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PERIODIC SOLUTIONS FOR SECOND ORDER
HAMILTONIAN SYSTEMS*

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Abstract. By using the least action principle and minimax methods in critical point theory, some existence theorems for periodic solutions of second order Hamiltonian systems are obtained.

Keywords: periodic solutions, minimax methods, second order Hamiltonian systems

MSC 2010: 34C25, 58E05, 70H05

1. INTRODUCTION

Consider periodic solutions of the Hamiltonian system

$$(1.1) \quad \begin{cases} \ddot{u}(t) = \nabla F(t, u(t)) & \text{a.e. } t \in [0, T], \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T), \end{cases}$$

where $T > 0$, $F: \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ satisfies the following assumption:

- (A) $F(t, x)$ is measurable in t for all $x \in \mathbb{R}^n$ and continuously differentiable in x for a.e. $t \in [0, T]$, and there exist $a \in C(\mathbb{R}^+, \mathbb{R}^+)$, $b \in L^1([0, T]; \mathbb{R}^+)$ such that

$$|F(t, x)| \leq a(|x|)b(t), \quad |\nabla F(t, x)| \leq a(|x|)b(t)$$

for all $x \in \mathbb{R}^n$ and a.e. $t \in [0, T]$.

The corresponding functional φ on H_T^1 given by

$$(1.2) \quad \varphi(u) = \frac{1}{2} \int_0^T |\dot{u}(t)|^2 dt + \int_0^T F(t, u(t)) dt$$

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is continuously differentiable and weakly lower semicontinuous on H_T^1 , and

$$\langle \varphi'(u), v \rangle = \int_0^T [(\dot{u}(t), \dot{v}(t)) + (\nabla F(t, u(t)), v(t))] dt$$

for any $u, v \in H_T^1$, where

$$H_T^1 = \{u: [0, T] \rightarrow \mathbb{R}^n: \\ u \text{ is absolutely continuous, } u(0) = u(T), \dot{u} \in L^2([0, T], \mathbb{R}^n)\}$$

is a Hilbert space with the norm defined by

$$\|u\| = \left(\int_0^T [|\dot{u}(t)|^2 + |u(t)|^2] dt \right)^{1/2}$$

for $u \in H_T^1$. It is well known that the solutions of problem (1.1) correspond to the critical points of φ .

It has been proved that problem (1.1) has at least one solution by the least action principle and the minimax methods; we refer the readers to [1]–[15] and the references therein. Particularly, when the nonlinearity $\nabla F(t, x)$ is bounded, that is, there exists $g(t) \in L^1([0, T], \mathbb{R}^+)$ such that

$$|\nabla F(t, x)| \leq g(t)$$

for all $x \in \mathbb{R}^n$ and a.e. $t \in [0, T]$, and that

$$\int_0^T F(t, x) dt \rightarrow \pm\infty \quad \text{as } |x| \rightarrow \infty,$$

Mawhin and Willem [3] proved that problem (1.1) admits a periodic solution. After that, Tang [9] generalized these results to the sublinear case. In detail, he assumed that the nonlinearity $\nabla F(t, x)$ satisfies the following conditions:

$$(1.3) \quad |\nabla F(t, x)| \leq p(t)|x|^\alpha + q(t) \quad \text{for all } x \in \mathbb{R}^n \text{ and a.e. } t \in [0, T],$$

and

$$(1.4) \quad |x|^{-2\alpha} \int_0^T F(t, x) dt \rightarrow \pm\infty \quad \text{as } |x| \rightarrow \infty,$$

where $p(t), q(t) \in L^1([0, T], \mathbb{R}^+)$ and $\alpha \in [0, 1)$. Under these conditions, periodic solutions of problem (1.1) have been obtained. Subsequently, Zhao in [14], [15]

considered problem (1.1) when $\nabla F(t, x)$ was linear. He completed the results in [9] corresponding to $\alpha = 1$. As pointed out in [16], there are functions not satisfying the conditions in [9], [14], [15], thus the authors improved the conditions in [9], [14], [15] and obtained some new results.

Recently, Zhang and Wang [17] used a control function $h(|x|)$ instead of $|x|^\alpha$ in (1.3) and (1.4) and got some new results which improved many existed works. More precisely, they obtained the following main results.

Theorem A ([17]). *Suppose that F satisfies assumption (A) and the following conditions:*

(H1) *There exist constants $C_0 > 0$, $K_1 > 0$, $K_2 > 0$, $\alpha \in [0, 1)$ and a nonnegative function $h \in C([0, \infty), [0, \infty))$ with the properties*

- (i) $h(s) \leq h(t) \quad \forall s \leq t, s, t \in [0, \infty)$,
- (ii) $h(s+t) \leq C_0(h(s) + h(t)) \quad \forall s, t \in [0, \infty)$,
- (iii) $0 \leq h(s) \leq K_1 s^\alpha + K_2 \quad \forall s, t \in [0, \infty)$,
- (iv) $h(s) \rightarrow \infty \quad \text{as } s \rightarrow \infty$.

Moreover, there exist $f, g \in L^1([0, T], \mathbb{R}^+)$ such that

$$(1.5) \quad |\nabla F(t, x)| \leq f(t)h(|x|) + g(t) \quad \text{for all } x \in \mathbb{R}^n \text{ and a.e. } t \in [0, T].$$

(H2) *There exists a function $h \in C([0, \infty), [0, \infty))$ which satisfies the conditions (i)–(iv) and*

$$(1.6) \quad h^{-2}(|x|) \int_0^T F(t, x) dt \rightarrow \infty \quad \text{as } |x| \rightarrow \infty.$$

Then (1.1) has at least one solution which minimizes the functional φ in H_T^1 .

Theorem B ([17]). *Suppose that (H1), assumption (A) and the following condition hold:*

$$(H3) \quad h^{-2}(|x|) \int_0^T F(t, x) dt \rightarrow -\infty \quad \text{as } |x| \rightarrow \infty.$$

Then (1.1) has at least one solution in H_T^1 .

Motivated by the ideas of [16], [17], we will use weaker conditions instead of (H2) and (H3). Here are our main results.

Theorem 1.1. Suppose that F satisfies (H1), (A) and the following condition:

(H2)' There exists a function $h \in C([0, \infty), [0, \infty))$ which satisfies the conditions (i)–(iv) and

$$(1.7) \quad \liminf_{|x| \rightarrow \infty} h^{-2}(|x|) \int_0^T F(t, x) dt > \frac{T^2 C_0^2}{8\pi^2} \int_0^T f^2(t) dt.$$

Then (1.1) has at least one solution which minimizes the functional φ in H_T^1 .

Theorem 1.2. Suppose that F satisfies (H1), (A) and the following condition:

(H3)' $\limsup_{|x| \rightarrow \infty} h^{-2}(|x|) \int_0^T F(t, x) dt < -\frac{3}{8}(T^2 C_0^2 / \pi^2) \int_0^T f^2(t) dt.$

Then (1.1) has at least one solution in H_T^1 .

Theorem 1.3. Suppose that F satisfies (H1) with $\alpha = 1$, assumption (A) and that

$$(1.8) \quad \int_0^T f(t) dt < \frac{12}{TK_1 C_0}$$

and

$$(1.9) \quad \liminf_{|x| \rightarrow \infty} h^{-2}(|x|) \int_0^T F(t, x) dt > \frac{3T^2 C_0^2 \int_0^T f^2(t) dt}{2\pi^2(12 - K_1 C_0 T \int_0^T f(t) dt)}.$$

Then (1.1) has at least one solution which minimizes the functional φ in H_T^1 .

Theorem 1.4. Suppose that F satisfies (H1) with $\alpha = 1$, assumption (A), (1.8), and the following condition:

$$(H4) \quad \limsup_{|x| \rightarrow \infty} h^{-2}(|x|) \int_0^T F(t, x) dt < -\frac{216T^2 C_0^2 \int_0^T f^2(t) dt}{\pi^2(24 - K_1 C_0 T \int_0^T f(t) dt)^2}.$$

Then (1.1) has at least one solution in H_T^1 .

2. PROOFS OF THEOREMS

For $u \in H_T^1$, let $\bar{u} = T^{-1} \int_0^T u(t) dt$ and $\tilde{u} = u(t) - \bar{u}$, then one has

$$\|\tilde{u}\|_\infty^2 \leq \frac{T}{12} \int_0^T |\dot{u}(t)|^2 dt \quad (\text{Sobolev's inequality}),$$

and

$$\|\tilde{u}\|_{L^2}^2 \leq \frac{T^2}{4\pi^2} \int_0^T |\dot{u}(t)|^2 dt \quad (\text{Wirtinger's inequality}),$$

where $\|\tilde{u}\|_\infty := \max_{0 \leq t \leq T} |\tilde{u}(t)|$. For the sake of convenience, we denote

$$M_1 = \left(\int_0^T f^2(t) dt \right)^{1/2}, \quad M_2 = \int_0^T f(t) dt, \quad M_3 = \int_0^T g(t) dt.$$

Proof of Theorem 1.1. Due to (1.7), we can choose $a_1 > T^2/4\pi^2$ such that

$$(2.1) \quad \liminf_{|x| \rightarrow \infty} h^{-2}(|x|) \int_0^T F(t, x) dt > \frac{a_1 C_0^2}{2} \int_0^T f^2(t) dt.$$

It follows from (H1), Sobolev's inequality and Wirtinger's inequality that

$$\begin{aligned} & \left| \int_0^T [F(t, u(t)) - F(t, \bar{u})] dt \right| \\ &= \left| \int_0^T \int_0^1 (\nabla F(t, \bar{u} + s\tilde{u}(t)), \tilde{u}(t)) ds dt \right| \\ &\leq \int_0^T \int_0^1 f(t) h(|\bar{u} + s\tilde{u}(t)|) |\tilde{u}(t)| ds dt + \int_0^T \int_0^1 g(t) |\tilde{u}(t)| ds dt \\ &\leq \int_0^T \int_0^1 f(t) h(|\bar{u}| + |\tilde{u}(t)|) |\tilde{u}(t)| ds dt + \int_0^T \int_0^1 g(t) |\tilde{u}(t)| ds dt \\ &\leq \int_0^T \int_0^1 C_0 f(t) (h(|\bar{u}|) + h(|\tilde{u}(t)|)) |\tilde{u}(t)| ds dt + M_3 \|\tilde{u}\|_\infty \\ &\leq C_0 h(|\bar{u}|) \left(\int_0^T f^2(t) \right)^{1/2} \left(\int_0^T |\tilde{u}(t)|^2 dt \right)^{1/2} \\ &\quad + C_0 \int_0^T f(t) h(|\tilde{u}(t)|) |\tilde{u}(t)| dt + M_3 \|\tilde{u}\|_\infty \\ &\leq C_0 M_1 h(|\bar{u}|) \|\tilde{u}\|_{L^2} + C_0 \int_0^T f(t) (K_1 |\tilde{u}(t)|^\alpha + K_2) |\tilde{u}(t)| dt + M_3 \|\tilde{u}\|_\infty \\ &\leq C_0 M_1 h(|\bar{u}|) \|\tilde{u}\|_{L^2} + C_0 M_2 K_1 \|\tilde{u}\|_\infty^{1+\alpha} + C_0 M_2 K_2 \|\tilde{u}\|_\infty + M_3 \|\tilde{u}\|_\infty \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{2a_1} \|\tilde{u}\|_{L^2}^2 + \frac{a_1(C_0M_1)^2}{2} h^2(|\bar{u}|) + C_0M_2K_1 \|\tilde{u}\|_\infty^{1+\alpha} \\
&\quad + C_0M_2K_2 \|\tilde{u}\|_\infty + M_3 \|\tilde{u}\|_\infty \\
&\leq \frac{T^2}{8\pi^2a_1} \|\dot{u}\|_{L^2}^2 + \frac{a_1(C_0M_1)^2}{2} h^2(|\bar{u}|) + C_0M_2K_1 \left(\frac{T}{12}\right)^{(1+\alpha)/2} \|\dot{u}\|_{L^2}^{1+\alpha} \\
&\quad + C_0M_2K_2 \left(\frac{T}{12}\right)^{1/2} \|\dot{u}\|_{L^2} + M_3 \left(\frac{T}{12}\right)^{1/2} \|\dot{u}\|_{L^2}.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\varphi(u) &= \frac{1}{2} \int_0^T |\dot{u}(t)|^2 dt + \int_0^T F(t, u(t)) dt \\
&= \frac{1}{2} \int_0^T |\dot{u}(t)|^2 dt + \int_0^T [F(t, u(t)) - F(t, \bar{u})] dt + \int_0^T F(t, \bar{u}) dt \\
&\geq \frac{1}{2} \|\dot{u}\|_{L^2}^2 - \frac{T^2}{8\pi^2a_1} \|\dot{u}\|_{L^2}^2 - \frac{a_1(C_0M_1)^2}{2} h^2(|\bar{u}|) \\
&\quad - C_0M_2K_1 \left(\frac{T}{12}\right)^{(1+\alpha)/2} \|\dot{u}\|_{L^2}^{1+\alpha} \\
&\quad - C_0M_2K_2 \left(\frac{T}{12}\right)^{1/2} \|\dot{u}\|_{L^2} - M_3 \left(\frac{T}{12}\right)^{1/2} \|\dot{u}\|_{L^2} + \int_0^T F(t, \bar{u}) dt \\
&= \left(\frac{1}{2} - \frac{T^2}{8\pi^2a_1}\right) \|\dot{u}\|_{L^2}^2 + h^2(|\bar{u}|) \left(h^{-2}(|\bar{u}|) \int_0^T F(t, \bar{u}) dt - \frac{a_1C_0^2M_1^2}{2}\right) \\
&\quad - \left(\frac{T}{12}\right)^{1/2} (C_0M_2K_2 + M_3) \|\dot{u}\|_{L^2} - C_0M_2K_1 \left(\frac{T}{12}\right)^{(1+\alpha)/2} \|\dot{u}\|_{L^2}^{1+\alpha}.
\end{aligned}$$

As $\|u\| \rightarrow \infty$ if and only if $(|\bar{u}|^2 + \|\dot{u}\|_{L^2}^2)^{1/2} \rightarrow \infty$, the above inequality and (2.1) imply that $\varphi(u) \rightarrow \infty$. Hence, by the least action principle, problem (1.1) has at least one solution which minimizes the functional $\varphi(u)$ in H_T^1 . \square

Proof of Theorem 1.2. First we prove that φ satisfies the (PS) condition. Assume that u_n is a (PS) sequence of φ , that is, $\varphi'(u_n) \rightarrow 0$ as $n \rightarrow \infty$ and $\{\varphi(u_n)\}$ is bounded. By (H3)', we can choose $a_2 > T^2/4\pi^2$ such that

$$(2.2) \quad \limsup_{|x| \rightarrow \infty} h^{-2}(|x|) \int_0^T F(t, x) dt < -\left(\frac{a_2}{2} + \frac{\sqrt{a_2}T}{2\pi}\right) C_0^2 \int_0^T f^2(t) dt.$$

In a way similar to the proof of Theorem 1.1, we have

$$\begin{aligned}
&\left| \int_0^T (\nabla F(t, u_n(t)), \tilde{u}_n(t)) dt \right| \\
&\leq \frac{T^2}{8\pi^2a_2} \|\dot{u}_n\|_{L^2} + \frac{a_2(C_0M_1)^2}{2} h^2(|\bar{u}_n|) + C_0M_2K_1 \left(\frac{T}{12}\right)^{(1+\alpha)/2} \|\dot{u}_n\|_{L^2}^{1+\alpha} \\
&\quad + \left(\frac{T}{12}\right)^{1/2} (C_0M_2K_2 + M_3) \|\dot{u}_n\|_{L^2}.
\end{aligned}$$

Hence, we get

$$\begin{aligned}
(2.3) \quad \|\tilde{u}_n\| &\geq \langle \varphi'(u_n), \tilde{u}_n \rangle \\
&= \|\dot{u}_n\|_{L^2}^2 + \int_0^T (\nabla F(t, u_n(t)), \tilde{u}_n(t)) \, dt \\
&\geq \left(1 - \frac{T^2}{8\pi^2 a_2}\right) \|\dot{u}_n\|_{L^2}^2 - \frac{a_2(C_0 M_1)^2}{2} h^2(|\bar{u}_n|) \\
&\quad - C_0 M_2 K_1 \left(\frac{T}{12}\right)^{(1+\alpha)/2} \|\dot{u}_n\|_{L^2}^{1+\alpha} \\
&\quad - \left(\frac{T}{12}\right)^{1/2} (C_0 M_2 K_2 + M_3) \|\dot{u}_n\|_{L^2}.
\end{aligned}$$

On the other hand, we find that

$$(2.4) \quad \|\tilde{u}_n\| \leq \frac{(T^2 + 4\pi^2)^{1/2}}{2\pi} \|\dot{u}_n\|_{L^2}.$$

From (2.3) and (2.4), we obtain

$$\begin{aligned}
(2.5) \quad &\frac{a_2 C_0^2 M_1^2}{2} h^2(|\bar{u}_n|) \\
&\geq \left(1 - \frac{T^2}{8\pi^2 a_2}\right) \|\dot{u}_n\|_{L^2}^2 - C_0 M_2 K_1 \left(\frac{T}{12}\right)^{(1+\alpha)/2} \|\dot{u}_n\|_{L^2}^{1+\alpha} \\
&\quad - \left(\frac{T}{12}\right)^{1/2} (C_0 M_2 K_2 + M_3) \|\dot{u}_n\|_{L^2} - \|\tilde{u}_n\| \\
&\geq \left(1 - \frac{T^2}{8\pi^2 a_2}\right) \|\dot{u}_n\|_{L^2}^2 - C_0 M_2 K_1 \left(\frac{T}{12}\right)^{(1+\alpha)/2} \|\dot{u}_n\|_{L^2}^{1+\alpha} \\
&\quad - \left[\frac{(T^2 + 4\pi^2)^{1/2}}{2\pi} + C_0 M_2 K_2 \left(\frac{T}{12}\right)^{1/2} + M_3 \left(\frac{T}{12}\right)^{1/2}\right] \|\dot{u}_n\|_{L^2}^2 \\
&\geq \frac{1}{2} \|\dot{u}_n\|_{L^2}^2 + C_1,
\end{aligned}$$

where

$$\begin{aligned}
C_1 = \min_{s \in [0, \infty)} &\left\{ \frac{4\pi^2 a_2 - T^2}{8\pi^2 a_2} s^2 - \left(\frac{T}{12}\right)^{(1+\alpha)/2} C_0 M_2 K_1 s^{1+\alpha} \right. \\
&\quad \left. - \left[\frac{(T^2 + 4\pi^2)^{1/2}}{2\pi} + C_0 M_2 K_2 \left(\frac{T}{12}\right)^{1/2} + M_3 \left(\frac{T}{12}\right)^{1/2}\right] s \right\}.
\end{aligned}$$

Notice that $a_2 > T^2/4\pi^2$ implies $-\infty < C_1 < 0$. Hence, it follows from (2.5) that

$$(2.6) \quad \|\dot{u}_n\|_{L^2}^2 \leq a_2 C_0^2 M_1^2 h^2(|\bar{u}_n|) - 2C_1,$$

and

$$(2.7) \quad \|\dot{u}_n\|_{L^2} \leq \sqrt{a_2} C_0 M_1 h(|\bar{u}_n|) + C_2,$$

where $C_2 > 0$. By the proof of Theorem 1.1, we have

$$(2.8) \quad \begin{aligned} & \left| \int_0^T [F(t, u_n(t)) - F(t, \bar{u}_n)] dt \right| \\ & \leq C_0 M_1 h(|\bar{u}_n|) \|\tilde{u}_n\|_{L^2} + C_0 M_2 K_1 \|\tilde{u}_n\|_\infty^{1+\alpha} \\ & \quad + (C_0 M_2 K_2 + M_3) \|\tilde{u}_n\|_\infty \\ & \leq \frac{\pi}{\sqrt{a_2} T} \|\tilde{u}_n\|_{L^2}^2 + \frac{\sqrt{a_2} T C_0^2 M_1^2}{4\pi} h^2(|\bar{u}_n|) \\ & \quad + C_0 M_2 K_1 \|\tilde{u}_n\|_\infty^{1+\alpha} + (C_0 M_2 K_2 + M_3) \|\tilde{u}_n\|_\infty \\ & \leq \frac{T}{4\pi\sqrt{a_2}} \|\dot{u}_n\|_{L^2}^2 + \frac{\sqrt{a_2} T C_0^2 M_1^2}{4\pi} h^2(|\bar{u}_n|) \\ & \quad + \left(\frac{T}{12}\right)^{(1+\alpha)/2} C_0 M_2 K_1 \|\dot{u}_n\|_\infty^{1+\alpha} \\ & \quad + \left(\frac{T}{12}\right)^{1/2} (C_0 M_2 K_2 + M_3) \|\dot{u}_n\|_{L^2}. \end{aligned}$$

It follows from the boundedness of $\varphi(u_n)$ and from (2.6)–(2.8) that

$$\begin{aligned} C_3 & \leq \varphi(u_n) \\ & = \frac{1}{2} \|\dot{u}_n\|_{L^2}^2 + \int_0^T [F(t, u_n(t)) - F(t, \bar{u}_n)] dt + \int_0^T F(t, \bar{u}_n) dt \\ & \leq \left(\frac{1}{2} + \frac{T}{4\pi\sqrt{a_2}}\right) \|\dot{u}_n\|_{L^2}^2 + \frac{\sqrt{a_2} T C_0^2 M_1^2}{4\pi} h^2(|\bar{u}_n|) \\ & \quad + \left(\frac{T}{12}\right)^{(1+\alpha)/2} C_0 M_2 K_1 \|\dot{u}_n\|_{L^2}^{1+\alpha} \\ & \quad + \left(\frac{T}{12}\right)^{1/2} (C_0 M_2 K_2 + M_3) \|\dot{u}_n\|_{L^2} + \int_0^T F(t, \bar{u}_n) dt \\ & \leq \left(\frac{1}{2} + \frac{T}{4\pi\sqrt{a_2}}\right) (a_2 C_0^2 M_1^2 h^2(|\bar{u}_n|) - 2C_1) + \frac{\sqrt{a_2} T C_0^2 M_1^2}{4\pi} h^2(|\bar{u}_n|) \\ & \quad + \left(\frac{T}{12}\right)^{(1+\alpha)/2} C_0 M_2 K_1 (\sqrt{a_2} C_0 M_1 h(|\bar{u}_n|) + C_2)^{1+\alpha} \\ & \quad + \left(\frac{T}{12}\right)^{1/2} (C_0 M_2 K_2 + M_3) (\sqrt{a_2} C_0 M_1 h(|\bar{u}_n|) + C_2) + \int_0^T F(t, \bar{u}_n) dt \end{aligned}$$

$$\begin{aligned}
&= \left(\frac{a_2}{2} + \frac{\sqrt{a_2}T}{2\pi}\right) C_0^2 M_1^2 h^2(|\bar{u}_n|) - \left(1 + \frac{T}{2\pi\sqrt{a_2}}\right) C_1 \\
&\quad + \int_0^T F(t, \bar{u}_n) dt + \left(\frac{T}{12}\right)^{(1+\alpha)/2} C_0 M_2 K_1 (\sqrt{a_2} C_0 M_1 h(|\bar{u}_n|) + C_2)^{1+\alpha} \\
&\quad + \left(\frac{T}{12}\right)^{1/2} (C_0 M_2 K_2 + M_3) (\sqrt{a_2} C_0 M_1 h(|\bar{u}_n|) + C_2) \\
&\leq \left(\frac{a_2}{2} + \frac{\sqrt{a_2}T}{2\pi}\right) C_0^2 M_1^2 h^2(|\bar{u}_n|) - \left(1 + \frac{T}{2\pi\sqrt{a_2}}\right) C_1 + \int_0^T F(t, \bar{u}_n) dt \\
&\quad + \left(\frac{T}{12}\right)^{(1+\alpha)/2} C_0 M_2 K_1 2^\alpha [(\sqrt{a_2} C_0 M_1 h(|\bar{u}_n|))^{(1+\alpha)} + C_2^{1+\alpha}] \\
&\quad + \left(\frac{T}{12}\right)^{1/2} (C_0 M_2 K_2 + M_3) (\sqrt{a_2} C_0 M_1 h(|\bar{u}_n|) + C_2) \\
&= h^2(|\bar{u}_n|) \left[h^{-2}(|\bar{u}_n|) \int_0^T F(t, \bar{u}_n) dt + \left(\frac{a_2}{2} + \frac{\sqrt{a_2}T}{2\pi}\right) C_0^2 M_1^2 \right. \\
&\quad + 2^\alpha \left(\frac{a_2 T}{12}\right)^{(1+\alpha)/2} C_0^{2+\alpha} M_2 M_1^{1+\alpha} K_1 h^{\alpha-1}(|\bar{u}_n|) \\
&\quad \left. + \left(\frac{T}{12}\right)^{1/2} (C_0 M_2 K_2 + M_3) \sqrt{a_2} C_0 M_1 h^{-1}(|\bar{u}_n|) \right] \\
&\quad + 2^\alpha \left(\frac{T}{12}\right)^{(1+\alpha)/2} C_0 M_2 K_1 C_2^{1+\alpha} + \left(\frac{T}{12}\right)^{1/2} (C_0 M_2 K_2 + M_3) C_2 \\
&\quad - \left(1 + \frac{T}{2\pi\sqrt{a_2}}\right) C_1.
\end{aligned}$$

The above inequality and (2.2) imply that $\{\bar{u}_n\}$ is bounded. Hence, $\{u_n\}$ is bounded by (2.6) and (H1). Arguing as in Proposition 4.1 in [3], we conclude that φ satisfies the (PS) condition.

In order to use the saddle point theorem ([6, Theorem 4.6]), we only need to verify the following conditions:

- (1) $\varphi(u) \rightarrow -\infty$ as $|u| \rightarrow \infty$ in \mathbb{R}^n .
- (2) $\varphi(u) \rightarrow \infty$ as $\|u\| \rightarrow \infty$ in \tilde{H}_T^1 , where $\tilde{H}_T^1 = \{u \in H_T^1 : \bar{u} = 0\}$.

In fact, from the property (iv) of (H1) and (H3)', we have

$$\int_0^T F(t, u) dt \rightarrow -\infty \quad \text{as } |u| \rightarrow \infty \text{ in } \mathbb{R}^n,$$

which together with (1.2) implies that

$$\varphi(u) = \int_0^T F(t, u) dt \rightarrow -\infty \quad \text{as } |u| \rightarrow \infty \text{ in } \mathbb{R}^n.$$

Hence, (1) holds.

Next, for all $u \in \tilde{H}_T^1$, by (H1) and Sobolev's inequality, we have

$$\begin{aligned}
& \left| \int_0^T [F(t, u(t)) - F(t, 0)] dt \right| \\
&= \left| \int_0^T \int_0^1 (\nabla F(t, su(t)), u(t)) ds dt \right| \\
&\leq \int_0^T f(t)h(|u(t)|)|u(t)| dt + \int_0^T g(t)|u(t)| dt \\
&\leq \int_0^T f(t)(K_1|u(t)|^\alpha + K_2)|u(t)| dt + M_3\|u\|_\infty \\
&\leq M_2\|u\|_\infty(K_1\|u\|_\infty^\alpha + K_2) + M_3\|u\|_\infty \\
&= M_2K_1\|u\|_\infty^{1+\alpha} + M_2K_2\|u\|_\infty + M_3\|u\|_\infty \\
&\leq \left(\frac{T}{12}\right)^{(1+\alpha)/2} M_2K_1\|\dot{u}\|_{L^2}^{1+\alpha} + \left(\frac{T}{12}\right)^{1/2} (M_2K_2 + M_3)\|\dot{u}\|_{L^2},
\end{aligned}$$

which implies that

$$\begin{aligned}
(2.9) \quad \varphi(u) &= \frac{1}{2} \int_0^T |\dot{u}(t)|^2 dt + \int_0^T [F(t, u(t)) - F(t, 0)] dt + \int_0^T F(t, 0) dt \\
&\geq \frac{1}{2} \|\dot{u}\|_{L^2}^2 - \left(\frac{T}{12}\right)^{(1+\alpha)/2} M_2K_1\|\dot{u}\|_{L^2}^{1+\alpha} \\
&\quad - \left(\frac{T}{12}\right)^{1/2} (M_2K_2 + M_3)\|\dot{u}\|_{L^2} + \int_0^T F(t, 0) dt
\end{aligned}$$

for all $u \in \tilde{H}_T^1$. By Wirtinger's inequality, $\|u\| \rightarrow \infty$ in \tilde{H}_T^1 if and only if $\|\dot{u}\|_{L^2} \rightarrow \infty$, so from (2.9) we obtain $\varphi(u) \rightarrow \infty$ as $\|u\| \rightarrow \infty$ in \tilde{H}_T^1 , i.e. (2) is verified. Hence, the proof of Theorem 1.2 is complete. \square

P r o o f of Theorem 1.3. By virtue of (1.8) and (1.9), we can choose a constant $a_3 \in \mathbb{R}$ such that

$$(2.10) \quad a_3 > \frac{3T^2}{\pi^2(12 - K_1TM_2C_0)} > 0,$$

and

$$(2.11) \quad \liminf_{|x| \rightarrow \infty} h^{-2}(|x|) \int_0^T F(t, x) dt > \frac{a_3}{2} M_1^2 C_0^2.$$

It follows from (H1) with $\alpha = 1$, Sobolev's inequality and Wirtinger's inequality that

$$\begin{aligned}
& \left| \int_0^T [F(t, u(t)) - F(t, \bar{u})] dt \right| \\
&= \left| \int_0^T \int_0^1 (\nabla F(t, \bar{u} + s\tilde{u}(t)), \tilde{u}(t)) dt \right| \\
&\leq \int_0^T \int_0^1 f(t)h(|\bar{u} + s\tilde{u}(t)|)|\tilde{u}(t)| ds dt + \int_0^T g(t)|\tilde{u}(t)| dt \\
&\leq \int_0^T \int_0^1 C_0 f(t)(h(|\bar{u}|) + sh(|\tilde{u}(t)|))|\tilde{u}(t)| ds dt + \int_0^T g(t)|\tilde{u}(t)| dt \\
&= \int_0^T C_0 f(t) \left(h(|\bar{u}|) + \frac{1}{2} h(|\tilde{u}(t)|) \right) |\tilde{u}(t)| dt + M_3 \|\tilde{u}\|_\infty \\
&\leq C_0 h(|\bar{u}|) \left(\int_0^T f^2(t) \right)^{1/2} \left(\int_0^T |\tilde{u}(t)| dt \right)^{1/2} \\
&\quad + \frac{1}{2} M_2 C_0 h(\|\tilde{u}\|_\infty) \|\tilde{u}\|_\infty + M_3 \|\tilde{u}\|_\infty \\
&= C_0 M_1 h(|\bar{u}|) \|\tilde{u}\|_{L^2} + \frac{1}{2} M_2 C_0 h(\|\tilde{u}\|_\infty) \|\tilde{u}\|_\infty + M_3 \|\tilde{u}\|_\infty \\
&\leq \frac{1}{2a_3} \|\tilde{u}\|_{L^2}^2 + \frac{a_3 M_1^2 C_0^2}{2} h^2(|\bar{u}|) \\
&\quad + \frac{1}{2} M_2 C_0 (K_1 \|\tilde{u}\|_\infty + K_2) \|\tilde{u}\|_\infty + M_3 \|\tilde{u}\|_\infty \\
&\leq \left(\frac{T^2}{8\pi^2 a_3} + \frac{T M_2 K_1 C_0}{24} \right) \|\dot{u}\|_{L^2}^2 + \left(\frac{T}{12} \right)^{1/2} \left(M_3 + \frac{M_2 K_2 C_0}{2} \right) \|\dot{u}\|_{L^2} \\
&\quad + \frac{a_3 M_1^2 C_0^2}{2} h^2(|\bar{u}|),
\end{aligned}$$

which implies that

$$\begin{aligned}
\varphi(u) &= \frac{1}{2} \int_0^T |\dot{u}(t)|^2 dt + \int_0^T [F(t, u(t)) - F(t, \bar{u})] dt + \int_0^T F(t, \bar{u}) dt \\
&\geq \left(\frac{1}{2} - \frac{T^2}{8\pi^2 a_3} - \frac{T M_2 K_1 C_0}{24} \right) \|\dot{u}\|_{L^2}^2 - \left(\frac{T}{12} \right)^{1/2} \left(M_3 + \frac{M_2 K_2 C_0}{2} \right) \|\dot{u}\|_{L^2} \\
&\quad - \frac{a_3 M_1^2 C_0^2}{2} h^2(|\bar{u}|) + \int_0^T F(t, \bar{u}) dt \\
&= \left(\frac{1}{2} - \frac{T^2}{8\pi^2 a_3} - \frac{T M_2 K_1 C_0}{24} \right) \|\dot{u}\|_{L^2}^2 - \left(\frac{T}{12} \right)^{1/2} \left(M_3 + \frac{M_2 K_2 C_0}{2} \right) \|\dot{u}\|_{L^2} \\
&\quad + h^2(|\bar{u}|) \left(h^{-2}(|\bar{u}|) \int_0^T F(t, \bar{u}) dt - \frac{a_3 M_1^2 C_0^2}{2} \right).
\end{aligned}$$

As $\|u\| \rightarrow \infty$ if and only if $(|\bar{u}|^2 + \|\dot{u}\|_{L^2}^2)^{1/2} \rightarrow \infty$, the above inequality and (2.10) and (2.11) imply that $\varphi(u) \rightarrow \infty$. Hence, by the least action principle, problem (1.1) has at least one solution which minimizes the functional $\varphi(u)$ in H_T^1 . \square

Proof of Theorem 1.4. First we prove that φ satisfies the (PS) condition. Assume that u_n is a (PS) sequence of φ , that is, $\varphi'(u_n) \rightarrow 0$ as $n \rightarrow \infty$ and $\{\varphi(u_n)\}$ is bounded. By (H4), we can choose $a_4 \in \mathbb{R}$ such that

$$(2.12) \quad a_4 > \frac{6T^2}{\pi^2(24 - TM_2K_1C_0)} > 0$$

and

$$(2.13) \quad \limsup_{|x| \rightarrow \infty} h^{-2}(|x|) \int_0^T F(t, x) dt < - \left[\frac{(12 + TM_2K_1C_0)a_4}{24 - TM_2K_1C_0} + \frac{T}{\pi} \sqrt{\frac{6a_4}{24 - TM_2K_1C_0}} \right] M_1^2 C_0^2.$$

In a way similar to the proof of Theorem 1.3, we obtain

$$\begin{aligned} & \left| \int_0^T (\nabla F(t, u_n(t)), \tilde{u}_n(t)) dt \right| \\ & \leq \left(\frac{T^2}{8\pi^2 a_4} + \frac{TM_2K_1C_0}{24} \right) \|\dot{u}_n\|_{L^2}^2 + \left(\frac{T}{12} \right)^{1/2} \left(M_3 + \frac{M_2K_2C_0}{2} \right) \|\dot{u}_n\|_{L^2} \\ & \quad + \frac{a_4 M_1^2 C_0^2}{2} h^2(|\bar{u}_n|). \end{aligned}$$

Hence, we have

$$\begin{aligned} \|\tilde{u}_n\| & \geq \langle \varphi'(u_n), \tilde{u}_n \rangle \\ & = \|\dot{u}_n\|_{L^2}^2 + \int_0^T (\nabla F(t, u_n(t)), \tilde{u}_n(t)) dt \\ & \geq \left(1 - \frac{T^2}{8\pi^2 a_4} - \frac{TM_2K_1C_0}{24} \right) \|\dot{u}_n\|_{L^2}^2 - \frac{a_4(C_0M_1)^2}{2} h^2(|\bar{u}_n|) \\ & \quad - \left(\frac{T}{12} \right)^{1/2} \left(\frac{C_0M_2K_2}{2} + M_3 \right) \|\dot{u}_n\|_{L^2}, \end{aligned}$$

which together with (2.4) implies that

$$\begin{aligned} \frac{a_4(C_0M_1)^2}{2} h^2(|\bar{u}_n|) & \geq \left(1 - \frac{T^2}{8\pi^2 a_4} - \frac{TM_2K_1C_0}{24} \right) \|\dot{u}_n\|_{L^2}^2 \\ & \quad - \left[\frac{(T^2 + 4\pi^2)^{1/2}}{2\pi} + \left(\frac{T}{12} \right)^{1/2} \left(\frac{C_0M_2K_2}{2} + M_3 \right) \right] \|\dot{u}_n\|_{L^2} \\ & \geq \frac{1}{2} \left(1 - \frac{TM_2K_1C_0}{24} \right) \|\dot{u}_n\|_{L^2}^2 + C_4, \end{aligned}$$

where

$$C_4 = \min_{s \in [0, \infty)} \left\{ \left(\frac{1}{2} - \frac{T^2}{8\pi^2 a_4} - \frac{TM_2 K_1 C_0}{48} \right) s^2 - \left[\frac{(T^2 + 4\pi^2)^{1/2}}{2\pi} + \left(\frac{T}{12} \right)^{1/2} \left(\frac{1}{2} C_0 M_2 K_2 + M_3 \right) \right] s \right\}.$$

It follows from (2.12) that $-\infty < C_4 < 0$, so we obtain

$$(2.14) \quad \|\dot{u}_n\|_{L^2}^2 \leq \frac{24 a_4 M_1^2 C_0^2}{24 - TM_2 K_1 C_0} h^2(|\bar{u}_n|) - \frac{48C_4}{24 - TM_2 K_1 C_0}$$

and

$$(2.15) \quad \|\dot{u}_n\|_{L^2} \leq \frac{\sqrt{24 a_4 M_1 C_0}}{\sqrt{24 - TM_2 K_1 C_0}} h(|\bar{u}_n|) + C_5.$$

By the proof of Theorem 1.3, we have

$$\begin{aligned} & \left| \int_0^T (F(t, u_n(t)) - F(t, \bar{u}_n)) dt \right| \\ & \leq C_0 M_1 h(|\bar{u}_n|) \|\tilde{u}_n\|_{L^2} + \frac{M_2 C_0}{2} h(\|\tilde{u}_n\|_\infty) \|\tilde{u}_n\|_\infty + M_3 \|\tilde{u}_n\|_\infty \\ & \leq \frac{\pi}{2T} \sqrt{\frac{24 - TM_2 K_1 C_0}{6a_4}} \|\tilde{u}_n\|_{L^2}^2 + \frac{T}{\pi} \sqrt{\frac{3a_4}{2(24 - TM_2 K_1 C_0)}} M_1^2 C_0^2 h^2(|\bar{u}_n|) \\ & \quad + \frac{M_2 C_0}{2} \|\tilde{u}_n\|_\infty (K_1 \|\tilde{u}_n\|_\infty + K_2) + M_3 \|\tilde{u}_n\|_\infty \\ & \leq \left(\frac{T}{8\pi} \sqrt{\frac{24 - TM_2 K_1 C_0}{6a_4}} + \frac{TM_2 K_1 C_0}{24} \right) \|\dot{u}_n\|_{L^2}^2 \\ & \quad + \frac{T}{\pi} \sqrt{\frac{3a_4}{2(24 - TM_2 K_1 C_0)}} M_1^2 C_0^2 h^2(|\bar{u}_n|) \\ & \quad + \left(\frac{T}{12} \right)^{1/2} \left(M_3 + \frac{K_2 M_2 C_0}{2} \right) \|\dot{u}_n\|_{L^2}. \end{aligned}$$

It follows from the boundedness of $\varphi(u_n)$, (2.14), (2.15), and the above inequality that

$$\begin{aligned} C_6 & \leq \varphi(u_n) \\ & = \frac{1}{2} \|\dot{u}_n\|_{L^2}^2 + \int_0^T [F(t, u_n(t)) - F(t, \bar{u}_n)] dt + \int_0^T F(t, \bar{u}_n) dt \end{aligned}$$

$$\begin{aligned}
&\leq \left[\frac{1}{2} + \frac{T}{8\pi} \sqrt{\frac{24 - TM_2K_1C_0}{6a_4}} + \frac{TM_2K_1C_0}{24} \right] \|\dot{u}_n\|_{L^2}^2 + \int_0^T F(t, \bar{u}_n) dt \\
&\quad + \frac{T}{\pi} \sqrt{\frac{3a_4}{2(24 - TM_2K_1C_0)}} M_1^2 C_0^2 h^2(|\bar{u}_n|) \\
&\quad + \left(\frac{T}{12} \right)^{1/2} \left(M_3 + \frac{K_2 M_2 C_0}{2} \right) \|\dot{u}_n\|_{L^2} \\
&\leq \left[\frac{1}{2} + \frac{T}{8\pi} \sqrt{\frac{24 - TM_2K_1C_0}{6a_4}} + \frac{TM_2K_1C_0}{24} \right] \\
&\quad \times \left(\frac{24a_4 M_1^2 C_0^2}{24 - TM_2K_1C_0} h^2(|\bar{u}_n|) - \frac{48C_4}{24 - TM_2K_1C_0} \right) \\
&\quad + \frac{T}{\pi} \sqrt{\frac{3a_4}{2(24 - TM_2K_1C_0)}} M_1^2 C_0^2 h^2(|\bar{u}_n|) + \int_0^T F(t, \bar{u}_n) dt \\
&\quad + \left(\frac{T}{12} \right)^{1/2} \left(M_3 + \frac{K_2 M_2 C_0}{2} \right) \left(\frac{\sqrt{24a_4} M_1 C_0}{\sqrt{24 - TM_2K_1C_0}} h(|\bar{u}_n|) + C_5 \right) \\
&= \left[\frac{(12 + TM_2K_1C_0)a_4}{24 - TM_2K_1C_0} + \frac{T}{\pi} \sqrt{\frac{6a_4}{24 - TM_2K_1C_0}} \right] M_1^2 C_0^2 h^2(|\bar{u}_n|) \\
&\quad + \int_0^T F(t, \bar{u}_n) dt + \left(\frac{T}{12} \right)^{1/2} \left(M_3 + \frac{K_2 M_2 C_0}{2} \right) \frac{\sqrt{24a_4} M_1 C_0}{\sqrt{24 - TM_2K_1C_0}} h(|\bar{u}_n|) \\
&\quad + \left(\frac{T}{12} \right)^{1/2} \left(M_3 + \frac{K_2 M_2 C_0}{2} \right) C_5 \\
&\quad - \frac{48C_4}{24 - TM_2K_1C_0} \left[\frac{1}{2} + \frac{T}{8\pi} \sqrt{\frac{24 - TM_2K_1C_0}{6a_4}} + \frac{TM_2K_1C_0}{24} \right] \\
&= h^2(|\bar{u}_n|) \left\{ h^{-2}(|\bar{u}_n|) \int_0^T F(t, \bar{u}_n) dt \right. \\
&\quad + \left[\frac{(12 + TM_2K_1C_0)a_4}{24 - TM_2K_1C_0} + \frac{T}{\pi} \sqrt{\frac{6a_4}{24 - TM_2K_1C_0}} \right] M_1^2 C_0^2 \\
&\quad + \left(\frac{T}{12} \right)^{1/2} \left(M_3 + \frac{K_2 M_2 C_0}{2} \right) \frac{\sqrt{24a_4} M_1 C_0}{\sqrt{24 - TM_2K_1C_0}} h^{-1}(|\bar{u}_n|) \left. \right\} \\
&\quad + \left(\frac{T}{12} \right)^{1/2} \left(M_3 + \frac{K_2 M_2 C_0}{2} \right) C_5 \\
&\quad - \frac{48C_4}{24 - TM_2K_1C_0} \left[\frac{1}{2} + \frac{T}{8\pi} \sqrt{\frac{24 - TM_2K_1C_0}{6a_4}} + \frac{TM_2K_1C_0}{24} \right].
\end{aligned}$$

The above inequality and (2.13) imply that $\{\bar{u}_n\}$ is bounded. Hence, $\{u_n\}$ is bounded by (H1) and (2.14).

Similarly to the proof of Theorem 1.2, we only need to verify (1) and (2). It is easy to check (1) by (H4). Now, we verify that (2) holds. For $u \in \tilde{H}_T^1$, by (H1) and

Sobolev's inequality, we have

$$\begin{aligned}
& \left| \int_0^T (F(t, u(t)) - F(t, 0)) dt \right| \\
&= \left| \int_0^T \int_0^1 (\nabla F(t, su(t)), u(t)) ds dt \right| \\
&\leq \int_0^T \int_0^1 f(t)h(s|u(t)|)|u(t)| ds dt + \int_0^T g(t)|u(t)| dt \\
&\leq \int_0^T \int_0^1 C_0 f(t)(h(s|u(t)|) + h(0))|u(t)| ds dt + \int_0^T g(t)|u(t)| dt \\
&\leq \int_0^T \int_0^1 C_0 f(t)(K_1 s|u(t)| + K_2)|u(t)| ds dt \\
&\quad + M_2 C_0 h(0) \|u\|_\infty + M_3 \|u\|_\infty \\
&= \frac{K_1 M_2 C_0}{2} \|u\|_\infty^2 + (M_2 C_0 h(0) + K_2 M_2 C_0 + M_3) \|u\|_\infty \\
&\leq \frac{TK_1 M_2 C_0}{24} \|\dot{u}\|_{L^2}^2 + \left(\frac{T}{12}\right)^{1/2} (M_2 C_0 h(0) + K_2 M_2 C_0 + M_3) \|\dot{u}\|_{L^2}.
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
\varphi(u) &= \frac{1}{2} \|\dot{u}\|_{L^2}^2 + \int_0^T (F(t, u(t)) - F(t, 0)) dt + \int_0^T F(t, 0) dt \\
&\geq \frac{12 - TM_2 K_1 C_0}{24} \|\dot{u}\|_{L^2}^2 - \left(\frac{T}{12}\right)^{1/2} (M_2 C_0 h(0) + K_2 M_2 C_0 + M_3) \|\dot{u}\|_{L^2} \\
&\quad + \int_0^T F(t, 0) dt
\end{aligned}$$

for all $u \in \tilde{H}_T^1$. By Wirtinger's inequality, $\|u\| \rightarrow \infty$ in \tilde{H}_T^1 if and only if $\|\dot{u}\|_{L^2} \rightarrow \infty$. So from the above inequality we have $\varphi(u) \rightarrow \infty$ as $\|u\| \rightarrow \infty$, i.e. (2) is verified. Hence, the proof of Theorem 1.4 is complete. \square

3. EXAMPLES

In this section, we give four examples to illustrate our results.

Example 3.1. Let

$$F(t, x) = \left(\frac{1}{2}T - t\right) \ln^{3/2}(1 + |x|^2) + \left(\frac{2}{3}T - t\right) \ln(1 + |x|^2).$$

It is easy to see that

$$|\nabla F(t, x)| \leq \frac{3}{2} \left| \frac{1}{2}T - t \right| \ln^{1/2}(1 + |x|^2) + \left| \frac{2}{3}T - t \right|$$

for all $x \in \mathbb{R}^n$ and a.e. $t \in [0, T]$. We can see that (1.5) holds with $f(t) = \frac{3}{2}|\frac{1}{2}T - t|$, $g(t) = |\frac{2}{3}T - t|$ and $h(|x|) = \ln^{1/2}(1 + |x|^2)$. Let $C_0 = 2$. It is easy to check that (H1) holds. However, $F(t, x)$ does not satisfy (1.6). In fact,

$$\frac{T^2 C_0^2}{8\pi^2} \int_0^T f^2(t) dt = \frac{9T^5}{96\pi^2}.$$

Let $T^3 < \frac{16}{9}\pi^2$, then we have

$$\liminf_{|x| \rightarrow \infty} h^{-2}(|x|) \int_0^T F(t, x) dt = \frac{T^2}{6} > \frac{9T^5}{96\pi^2} = \frac{T^2 C_0^2}{8\pi^2} \int_0^T f^2(t) dt.$$

Hence, according to Theorem 1.1, problem (1.1) has at least one solution which minimizes the functional φ in H_T^1 .

Example 3.2. Let

$$F(t, x) = \frac{\sin(2\pi t/T)}{2} \ln^{3/2}(1 + |x|^2) + \left(\frac{2}{5}T - t\right) \ln(1 + |x|^2).$$

It is easy to see that

$$|\nabla F(t, x)| \leq \frac{3}{4} \left| \sin \frac{2\pi t}{T} \right| \ln^{1/2}(1 + |x|^2) + \left| \frac{2}{5}T - t \right|$$

for all $x \in \mathbb{R}^n$ and a.e. $t \in [0, T]$. We can see that (1.5) holds with $f(t) = \frac{3}{4}|\sin(2\pi t/T)|$, $g(t) = |\frac{2}{5}T - t|$ and $h(|x|) = \ln^{1/2}(1 + |x|^2)$. Let $C_0 = 2$. It is easy to check that (H1) holds. However, $F(t, x)$ does not satisfy (H3). In fact,

$$\frac{3T^2 C_0^2}{8\pi^2} \int_0^T f^2(t) dt = \frac{27T^3}{64\pi^2}.$$

Let $T < \frac{32}{135}\pi^2$, then we have

$$\limsup_{|x| \rightarrow \infty} h^{-2}(|x|) \int_0^T F(t, x) dt = -\frac{T^2}{10} < -\frac{27T^3}{64\pi^2} = -\frac{3T^2 C_0^2}{8\pi^2} \int_0^T f^2(t) dt.$$

Hence, by Theorem 1.2, problem (1.1) has at least one solution in H_T^1 .

Example 3.3. Let

$$F(t, x) = \left(\frac{5}{6}T - t\right) \ln^2(1 + |x|^2) + c(t) \ln(1 + |x|^2),$$

where $c(t) \in L^1([0, T], \mathbb{R}^+)$. It is easy to see that

$$|\nabla F(t, x)| \leq 2 \left| \frac{5}{6}T - t \right| \ln(1 + |x|^2) + |c(t)|$$

for all $x \in \mathbb{R}^n$ and a.e. $t \in [0, T]$. Let $f(t) = 2 \left| \frac{5}{6}T - t \right|$, $g(t) = |c(t)|$, $h(|x|) = \ln(1 + |x|^2)$, $C_0 = 2$, $K_1 = 1$, $K_2 = 10$. Then we have

- (i) $h(s) \leq h(t) \quad \forall s \leq t, s, t \in [0, \infty)$,
- (ii) $h(s + t) \leq 2(h(s) + h(t)) \quad \forall s, t \in [0, \infty)$,
- (iii) $0 \leq h(s) \leq s + 10 \quad \forall s, t \in [0, \infty)$,
- (iv) $h(s) \rightarrow \infty$ as $s \rightarrow \infty$.

By a direct computation, we get

$$\int_0^T f(t) dt = 2 \int_0^T \left| \frac{5}{6}T - t \right| dt = \frac{13T^2}{18}$$

and

$$\int_0^T f^2(t) dt = 4 \int_0^T \left| \frac{5}{6}T - t \right|^2 dt = \frac{7T^3}{9}.$$

Let $T^3 < 108\pi^2/(126 + 13\pi^2)$. Then we have

$$\int_0^T f(t) dt = \frac{13T^2}{18} < \frac{6}{T} = \frac{12}{TK_1C_0}$$

and

$$\begin{aligned} \liminf_{|x| \rightarrow \infty} h^{-2}(|x|) \int_0^T F(t, x) dt &= \frac{T^2}{3} > \frac{3T^2 \times 4 \times 7T^3/9}{2\pi^2(12 - 2T \times 13T^2/18)} \\ &= \frac{3T^2 C_0^2 \int_0^T f^2(t) dt}{2\pi^2(12 - K_1 C_0 T \int_0^T f(t) dt)}. \end{aligned}$$

Hence, by Theorem 1.1, problem (1.1) has at least one solution which minimizes the functional φ in H_T^1 .

Example 3.4. Let

$$F(t, x) = \left(\frac{2}{5}T - t \right) \ln^2(1 + |x|^2) + c(t) \ln(1 + |x|^2),$$

where $c(t) \in L^1([0, T], \mathbb{R}^+)$. It is easy to see that

$$|\nabla F(t, x)| \leq 2 \left| \frac{2}{5}T - t \right| \ln(1 + |x|^2) + |c(t)|$$

for all $x \in \mathbb{R}^n$ and a.e. $t \in [0, T]$. Let $f(t) = 2|\frac{2}{5}T - t|$, $g(t) = |c(t)|$, $h(|x|) = \ln(1 + |x|^2)$, $C_0 = 2$, $K_1 = 1$, $K_2 = 10$. As shown in Example 3.3, we know that $h(s)$ satisfies (i)–(iv). By a direct computation, we have

$$\int_0^T f(t) dt = 2 \int_0^T \left| \frac{2}{5}T - t \right| dt = \frac{13T^2}{25}$$

and

$$\int_0^T f^2(t) dt = 4 \int_0^T \left| \frac{5}{6}T - t \right|^2 dt = \frac{28T^3}{75}.$$

Let $T^3 < \frac{3}{2}$. Then we have

$$\int_0^T f(t) dt = \frac{13T^2}{18} < \frac{6}{T} = \frac{12}{TK_1C_0}$$

and

$$\begin{aligned} \limsup_{|x| \rightarrow \infty} h^{-2}(|x|) \int_0^T F(t, x) dt \\ &= -\frac{T^2}{10} < -\frac{8064T^5}{25\pi^2(24 - 1.04T^3)^2} \\ &= -\frac{216T^2C_0^2 \int_0^T f^2(t) dt}{\pi^2(24 - K_1C_0T \int_0^T f(t) dt)^2}. \end{aligned}$$

Hence, by Theorem 1.1, problem (1.1) has at least one solution in H_T^1 .

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