb$ $(c>1)$. *Then every solution* $f\not\equiv 0$ *of equation* (1.3) *is of infinite order.*

Very recently in [18], H. Y. Xu and T. B. Cao have investigated the growth of solutions of some higher order nonhomogeneous linear differential equations and have obtained the following result.

Theorem D ([18]). *Let* $P(z)=\sum\limits_{i=0}^{n}a_iz^i$ *and* $Q(z)=\sum\limits_{i=0}^{n}b_iz^i$ *be nonconstant polynomials, where* a_i, b_i $(i=0,1,\ldots,n)$ *are complex numbers,* $a_nb_n(a_n-b_n)\neq 0$. *Suppose that* $h_i(z)$ $(2\leqslant i\leqslant k-1)$ *are polynomials of degree not greater than* $n-1$ *in* z, $A_j(z)\not\equiv 0$ $(j=0,1)$ *and* $H(z)\not\equiv 0$ *are entire functions with* $\max\{\varrho(A_j)\ (j=0,1),\ \varrho(H)\}<n$, *and* φ *is an entire function of finite order. Then every nontrivial solution* f *of the equation*

$$f^{(k)}+h_{k-1}f^{(k-1)}+\ldots+h_2f''+A_1(z)\mathrm{e}^{P(z)}f'+A_0(z)\mathrm{e}^{Q(z)}f=H \tag{1.4}$$

satisfies $\varrho(f)=\lambda(f)=\overline{\lambda}(f)=\overline{\lambda}(f-\varphi)=\infty$ *and* $\varrho_2(f)=\lambda_2(f)=\overline{\lambda}_2(f)=\overline{\lambda}_2(f-\varphi)\leqslant n$.

R e m a r k 1.1. In the original statement of Theorem D (see [18]), the condition $H\not\equiv 0$ must be added. Indeed, if $H\equiv 0$, then the conclusions of Theorem D are false. For example, the equation $f'''-f''-2\mathrm{e}^zf'-\mathrm{e}^{3z}f=0$ possesses the solution $f(z)=\mathrm{e}^{\mathrm{e}^z}$ with $\varrho(f)=\infty$ and $\lambda(f)=0$.

It is natural to ask whether the polynomials $h_{k-1}(z),\ldots,h_2(z)$ in (1.4) can be replaced by entire functions of orders that are less than 1. The main purpose of this paper is to study the growth and the oscillation of solutions of the linear differential equation

$$\begin{aligned}f^{(k)}+A_{k-1}f^{(k-1)}+\ldots+A_2f''+(D_1(z)+A_1(z)\mathrm{e}^{az})f'\\+(D_0(z)+A_0(z)\mathrm{e}^{bz})f=F\quad(k\geqslant 2).\end{aligned} \tag{1.5}$$

We obtain the following results.

Theorem 1.1. *Let a and b be complex numbers that satisfy $ab(a-b) \neq 0$. Suppose that $A_j(z)$ $(j=0,1,\ldots,k-1)$, $A_j(z) \not\equiv 0$ $(j=0,1)$, $D_j(z)$ $(j=0,1)$ and $F(z)$ are entire functions with $\max\{\varrho(A_j)\ (j=0,1,\ldots,k-1),\ \varrho(D_j)\ (j=0,1),\ \varrho(F)\} < 1$, and let $\varphi(z) \not\equiv 0$ be an entire function of finite order. Then every solution $f \not\equiv 0$ of equation* (1.5) *satisfies*

$$\overline{\lambda}(f-\varphi) = \varrho(f) = \infty, \quad \overline{\lambda}_2(f-\varphi) = \varrho_2(f) \leqslant 1. \tag{1.6}$$

Furthermore, if $F \not\equiv 0$, then every solution f of equation (1.5) *satisfies*

$$\lambda(f) = \overline{\lambda}(f) = \overline{\lambda}(f-\varphi) = \varrho(f) = \infty \tag{1.7}$$

and

$$\lambda_2(f) = \overline{\lambda}_2(f) = \overline{\lambda}_2(f-\varphi) = \varrho_2(f) \leqslant 1. \tag{1.8}$$

R e m a r k 1.2. The proof of Theorem 1.1 in which every solution f of equation (1.5) has infinite order is quite different from that of Theorem D (see [18]). The main ingredient in the proof is Lemma 2.9.

R e m a r k 1.3. In [18], J. H. Y. Xu and T. B. Cao studied equation (1.5) and obtained the same result as in Theorem 1.1 but under the restriction that the complex constants a, b satisfy $ab \neq 0$ and $ab < 0$ and $A_j(z)$ $(j = 2,\ldots,k-1)$ are polynomials of degree not greater than $n-1$ in z.

Setting $D_j \equiv 0$ $(j=0,1)$ in Theorem 1.1, we obtain the following corollary.

Corollary 1.1. *Let a, b be complex numbers that satisfy $ab(a-b) \neq 0$. Suppose that $A_j(z)$ $(j=0,1,\ldots,k-1)$, $A_j(z) \not\equiv 0$ $(j=0,1)$ and $F(z)$ are entire functions with $\max\{\varrho(A_j)\ (j=0,1,\ldots,k-1),\ \varrho(F)\} < 1$, and let $\varphi(z) \not\equiv 0$ be an entire function of finite order. Then every solution $f \not\equiv 0$ of the equation*

$$f^{(k)} + A_{k-1}f^{(k-1)} + \ldots + A_2 f'' + A_1(z)\mathrm{e}^{az} f' + A_0(z)\mathrm{e}^{bz} f = F \quad (k \geqslant 2) \tag{1.9}$$

satisfies (1.6). *Furthermore, if $F \not\equiv 0$, then every solution f of equation* (1.9) *satisfies* (1.7) *and* (1.8).

R e m a r k 1.4. If $\varrho(F) \geqslant 1$, then equation (1.5) can possess solution of a finite order. For instance the equation

$$f''' - f'' + (\mathrm{e}^{-z} - 1)f' + \mathrm{e}^{z} f = \mathrm{e}^{z}$$

satisfies $\varrho(F) = \varrho(\mathrm{e}^z) = 1$ and has a finite order solution $f(z) = 1$.

Theorem 1.2. *Let a, b, $A_j(z)$ $(j=0,1,\ldots,k-1)$, $D_j(z)$ $(j=0,1)$, and $\varphi(z)$ satisfy the additional hypotheses of Theorem* 1.1, *and let $F(z)$ be an entire function such that $\varrho(F)\geqslant 1$. Then every solution f of equation* (1.5) *satisfies* (1.7) *and* (1.8) *with at most one finite order solution f_0. For the exceptional solution f_0, if $\varrho(F)>1$, then $\varrho(f_0)=\varrho(F)$ and if $\varrho(F)=1$, then $\varrho(f_0)\leqslant 1$.*

Corollary 1.2. *Let a, b, $A_j(z)$ $(j=0,1)$, $D_j(z)$ $(j=0,1)$, and $\varphi(z)$ satisfy the additional hypotheses of Theorem* 1.1, *and let $F(z)$ be an entire function. Then the following statements hold:*

(i) *If $\varrho(F)<1$, then every solution $f\not\equiv 0$ of the equation*

$$f''+(D_1(z)+A_1(z)\mathrm{e}^{az})f'+(D_0(z)+A_0(z)\mathrm{e}^{bz})f=F \quad (k\geqslant 2) \tag{1.10}$$

has infinite order and satisfies (1.6)*. Furthermore, if $F\not\equiv 0$, then every solution f of equation* (1.10) *satisfies* (1.7) *and* (1.8).

(ii) *If $\varrho(F)=1$, then every solution f of equation* (1.10) *has infinite order and satisfies* (1.7) *and* (1.8), *with at most one finite order solution f_0 satisfying $\varrho(f_0)\leqslant 1$.*

(iii) *If $\varrho(F)>1$, then every solution f of equation* (1.10) *has infinite order and satisfies* (1.7) *and* (1.8), *with at most one finite order solution f_0 satisfying $\varrho(f_0)=\varrho(F)$.*

2. Preliminary lemmas

Our proofs depend mainly upon the following lemmas. Before starting these lemmas, we recall the concept of the logarithmic density of subsets of $(1,\infty)$. For $E\subset(1,\infty)$, we define the logarithmic measure of a set E by $\mathrm{lm}(E)=\int_1^\infty \chi_E(t)/t\,\mathrm{d}t$, where χ_E is the characteristic function of E. The upper logarithmic density and the lower logarithmic density of E are defined by

$$\overline{\log\mathrm{dens}}(E)=\limsup_{r\to\infty}\frac{\mathrm{lm}(E\cap[1,r])}{\log r}$$

and

$$\underline{\log\mathrm{dens}}(E)=\liminf_{r\to\infty}\frac{\mathrm{lm}(E\cap[1,r])}{\log r}.$$

Lemma 2.1 ([10]). *Let f be a transcendental meromorphic function of finite order ϱ, let $\Gamma = \{(k_1, j_1), (k_2, j_2), \ldots, (k_m, j_m)\}$ denote a finite set of distinct pairs of integers that satisfy $k_i > j_i \geqslant 0$ for $i = 1, \ldots, m$ and let $\varepsilon > 0$ be a given constant. Then there exists a set $E_1 \subset [0, 2\pi)$ that has linear measure zero, such that if $\psi \in [0, 2\pi) - E_1$, then there is a constant $R_1 = R_1(\psi) > 1$ such that for all z satisfying $\arg z = \psi$ and $|z| \geqslant R_1$ and for all $(k, j) \in \Gamma$ we have*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leqslant |z|^{(k-j)(\varrho-1+\varepsilon)}. \tag{2.1}$$

The next lemma describing the behavior of $\mathrm{e}^{P(z)}$, where $P(z)$ is a linear polynomial, is a special case of a more general result in [14, p. 254].

Lemma 2.2 ([14]). *Let $P(z) = (\alpha + \mathrm{i}\beta)z$, $(\alpha + \mathrm{i}\beta \neq 0)$, and let $A(z)\,(\not\equiv 0)$ be a meromorphic function with $\varrho(A) < 1$. Set $f(z) = A(z)\mathrm{e}^{P(z)}$, $z = r\mathrm{e}^{\mathrm{i}\theta}$, $\delta(P, \theta) = \alpha\cos\theta - \beta\sin\theta$. Then for any given $\varepsilon > 0$ there exists a set $E_2 \subset [0, 2\pi)$ that has linear measure zero such that if $\theta \in [0, 2\pi)\setminus(E_2 \cup E_3)$, where $E_3 = \{\theta \in [0, 2\pi)\colon\ \delta(P, \theta) = 0\}$ is a finite set, then for sufficiently large $|z| = r$, we have*

(i) *If $\delta(P, \theta) > 0$, then*

$$\exp\{(1-\varepsilon)\delta(P,\theta)r\} \leqslant |f(z)| \leqslant \exp\{(1+\varepsilon)\delta(P,\theta)r\}. \tag{2.2}$$

(ii) *If $\delta(P, \theta) < 0$, then*

$$\exp\{(1+\varepsilon)\delta(P,\theta)r\} \leqslant |f(z)| \leqslant \exp\{(1-\varepsilon)\delta(P,\theta)r\}. \tag{2.3}$$

Lemma 2.3 ([4]). *Let $A_0, A_1, \ldots, A_{k-1}, F \not\equiv 0$ be finite order meromorphic functions. If f is a meromorphic solution with $\varrho(f) = \infty$ of the equation*

$$f^{(k)} + A_{k-1}f^{(k-1)} + \ldots + A_1 f' + A_0 f = F, \tag{2.4}$$

then $\overline{\lambda}(f) = \lambda(f) = \varrho(f) = \infty$.

Lemma 2.4 ([2]). *Let $A_0, A_1, \ldots, A_{k-1}, F \not\equiv 0$ be finite order meromorphic functions. If f is a meromorphic solution with $\varrho(f) = \infty$ and $\varrho_2(f) = \varrho$ of equation (2.4), then $\overline{\lambda}(f) = \lambda(f) = \varrho(f) = \infty$ and $\overline{\lambda}_2(f) = \lambda_2(f) = \varrho_2(f) = \varrho$.*

Lemma 2.5 ([3]). *Let a and b be complex numbers, $ab \neq 0$ such that $\arg a \neq \arg b$ or $a = cb$ $(0 < c < 1)$. We denote index sets by*

$$\Lambda_1 = \{0, a\},$$
$$\Lambda_2 = \{0, a, b, 2a, a + b\}.$$

(i) *If H_j $(j \in \Lambda_1)$ and $H_b \not\equiv 0$ are meromorphic functions of orders that are less than 1, setting $\Psi_1(z) = \sum_{j \in \Lambda_1} H_j(z)\mathrm{e}^{jz}$, then $\Psi_1(z) + H_b\mathrm{e}^{bz} \not\equiv 0$.*

(ii) *If H_j $(j \in \Lambda_2)$ and $H_{2b} \not\equiv 0$ are meromorphic functions of orders that are less than 1, setting $\Psi_2(z) = \sum_{j \in \Lambda_2} H_j(z)\mathrm{e}^{jz}$, then $\Psi_2(z) + H_{2b}\mathrm{e}^{2bz} \not\equiv 0$.*

By interchanging a and b in Lemma 2.5, we easily obtain the following lemma.

Lemma 2.6. *Let a and b be complex numbers, $ab \neq 0$, such that $a = cb$ $(c > 1)$. We denote the index set by*

$$\Lambda_3 = \{0, b\}.$$

If H_j $(j \in \Lambda_3)$ and $H_a \not\equiv 0$ are meromorphic functions of orders that are less than 1, setting $\Psi_3(z) = \sum_{j \in \Lambda_3} H_j(z)\mathrm{e}^{jz}$, then $\Psi_3(z) + H_a\mathrm{e}^{az} \not\equiv 0$.

Lemma 2.7 ([6]). *Let $f(z)$ be a transcendental entire function. Then there is a set $E_4 \subset (1, \infty)$ that has finite logarithmic measure, such that for all z with $|z| = r \notin [0, 1] \cup E_4$ at which $|f(z)| = M(r, f)$, we have*

$$\left|\frac{f(z)}{f^{(s)}(z)}\right| \leqslant 2r^s \quad (s \in \mathbb{N}). \tag{2.5}$$

Lemma 2.8 ([17]). *Let $f(z)$ and $g(z)$ be two nonconstant entire functions with $\varrho(g) < \varrho(f) < \infty$. Given ε with $0 < 4\varepsilon < \varrho(f) - \varrho(g)$ and $0 < \delta < \frac{1}{4}$, there exists a set E_5 with $\overline{\log\operatorname{dens}}(E_5) > 0$ such that*

$$\left|\frac{g(z)}{f(z)}\right| \leqslant \exp\{-r^{\varrho(f)-2\varepsilon}\} \tag{2.6}$$

for all z such that $|z| = r \in E_5$ is sufficiently large and that $|f(z)| \geqslant M(r, f)\nu_f(r)^{\delta - \frac{1}{4}}$.

Lemma 2.9. *Let a and b be complex numbers that satisfy $ab(a-b)\neq 0$. Suppose that $A_j(z)$ $(j=0,1,\dots,k-1)$, $A_j(z)\not\equiv 0$ $(j=0,1)$ and $D_j(z)$ $(j=0,1)$ are entire functions with $\max\{\varrho(A_j)\ (j=0,1,\dots,k-1),\ \varrho(D_j)\ (j=0,1)\}<1$. We denote*

$$L_f = f^{(k)} + A_{k-1}f^{(k-1)} + \ldots + A_2 f'' + (D_1(z)+A_1(z)\mathrm{e}^{az})f' + (D_0(z)+A_0(z)\mathrm{e}^{bz})f. \tag{2.7}$$

If $f\not\equiv 0$ is a finite order entire function, then we have

$$\varrho(L_f)=\max\{1,\varrho(f)\}.$$

Proof. Let $f\not\equiv 0$ be a finite order entire function. First, if $f(z)\equiv C\neq 0$, then

$$L_f=(D_0(z)+A_0(z)\mathrm{e}^{bz})C.$$

Hence, $\varrho(L_f)=1$ and Lemma 2.9 holds.

We suppose $f\not\equiv C$. Then, by (2.7), we have $\varrho(L_f)\leqslant\max\{1,\varrho(f)\}$.

(i) If $\varrho(f)=\varrho<1$, then $\varrho(L_f)\leqslant 1$. Suppose that $\varrho(L_f)<1$. By (2.7), we easily obtain a contradiction by Lemma 2.5 (i) or Lemma 2.6. Thus $\varrho(L_f)=1$.

(ii) If $\varrho(f)=\varrho\geqslant 1$, then $\varrho(L_f)\leqslant\varrho(f)$. Suppose that $\varrho(L_f)<\varrho(f)$. We can rewrite (2.7) as

$$\frac{L_f}{f}=\frac{f^{(k)}}{f}+A_{k-1}\frac{f^{(k-1)}}{f}+\ldots+A_2\frac{f''}{f} + (D_1(z)+A_1(z)\mathrm{e}^{az})\frac{f'}{f}+D_0(z)+A_0(z)\mathrm{e}^{bz}. \tag{2.8}$$

We divide the proof in three cases.

Case 1. Suppose first that $\arg a\neq\arg b$. Set

$$\max\{\varrho(A_j)\ (j=0,1,\dots,k-1),\ \varrho(D_j)\ (j=0,1)\}=\beta<1.$$

Then, for any given ε $(0<\varepsilon<\min(1-\beta,\frac14(\varrho(f)-\varrho(L_f)))$, we have for sufficiently large r

$$|D_j(z)|\leqslant\exp\{r^{\beta+\varepsilon}\}\quad (j=0,1),\qquad |A_j(z)|\leqslant\exp\{r^{\beta+\varepsilon}\}\quad (j=0,1,\dots,k-1). \tag{2.9}$$

By Lemma 2.8, we know that there exists a set E_5 with $\overline{\log\operatorname{dens}}(E_5)>0$ such that

$$\left|\frac{L_f}{f}\right|\leqslant\exp\{-r^{\varrho(f)-2\varepsilon}\}\leqslant 1 \tag{2.10}$$

for all z such that $|z| = r \in E_5$ is sufficiently large and that $|f(z)| \geqslant M(r,f)\nu_f(r)^{\delta-\frac{1}{4}}$. Also, by Lemma 2.1, for the above ε there exists a set $E_1 \subset [0,2\pi)$ that has linear measure zero, such that if $\theta \in [0,2\pi) - E_1$, then there is a constant $R_1 = R_1(\theta) > 1$ such that for all z satisfying $\arg z = \theta$ and $|z| \geqslant R_1$, we have

$$\Big|\frac{f^{(i)}(z)}{f(z)}\Big| \leqslant |z|^{i(\varrho-1+\varepsilon)} \quad (i = 1,\dots,k). \tag{2.11}$$

By Lemma 2.2, there exists a ray $\arg z = \theta \in [0,2\pi) \setminus E_1 \cup E_2 \cup E_3$, $E_3 = \{\theta \in [0,2\pi)\colon\ \delta(az,\theta) = 0 \text{ or } \delta(bz,\theta) = 0\} \subset [0,2\pi)$, $E_1 \cup E_2$ having linear measure zero, E_3 being a finite set, such that

$$\delta(az,\theta) < 0, \quad \delta(bz,\theta) > 0$$

and for any given ε $(0 < \varepsilon < \min(1-\beta, \frac{1}{4}(\varrho(f) - \varrho(L_f))))$, by (2.9), (2.11) we have for sufficiently large $|z| = r$

$$|A_0 e^{bz}| \geqslant \exp\{(1-\varepsilon)\delta(bz,\theta)r\}, \tag{2.12}$$

$$\begin{aligned}
\Big|\frac{f^{(k)}}{f} A_{k-1}\frac{f^{(k-1)}}{f} + \ldots + A_2\frac{f''}{f} + D_0(z)\Big| \qquad\qquad (2.13)\\
&\leqslant \Big|\frac{f^{(k)}}{f}\Big| + |A_{k-1}|\Big|\frac{f^{(k-1)}}{f}\Big| + \ldots + |A_2|\Big|\frac{f''}{f}\Big| + |D_0(z)| \\
&\leqslant r^{k(\varrho-1+\varepsilon)} + r^{(k-1)(\varrho-1+\varepsilon)}\exp\{r^{\beta+\varepsilon}\} + \ldots \\
&\quad + r^{2(\varrho-1+\varepsilon)}\exp\{r^{\beta+\varepsilon}\} + \exp\{r^{\beta+\varepsilon}\} \\
&\leqslant kr^{k(\varrho-1+\varepsilon)}\exp\{r^{\beta+\varepsilon}\},
\end{aligned}$$

$$\begin{aligned}
\Big|(D_1(z) + A_1(z)e^{az})\frac{f'}{f}\Big| \qquad\qquad (2.14)\\
&\leqslant r^{\varrho-1+\varepsilon}(\exp\{(1-\varepsilon)\delta(az,\theta)r\} + \exp\{r^{\beta+\varepsilon}\}) \\
&\leqslant r^{\varrho-1+\varepsilon}(1 + \exp\{r^{\beta+\varepsilon}\}).
\end{aligned}$$

By (2.8), (2.10), and (2.12)–(2.14) we have

$$\exp\{(1-\varepsilon)\delta(bz,\theta)r\} \leqslant |A_0 e^{bz}| \leqslant Kr^{k(\varrho-1+\varepsilon)}\exp\{r^{\beta+\varepsilon}\},$$

where $K > 0$ is a real constant. This is a contradiction by $\beta + \varepsilon < 1$. Hence, $\varrho(L_f) = \varrho(f)$.

Case 2. Suppose now $a = cb$ $(0 < c < 1)$. Then for any ray $\arg z = \theta$ we have

$$\delta(az,\theta) = c\delta(bz,\theta).$$

Then, by Lemma 2.2, for any given ε $(0 < \varepsilon < \min((1-c)/(2(1+c)), 1-\beta, \frac{1}{4}(\varrho(f)-\varrho(L_f))))$ there exist $E_j \subset [0, 2\pi)$ $(j = 1, 2, 3)$ such that E_1, E_2 have linear measure zero and E_3 is a finite set, where E_1, E_2, and E_3 are defined as in Case 1, respectively. We take the ray $\arg z = \theta \in [0, 2\pi) \setminus E_1 \cup E_2 \cup E_3$ such that $\delta(bz, \theta) > 0$ and for sufficiently large $|z| = r$, we have (2.12), (2.13), and

$$\left|(D_1(z) + A_1(z)\mathrm{e}^{az})\frac{f'}{f}\right| \leqslant r^{\varrho-1+\varepsilon}(\exp\{r^{\beta+\varepsilon}\} + \exp\{(1+\varepsilon)c\delta(bz,\theta)r\}). \tag{2.15}$$

Thus by (2.8), (2.10), (2.12), (2.13), and (2.15) we obtain

$$\begin{aligned}\exp\{(1-\varepsilon)\delta(bz,\theta)r\} &\leqslant |A_0\mathrm{e}^{bz}| \\ &\leqslant kr^{k(\varrho-1+\varepsilon)}\exp\{r^{\beta+\varepsilon}\} \\ &\quad + r^{\varrho-1+\varepsilon}(\exp\{r^{\beta+\varepsilon}\} + \exp\{(1+\varepsilon)c\delta(bz,\theta)r\}) + 1 \\ &\leqslant (k+1)r^{k(\varrho-1+\varepsilon)}\exp\{r^{\beta+\varepsilon}\} \\ &\quad + r^{\varrho-1+\varepsilon}\exp\{(1+\varepsilon)c\delta(bz,\theta)r\} + 1.\end{aligned} \tag{2.16}$$

For ε $(0 < \varepsilon < \min((1-c)/(2(1+c)), 1-\beta, \frac{1}{4}(\varrho(f)-\varrho(L_f))))$, we have as $r \to \infty$

$$\frac{(k+1)r^{k(\varrho-1+\varepsilon)}\exp\{r^{\beta+\varepsilon}\}}{\exp\{(1-\varepsilon)\delta(bz,\theta)r\}} \to 0, \tag{2.17}$$

$$\frac{r^{\varrho-1+\varepsilon}\exp\{(1+\varepsilon)c\delta(bz,\theta)r\}}{\exp\{(1-\varepsilon)\delta(bz,\theta)r\}} \to 0, \tag{2.18}$$

$$\frac{1}{\exp\{(1-\varepsilon)\delta(bz,\theta)r\}} \to 0. \tag{2.19}$$

By (2.16)–(2.19), we get $1 \leqslant 0$. This is a contradiction. Hence, $\varrho(L_f) = \varrho(f)$.

Case 3. Finally, we suppose $a = cb$ $(c > 1)$. We can rewrite (2.7) as

$$\begin{aligned}\frac{L_f}{f}\frac{f}{f'} &= \frac{f^{(k)}}{f'} + A_{k-1}\frac{f^{(k-1)}}{f'} + \ldots + A_2\frac{f''}{f'} \\ &\quad + (D_0(z) + A_0(z)\mathrm{e}^{bz})\frac{f}{f'} + D_1(z) + A_1(z)\mathrm{e}^{az}.\end{aligned} \tag{2.20}$$

By Lemma 2.7, there is a set $E_4 \subset (1, \infty)$ that has finite logarithmic measure such that for all z with $|z| = r \notin [0, 1] \cup E_4$ at which $|f(z)| = M(r, f)$ we have

$$\left|\frac{f(z)}{f'(z)}\right| \leqslant 2r. \tag{2.21}$$

By Lemma 2.8, for ε $(0 < \varepsilon < \min((c-1)/(2(c+1)), 1-\beta, \frac{1}{4}(\varrho(f)-\varrho(L_f))))$, we know that there exists a set E_5 with $\overline{\log\operatorname{dens}}(E_5) > 0$ such that

$$\left|\frac{L_f}{f}\right| \leqslant \exp\{-r^{\varrho(f)-2\varepsilon}\} \leqslant 1 \tag{2.22}$$

for all z such that $|z| = r \in E_5$ is sufficiently large and that $|f(z)| \geqslant M(r,f)\nu_f(r)^{\delta-\frac{1}{4}}$. Since $E_4 \subset (1,\infty)$ has finite logarithmic measure and E_5 satisfies $\overline{\log\text{dens}}(E_5) > 0$, we have $\overline{\log\text{dens}}(E_5 - ([0,1] \cup E_4)) > 0$. By (2.21) and (2.22) for sufficiently large $|z| = r$ we get

$$\Big|\frac{L_f}{f'}\Big| = \Big|\frac{L_f}{f}\,\frac{f}{f'}\Big| \leqslant 2r\exp\{-r^{\varrho(f)-2\varepsilon}\} \leqslant 2r. \tag{2.23}$$

For any ray $\arg z = \theta$, we have

$$\delta(az,\theta) = c\delta(bz,\theta).$$

By Lemma 2.2, there exists a ray $\arg z = \theta \in [0,2\pi) \setminus E_1 \cup E_2 \cup E_3$, $E_3 = \{\theta \in [0,2\pi)\colon\ \delta(az,\theta) = 0 \text{ or } \delta(bz,\theta) = 0\} \subset [0,2\pi)$, $E_1 \cup E_2$ having linear measure zero, E_3 being a finite set, such that

$$\delta(az,\theta) = c\delta(bz,\theta) > 0$$

and by (2.9), (2.11), and (2.21), for sufficiently large $|z| = r$ we have

$$|A_1 e^{az}| \geqslant \exp\{(1-\varepsilon)c\delta(bz,\theta)r\}, \tag{2.24}$$

$$\Big|(D_0(z) + A_0(z)e^{bz})\frac{f}{f'}\Big| \leqslant 2r\exp\{r^{\beta+\varepsilon}\} + 2r\exp\{(1+\varepsilon)\delta(bz,\theta)r\}, \tag{2.25}$$

$$\begin{aligned} &\Big|\frac{f^{(k)}}{f'} + A_{k-1}\frac{f^{(k-1)}}{f'} + \ldots + A_2\frac{f''}{f'} + D_1\Big| \\ &\quad\leqslant \Big|\frac{f(z)}{f'(z)}\Big|\Big(\Big|\frac{f^{(k)}}{f}\Big| + |A_{k-1}|\Big|\frac{f^{(k-1)}}{f}\Big| + \ldots + |A_2|\Big|\frac{f''}{f}\Big|\Big) + |D_1| \\ &\quad\leqslant 2r(k-1)r^{k(\varrho-1+\varepsilon)}\exp\{r^{\beta+\varepsilon}\} + \exp\{r^{\beta+\varepsilon}\} \\ &\quad\leqslant 2kr^{k(\varrho-1+\varepsilon)+1}\exp\{r^{\beta+\varepsilon}\}. \end{aligned} \tag{2.26}$$

By (2.20), (2.23), and (2.24)–(2.26), we have

$$\begin{aligned} \exp\{(1-\varepsilon)c\delta(bz,\theta)r\} &\leqslant |A_1 e^{az}| \\ &\leqslant 2kr^{k(\varrho-1+\varepsilon)+1}\exp\{r^{\beta+\varepsilon}\} \\ &\qquad + 2r\exp\{r^{\beta+\varepsilon}\} + 2r\exp\{(1+\varepsilon)\delta(bz,\theta)r\} + 2r \\ &\leqslant 2(k+1)r^{k(\varrho-1+\varepsilon)+1}\exp\{r^{\beta+\varepsilon}\} \\ &\qquad + 2r\exp\{(1+\varepsilon)\delta(bz,\theta)r\} + 2r. \end{aligned} \tag{2.27}$$

For ε $(0 < \varepsilon < \min((c-1)/(2(c+1)), 1-\beta, \frac{1}{4}(\varrho(f) - \varrho(L_f))))$, we have as $r \to \infty$

$$\frac{2(k+1)r^{k(\varrho-1+\varepsilon)+1}\exp\{r^{\beta+\varepsilon}\}}{\exp\{(1-\varepsilon)c\delta(bz,\theta)r\}} \to 0, \tag{2.28}$$

$$\frac{2r\exp\{(1+\varepsilon)\delta(bz,\theta)r\}}{\exp\{(1-\varepsilon)c\delta(bz,\theta)r\}} \to 0, \tag{2.29}$$

$$\frac{2r}{\exp\{(1-\varepsilon)c\delta(bz,\theta)r\}} \to 0. \tag{2.30}$$

By (2.27)–(2.30), we get $1 \leqslant 0$. This is a contradiction. Hence, $\varrho(L_f) = \varrho(f)$. □

By using the Wiman-Valiron theory [12], we easily obtain the following result of which we omit the proof.

Lemma 2.10. *Let $A_0(z), \ldots, A_{k-1}(z), F(z)$ be entire functions of finite order. If f is a solution of the equation*

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \ldots + A_1(z)f' + A_0(z)f = F, \tag{2.31}$$

then $\varrho_2(f) \leqslant \max\{\varrho(A_0), \ldots, \varrho(A_{k-1}), \varrho(F)\}$.

3. Proof of Theorem 1.1

Assume that $f \not\equiv 0$ is a solution of equation (1.5). We prove that f is of infinite order. We suppose the contrary: $\varrho(f) < \infty$. By Lemma 2.9, we have $1 \leqslant \varrho(L_f) = \varrho(F) < 1$ and this is a contradiction. Hence, every solution of equation (1.5) is of infinite order and by Lemma 2.10 we have $\varrho_2(f) \leqslant 1$. Suppose that $\varphi(z) \not\equiv 0$ is an entire function of finite order. Set $g = f - \varphi$, then $f = g + \varphi$ and due to $\varrho(\varphi) < \infty$ we have $\varrho(f) = \varrho(g) = \infty$, $\varrho_2(f) = \varrho_2(g) \leqslant 1$. Thus, g is a solution of the equation

$$g^{(k)} + A_{k-1}g^{(k-1)} + \ldots + A_2g'' + (D_1 + A_1\mathrm{e}^{az})g' + (D_0 + A_0\mathrm{e}^{bz})g = H,$$

where

$$H = F - (\varphi^{(k)} + A_{k-1}\varphi^{(k-1)} + \ldots + A_2\varphi'' + (D_1 + A_1\mathrm{e}^{az})\varphi' + (D_0 + A_0\mathrm{e}^{bz})\varphi).$$

By $\varphi(z) \not\equiv 0$ and $\varrho(\varphi) < \infty$ we have $H \not\equiv 0$. Since $\varrho(H) < \infty$, by Lemma 2.3 and Lemma 2.4 we get

$$\overline{\lambda}(f-\varphi) = \varrho(f-\varphi) = \varrho(f) = \infty, \quad \overline{\lambda}_2(f-\varphi) = \varrho_2(f-\varphi) = \varrho_2(f) \leqslant 1.$$

Furthermore, if $F \not\equiv 0$, then by the fact that f is an infinite order solution of equation (1.5), and due to Lemma 2.3 and Lemma 2.4 we have

$$\lambda(f) = \overline{\lambda}(f) = \overline{\lambda}(f - \varphi) = \varrho(f) = \infty,$$
$$\lambda_2(f) = \overline{\lambda}_2(f) = \overline{\lambda}_2(f - \varphi) = \varrho_2(f) \leqslant 1.$$

□

4. Proof of Theorem 1.2

Assume that f_0 is a solution of (1.5) with $\varrho(f_0) = \varrho < \infty$. If f_1 is another finite order solution of (1.5), then $\varrho(f_1 - f_0) < \infty$, and $f_1 - f_0$ is a solution of the corresponding homogeneous equation of (1.5), but $\varrho(f_1 - f_0) = \infty$ by virtue of Theorem 1.1, which is a contradiction. Hence, (1.5) has at most one finite order solution f_0 and all other solutions f_1 of (1.5) are of infinite order and satisfy (1.7) and (1.8). If $\varrho(F) > 1$, suppose there exists a solution f_0 of (1.5) with $\varrho(f_0) < \infty$. Then, we have $\varrho(f_0) > 1$ and by Lemma 2.9 we get $\varrho(L_f) = \varrho(f_0) = \varrho(F)$. Suppose that $\varrho(F) = 1$. If there exists a solution f_0 of (1.5) with $\varrho(f_0) < \infty$, then $\varrho(f_0) \leqslant 1$. Indeed, if we suppose that $\varrho(f_0) > 1$, then by Lemma 2.9 we get $\varrho(L_f) = \varrho(f_0) = \varrho(F) > 1$ and this is a contradiction. □

5. Proof of Corollary 1.2

By using Theorem 1.1 and Theorem 1.2, we obtain Corollary 1.2. □

Acknowledgment. The authors would like to thank the referee for his/her helpful remarks and suggestions improving the paper.

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