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A STUDY OF THE NUMBER OF SOLUTIONS OF THE SYSTEM OF THE LOG-LIKELIHOOD EQUATIONS FOR THE 3-PARAMETER WEIBULL DISTRIBUTION

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Abstract. The maximum likelihood estimators of the parameters for the 3-parameter Weibull distribution do not always exist. Furthermore, computationally it is difficult to find all the solutions. Thus, the case of missing some solutions and among them the maximum likelihood estimators cannot be excluded. In this paper we provide a simple rule with help of which we are able to know if the system of the log-likelihood equations has even or odd number of solutions. It is a useful tool for the detection of all the solutions of the system.

Keywords: Weibull distribution, Hessian matrix, maximum likelihood estimator, stationary value

MSC 2010: 62F10, 62F99

1. Introduction

The Weibull distribution is one of the most popular and widely used models for failure time in life testing and reliability theory. The probability density function of the three parameter Weibull distribution is expressed by

$$f(x;\xi,\alpha,c)=\frac{c}{\alpha}\Big(\frac{x-\xi}{\alpha}\Big)^{c-1}\exp\Big\{-\Big(\frac{x-\xi}{\alpha}\Big)^{c}\Big\} \tag{1.1}$$

with $x>\xi$, $c>0$, $\alpha>0$, $-\infty<\xi<\infty$. Here c is the shape parameter, α the scale parameter, and ξ the location parameter. Several methods have been proposed for the estimation of the parameters ξ, α, c. The books [1] and [4] summarize most of them. The method of maximum likelihood is the most popular because of its properties of asymptotic normality and efficiency [15]. However, finding the maximum likelihood estimator (MLE) demands the solution of a non linear system of equations which is not an easy task. More specifically, for an independent sample $\underline{x} = (x_1, x_2, \ldots, x_n)$ with $m = \min\{x_1, x_2, \ldots, x_n\}$, the system of the partial derivatives of the log-likelihood function

$$L(\xi, \alpha, c; \underline{x}) = n \ln c - nc \ln \alpha + (c-1) \sum_{i=1}^{n} \ln(x_i - \xi) - \frac{1}{\alpha^c} \sum_{i=1}^{n} (x_i - \xi)^c \tag{1.2}$$

with domain $(-\infty, m) \times (0, \infty) \times (0, \infty)$, results in the following system of estimating functions:

$$g_0(\xi, \alpha, c) = \alpha^c - \frac{1}{n} \sum_{i=1}^{n} (x_i - \xi)^c = 0, \tag{1.3}$$

$$g_1(\xi, c) = \frac{1}{c} + \frac{1}{n} \sum_{i=1}^{n} \ln(x_i - \xi) - \sum_{i=1}^{n} (x_i - \xi)^c \ln(x_i - \xi) \Big/ \sum_{i=1}^{n} (x_i - \xi)^c = 0, \tag{1.4}$$

$$g_2(\xi, c) = c \sum_{i=1}^{n} (x_i - \xi)^{c-1} - (c-1) \frac{1}{n} \sum_{i=1}^{n} (x_i - \xi)^c \sum_{i=1}^{n} \frac{1}{x_i - \xi} = 0, \tag{1.5}$$

where the functions $g_1(\xi, c)$ and $g_2(\xi, c)$ are $\partial L / \partial c$ and $\partial L / \partial \xi$, respectively, after removing α from (1.3). Qiao and Tsokos [11], Smith and Naylor [14], [15], and Gourdin et al. [3] proposed effective algorithms for finding the maximum likelihood estimators for the three-parameter Weibull distribution. The aim of this paper is not to propose a new algorithm but to study the number of solutions of the system (1.3), (1.4), (1.5). If ξ is known, then there is a unique solution of the system of equations (1.3) and (1.4) ([8], [9]). Similar conclusions are drawn when α is known [10]. When all three parameters are unknown, the number of solutions of the system (1.3)–(1.5) is not known. Rockette et al. [12] argue that if a local maximum exists then a saddle point must also exists. Lockhart and Stephens [5], [6] suggest that the functions $c_1(\xi)$ and $c_2(\xi)$ defined implicitly from the equation (1.4) and (1.5) respectively are of the form

$$c(\xi) = L\xi + b_0 + \frac{b_1}{\xi}. \tag{1.6}$$

Their argument is supported by simulation results. With help of the above expressions they draw conclusions for the solutions of the system (1.3)–(1.5). The quantity L is the solution of the equation

$$L = \frac{L_0}{L_1}, \tag{1.7}$$

where

$$L_k = \sum_{i=1}^{n} \mathrm{e}^{L(x_i - \overline{x})} (x_i - \overline{x})^k \tag{1.8}$$

and $\overline{x}$ is the arithmetic mean $\overline{x} = \sum x_i / n$.

In this paper we study the number of solutions of the system of equations (1.4) and (1.5). The procedure is the following: we find the implicit function $c = c(\xi)$ determined from (1.5). More specifically we prove that $c(\xi)$ is expressed in the form

$$c(\xi) = L(\overline{x} - \xi) + a_0 + o(1) \quad \text{for } \xi \to -\infty. \tag{1.9}$$

Next we insert (1.9) in the equation (1.4). In this way we study $g_1(\xi, c(\xi)) = 0$ which is an equation of one unknown parameter. With help of the sign of $g_1(\xi, c(\xi))$ for $\xi \to -\infty$ and $\xi \to m-$ we can draw conclusions concerning the number of solutions of $g_1(\xi, c(\xi))$. Furthermore, we prove that a local maximum cannot be located close to the asymptote $\xi = m$. We use the notation $x_{(n)} = \max\{x_1, x_2, \dots, x_n\}$ and $s^2 = \overline{(x - \overline{x})^2}$.

The paper is organized in the following way. In Section 2 we present some preliminary results and formulae which are necessary for the development of the main results. In Section 3 we prove the main results. In Section 4 we discuss the results and present several examples which support our findings.

2. Some preliminary results

Let us write c instead of $c(\xi)$. Equation (1.5) can be written as

$$c = \frac{\frac{1}{n}\sum\limits_{i=1}^{n}(x_i - \xi)^c \sum\limits_{i=1}^{n}\frac{1}{x_i - \xi}}{\frac{1}{n}\sum\limits_{i=1}^{n}(x_i - \xi)^c \sum\limits_{i=1}^{n}\frac{1}{x_i - \xi} - \sum\limits_{i=1}^{n}(x_i - \xi)^{c-1}}. \tag{2.1}$$

It is obvious that always $c(\xi) > 1$. Therefore, the domain of $L(\xi, \alpha, c, \underline{x})$ is restricted to $(-\infty, m) \times (0, \infty) \times (1, \infty)$ and in this specific domain we are looking for a solution of the system (1.3)–(1.5). Another equivalent expression of (2.1) is

$$c = \frac{\frac{1}{n}\sum\limits_{i=1}^{n}\left(1 + \frac{x_i - \overline{x}}{\overline{x} - \xi}\right)^c \sum\limits_{i=1}^{n}\left(1 + \frac{x_i - \overline{x}}{\overline{x} - \xi}\right)^{-1}}{\frac{1}{n}\sum\limits_{i=1}^{n}\left(1 + \frac{x_i - \overline{x}}{\overline{x} - \xi}\right)^c \sum\limits_{i=1}^{n}\left(1 + \frac{x_i - \overline{x}}{\overline{x} - \xi}\right)^{-1} - \sum\limits_{i=1}^{n}\left(1 + \frac{x_i - \overline{x}}{\overline{x} - \xi}\right)^c\left(1 + \frac{x_i - \overline{x}}{\overline{x} - \xi}\right)^{-1}}. \tag{2.2}$$

From the expression (2.2) we conclude that

$$\lim_{\xi \to -\infty} c(\xi) = \infty.$$

Utilizing the expression

$$\left(1 + \frac{x_i - \overline{x}}{\overline{x} - \xi}\right)^{-1} = 1 - \frac{x_i - \overline{x}}{\overline{x} - \xi} + \left(\frac{x_i - \overline{x}}{\overline{x} - \xi}\right)^2 + o\left(\frac{1}{\overline{x} - \xi}\right)^2,$$

from (2.2) we obtain the expression

$$c = \frac{\sum_{i=1}^{n} \bigl(1 + \frac{x_i - \overline{x}}{\overline{x} - \xi}\bigr)^c \bigl(1 + \frac{s^2}{(\overline{x} - \xi)^2} + o\bigl(\frac{1}{\overline{x} - \xi}\bigr)^2\bigr)}{\sum_{i=1}^{n} \bigl(1 + \frac{x_i - \overline{x}}{\overline{x} - \xi}\bigr)^c \bigl(\frac{x_i - \overline{x}}{\overline{x} - \xi} + \frac{s^2 - (x_i - \overline{x})^2}{(\overline{x} - \xi)^2} + o\bigl(\frac{1}{\overline{x} - \xi}\bigr)^2\bigr)} \tag{2.3}$$

or equivalently

$$c = \frac{(\overline{x} - \xi) \sum_{i=1}^{n} \bigl(1 + \frac{x_i - \overline{x}}{\overline{x} - \xi}\bigr)^c \bigl(1 + \frac{s^2}{(\overline{x} - \xi)^2} + o\bigl(\frac{1}{\overline{x} - \xi}\bigr)^2\bigr)}{\sum_{i=1}^{n} \bigl(1 + \frac{x_i - \overline{x}}{\overline{x} - \xi}\bigr)^c \bigl(x_i - \overline{x} + \frac{s^2 - (x_i - \overline{x})^2}{(\overline{x} - \xi)} + o\bigl(\frac{1}{\overline{x} - \xi}\bigr)\bigr)}. \tag{2.4}$$

From the expression (2.4) we get that

$$\frac{c}{\overline{x} - \xi} = \frac{\sum_{i=1}^{n} \bigl(1 + \frac{x_i - \overline{x}}{\overline{x} - \xi}\bigr)^c \bigl(1 + \frac{s^2}{(\overline{x} - \xi)^2} + o\bigl(\frac{1}{\overline{x} - \xi}\bigr)^2\bigr)}{\sum_{i=1}^{n} \bigl(1 + \frac{x_i - \overline{x}}{\overline{x} - \xi}\bigr)^c \bigl(x_i - \overline{x} + \frac{s^2 - (x_i - \overline{x})^2}{\overline{x} - \xi} + o\bigl(\frac{1}{\overline{x} - \xi}\bigr)\bigr)}. \tag{2.5}$$

Proposition 2.1. *The equation*

$$x = \sum_{i=1}^{n} \mathrm{e}^{x(x_i - \overline{x})} \Big/ \sum_{i=1}^{n} \mathrm{e}^{x(x_i - \overline{x})} (x_i - \overline{x})$$

has a unique solution on the interval $(0, \infty)$.

Proof. Taking the derivative we can prove that the function

$$t(x) = \sum_{i=1}^{n} \mathrm{e}^{x(x_i - \overline{x})} \Big/ \sum_{i=1}^{n} \mathrm{e}^{x(x_i - \overline{x})} (x_i - \overline{x})$$

is strictly decreasing. Furthermore, we can see that

$$\lim t(x) = \infty \quad \text{for } x \to 0$$

as well as

$$\lim t(x) = \frac{1}{x_{(n)} - \overline{x}} \quad \text{for } x \to \infty.$$

Thus the functions $t(x)$ and $y = x$ have only one common point. □

Proposition 2.2. *If*

$$f(x) = c_1 x + \frac{c_2}{x^n} + o\Big(\frac{1}{x^n}\Big) \quad \text{for } n \geqslant 0 \text{ and } c_1 > 0 \text{ as } x \to \infty,$$

then

$$\frac{1}{f(x)} = \frac{1}{c_1 x} - \frac{c_2}{c_1^2 x^{n+2}} + o\Big(\frac{1}{x^{n+2}}\Big).$$

P r o o f. The result follows by writing

$$\frac{1}{f(x)} = \frac{1}{c_1 x} \frac{1}{1 + c_2 c_1^{-1} x^{-(n+1)} + o(x^{-(n+1)})}$$

and then using the geometric expansion. □

3. Main results

3.1. The formula for the implicit function $c = c(\xi)$

We prove the relation (1.9) and determine the quantities L and a_0.

Theorem 3.1. *We have*

$$\lim_{\xi \to -\infty} \frac{c}{\overline{x} - \xi} = L, \tag{3.1}$$

where L satisfies the relation (1.7).

P r o o f. Let us assume that there is a sequence ξ_k with $\xi_k \to -\infty$ and

$$\lim_{\xi_k \to -\infty} \frac{c_k}{\overline{x} - \xi_k} = \infty. \tag{3.2}$$

Dividing the enumerator and denominator of (2.5) by

$$\Big(1 + \frac{x_{(n)} - \overline{x}}{\overline{x} - \xi_k}\Big)^{c_k/(\overline{x} - \xi_k)}$$

we have the relation

$$\frac{c_k}{\overline{x} - \xi_k} = \frac{h_1(\xi_k)}{h_2(\xi_k)},$$

where

$$h_1(\xi_k) = 1 + o(1) + \sum_{i=1}^{n-1} \Bigg[\Bigg(\frac{1 + (x_{(i)} - \overline{x})/(\overline{x} - \xi_k)}{1 + (x_{(n)} - \overline{x})/(\overline{x} - \xi_k)}\Bigg)^{\overline{x} - \xi_k}\Bigg]^{c_k/(\overline{x} - \xi_k)}$$

and

$$h_2(\xi_k) = x_{(n)} - \overline{x} + o(1) + \sum_{i=1}^{n-1} \left[\left(\frac{1 + (x_{(i)} - \overline{x})/(\overline{x} - \xi_k)}{1 + (x_{(n)} - \overline{x})/(\overline{x} - \xi_k)} \right)^{\overline{x} - \xi_k} \right]^{c_k/(\overline{x} - \xi_k)} \times (x_{(i)} - \overline{x} + o(1)).$$

From the last two relations we can prove that

$$\lim_{\xi_k \to -\infty} \frac{h_1(\xi_k)}{h_2(\xi_k)} = \frac{1}{x_{(n)} - \overline{x}} < \infty.$$

Since the left-hand side of (2.5) converges to infinity while the right-hand side to a finite number, we conclude that assumption (3.2) leads to a contradiction. Thus

$$\lim_{\xi_k \to -\infty} \frac{c_k}{\overline{x} - \xi_k} < \infty.$$

On the other hand,

$$\lim_{\xi \to m-} \frac{c}{\overline{x} - \xi} = \frac{1}{\overline{x} - m}.$$

So the function $c/(\overline{x} - \xi)$ is bounded in the domain $(-\infty, m)$. Let now ξ_k with $\xi_k \to -\infty$ be any sequence such that

$$\lim_{\xi_k \to -\infty} \frac{c_k}{\overline{x} - \xi_k} = L.$$

Then due to (2.5) the number L must satisfy the relation (1.7) which from Proposition 2.1 has a unique solution. We have proved that for all sequences ξ_k with $\xi_k \to -\infty$ the limit of $c_k/(\overline{x} - \xi_k)$ is the same. So

$$\lim_{\xi \to -\infty} \frac{c}{\overline{x} - \xi} = L. \tag{3.3}$$

□

Relation (3.3) implies the important relation

$$c(\xi) = L(\overline{x} - \xi) + R(\xi), \quad \text{where } R(\xi) = o(\overline{x} - \xi). \tag{3.4}$$

Proposition 3.1. *The following relation holds for $\xi \to -\infty$:*

$$\left(1 + \frac{x_i - \overline{x}}{\overline{x} - \xi}\right)^c = \mathrm{e}^{L(x_i - \overline{x})} \left[1 + \left(R(x_i - \overline{x}) - \frac{L}{2}(x_i - \overline{x})^2\right) \frac{1}{\overline{x} - \xi} + o\left(\frac{1}{\overline{x} - \xi}\right)\right]. \tag{3.5}$$

Proof.

$$\Big(1+\frac{x_i-\overline{x}}{\overline{x}-\xi}\Big)^c = \exp\Big[(L(\overline{x}-\xi)+R)\Big(\frac{x_i-\overline{x}}{\overline{x}-\xi}-\frac{1}{2}\frac{(x_i-\overline{x})^2}{(\overline{x}-\xi)^2}+o\Big(\frac{1}{(\overline{x}-\xi)^2}\Big)\Big)\Big]$$
$$= \exp[L(x_i-\overline{x})]\exp\Big[-\frac{L}{2}\frac{(x_i-\overline{x})^2}{\overline{x}-\xi}+R\frac{(x_i-\overline{x})}{\overline{x}-\xi}+o\Big(\frac{1}{\overline{x}-\xi}\Big)\Big]$$
$$= \mathrm{e}^{L(x_i-\overline{x})}\Big[1+\Big(R(x_i-\overline{x})-\frac{L}{2}(x_i-\overline{x})^2\Big)\frac{1}{\overline{x}-\xi}+o\Big(\frac{1}{\overline{x}-\xi}\Big)\Big].$$

□

Theorem 3.2. *The relation*

$$R(\xi)=a_0+o(1)\quad \text{as } \xi\to-\infty \tag{3.6}$$

holds with

$$a_0=\frac{1}{2}+\frac{1}{2}L\frac{L_3}{L_2}-\frac{L_0}{L_2}s^2. \tag{3.7}$$

Proof. From (2.5) and (3.4) after some calculations we obtain

$$R=\frac{\sum\limits_{i=1}^{n}\big(1+\frac{x_i-\overline{x}}{\overline{x}-\xi}\big)^c(\overline{x}-\xi)(1-L(x_i-\overline{x}))-LL_0s^2+LL_2+o(1)}{L_1+\frac{1}{\overline{x}-\xi}\sum\limits_{i=1}^{n}\big(1+\frac{x_i-\overline{x}}{\overline{x}-\xi}\big)^c(s^2-(x_i-\overline{x})^2)+o\big(\frac{1}{x_i-\overline{x}}\big)}. \tag{3.8}$$

On the other hand, from Proposition 3.1 we obtain that

$$\sum_{i=1}^{n}\Big(1+\frac{x_i-\overline{x}}{\overline{x}-\xi}\Big)^c(\overline{x}-\xi)(1-L(x_i-\overline{x})) = R(L_1-LL_2)-\frac{L}{2}(L_2-LL_3)+o(1). \tag{3.9}$$

With the use of (3.9), relation (3.8) gives

$$RL_1+o(1)=RL_1-RLL_2-\frac{L}{2}L_2+\frac{L^2}{2}L_3-LL_0s^2+LL_2+o(1),$$

or

$$R=\frac{1}{2}+\frac{1}{2}L\frac{L_3}{L_2}-\frac{L_0}{L_2}s^2+o(1). \tag{3.10}$$

In other words, we have

$$R(\xi)=a_0+o(1)$$

with a_0 as in (3.7). □

From Proposition 3.1 and relation (1.9) we obtain the expression

$$\sum_{i=1}^{n}\Big(1+\frac{x_i-\overline{x}}{\overline{x}-\xi}\Big)^c = L_0 + \Big(a_0L_1 - \frac{LL_2}{2}\Big)\frac{1}{\overline{x}-\xi} + o\Big(\frac{1}{\overline{x}-\xi}\Big). \tag{3.11}$$

3.2. The determination of the number of solutions

Let us now substitute c from (1.9) into the equation (1.4). We have

$$g(\xi) = g_1(\xi, c(\xi)). \tag{3.12}$$

The number of roots of the above function determines the number of solutions of the log-likelihood system of equations.

Theorem 3.3. *We have*

(1) $\lim g(\xi) = -\infty$ *for* $\xi \to m-$,

(2) $g(\xi) = \dfrac{1}{2}\Big(s^2 - \dfrac{L_2}{L_0}\Big)\dfrac{1}{(\overline{x}-\xi)^2} + o\Big(\dfrac{1}{(\overline{x}-\xi)^2}\Big)$ *for* $\xi \to -\infty$.

Proof. (1) Since $c(\xi) > 1$, we have for ξ close to m

$$(m-\xi)\ln(m-\xi) < (m-\xi)^c\ln(m-\xi) < 0$$

from which we conclude that

$$\lim_{\xi\to m-}(m-\xi)^c\ln(m-\xi) = 0.$$

Now it is apparent from (1.4) that

$$\lim_{\xi\to m-} g(\xi) = -\infty.$$

(2) The $g(\xi)$ function can be written as

$$g(\xi) = \frac{1}{c} + \frac{1}{n}\sum_{i=1}^{n}\ln\Big(1+\frac{x_i-\overline{x}}{\overline{x}-\xi}\Big) - \sum_{i=1}^{n}(x_i-\xi)^c\ln\Big(1+\frac{x_i-\overline{x}}{\overline{x}-\xi}\Big)\Big/\sum_{i=1}^{n}(x_i-\xi)^c.$$

Expanding the function $\sum\limits_{i=1}^{n}\ln(1+(x_i-\overline{x})/(\overline{x}-\xi))$ and utilizing Proposition 2.2 for the function $c(\xi)$, $g(\xi)$ can be written as

$$g(\xi) = \frac{1}{L}\frac{1}{\overline{x}-\xi} - \Big(\frac{a_0}{L^2}+\frac{s^2}{2}\Big)\frac{1}{(\overline{x}-\xi)^2} - \sum_{i=1}^{n}\Big(1+\frac{x_i-\overline{x}}{\overline{x}-\xi}\Big)^c\ln\Big(1+\frac{x_i-\overline{x}}{\overline{x}-\xi}\Big)\Big/\sum_{i=1}^{n}\Big(1+\frac{x_i-\overline{x}}{\overline{x}-\xi}\Big)^c + o\Big(\frac{1}{(\overline{x}-\xi)^2}\Big).$$

With the use of Proposition (3.1) and expanding the function ln we obtain

$$\tag{3.13} \begin{aligned} &\sum_{i=1}^{n}\Big(1+\frac{x_i-\overline{x}}{\overline{x}-\xi}\Big)^{c}\ln\Big(1+\frac{x_i-\overline{x}}{\overline{x}-\xi}\Big) \\ &\qquad = L_1\frac{1}{\overline{x}-\xi}+\Big(a_0L_2-\frac{L}{2}\,L_3-\frac{L_2}{2}\Big)\frac{1}{(\overline{x}-\xi)^2}+o\Big(\frac{1}{(\overline{x}-\xi)^2}\Big). \end{aligned}$$

Making use of Proposition 2.2 and (3.11), we find that

$$\tag{3.14} \begin{aligned} \Big(\sum_{i=1}^{n}(\frac{x_i-\overline{x}}{\overline{x}-\xi})^{c}\Big)^{-1} &= \frac{1}{L_0}\,\Big(1+\Big(a_0\frac{L_1}{L_0}-\frac{L}{2}\,\frac{L_2}{L_0}\Big)\frac{1}{\overline{x}-\xi}+o\Big(\frac{1}{\overline{x}-\xi}\Big)\Big)^{-1} \\ &= \frac{1}{L_0}\,\Big[1-\Big(a_0\frac{L_1}{L_0}-\frac{L}{2}\,\frac{L_2}{L_0}\Big)\frac{1}{\overline{x}-\xi}+o\Big(\frac{1}{\overline{x}-\xi}\Big)\Big]. \end{aligned}$$

By virtue of (3.13) and (3.14) the function $g(\xi)$ takes the form

$$g(\xi)=\Big(-a_0\frac{L_2}{L_0}-\frac{s^2}{2}+\frac{LL_3}{2L_0}\Big)\frac{1}{(\overline{x}-\xi)^2}+o\Big(\frac{1}{(\overline{x}-\xi)^2}\Big).$$

Replacing the expression for a_0 by (3.7) we have

$$g(\xi)=\frac{1}{2}\Big(s^2-\frac{L_2}{L_0}\Big)\frac{1}{(\overline{x}-\xi)^2}+o\Big(\frac{1}{(\overline{x}-\xi)^2}\Big).$$

□

From the structure of $g(\xi)$ we observe that for $\xi\to-\infty$, it is dominated by the term

$$\frac{\delta}{2}\,\frac{1}{(\overline{x}-\xi)^2},$$

where

$$\delta=s^2-\frac{L_2}{L_0}.$$

Thus the g function becomes negative or positive as $\xi\to\infty$, if $\delta<0$ or $\delta>0$, respectively.

4. Discussion

The number of roots of $g(\xi)$ are not known. If $\delta < 0$, Theorem 3 claims that the function $g(\xi)$ becomes negative when $\xi \to m-$ and $\xi \to -\infty$. Thus, the roots of $g(\xi)$—if any—must come in pairs, providing that all the roots are simple. Similarly if $\delta > 0$, the function $g(\xi)$ becomes negative when $\xi \to m-$ and positive for $\xi \to -\infty$. Thus it has always a root and more specifically, it has an odd number of roots. It is widely accepted that the log-likelihood system of equations (3)–(5) possesses at most two solutions. There is no theoretical justification for this. This rule is supported by simulation results. In fact there is no counter-example in bibliography violating this rule and therefore it is adopted by many authors. Lockhart and Stephens [5], [6] use three data sets. Data set 1 from [2] is the case where the system of log-likelihood equations has two solutions.

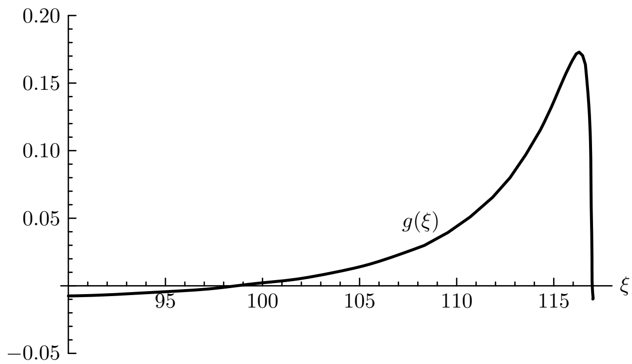

Figure 1. Graph of $g(\xi)$ with two solutions.

In this case we have $\delta = -598.195$. Data set 2 of [10] corresponds to the case where the log-likelihood system of equation has no solution.

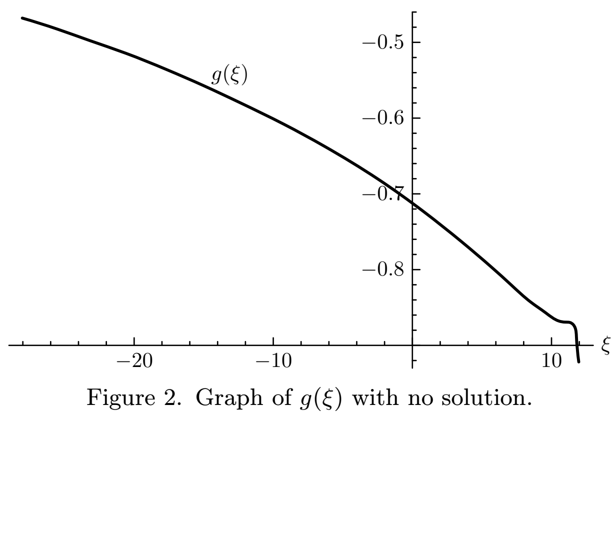

Figure 2. Graph of $g(\xi)$ with no solution.

In this case $\delta = -47168.7$. Data set 3 of artificial data corresponds to the case where the system has one solution. In this case $\delta = 0.00804$.

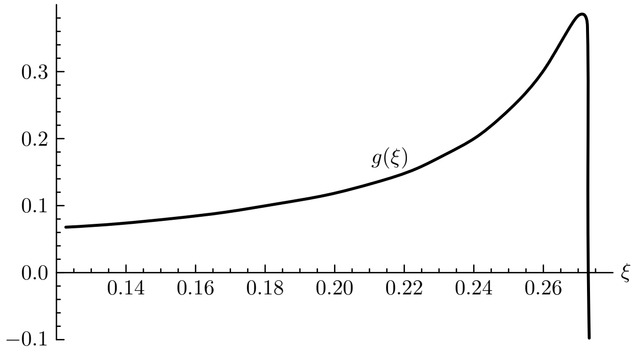

Figure 3. Graph of $g(\xi)$ with one solution.

The sign of the quantity δ determines the number of solutions of the equation $g(\xi) = 0$. In the case $\delta < 0$ we expect one solution and it cannot be a local maximum [12]. Thus, in this case there is no maximum likelihood estimator. In the case $\delta > 0$ we have two possibilities: either no solution, or two solutions. The case of no solution appears in the case of small samples. The usual case is that of two solutions. However, even in this case there is a difficulty in distinguishing between one or two solutions case because many times the second solution is hidden very close to the asymptote $\xi = m$ and is hard to be found. See for example Fig. 1. In such cases the sign of δ indicates that another solution must exist. Fortunately, close to the asymptote the log-likelihood function cannot have a local maximum. To see this we examine the behavior of the Hessian matrix of the log-likelihood function (1.2). If $\underline{\theta} = (\alpha, c, \xi)$, the Hessian matrix $H(\alpha, c, \xi)$ is defined as

$$H(\alpha, c, \xi) = \frac{\partial^2 L(\underline{\theta}; \underline{x})}{\partial \underline{\theta} \partial^t \underline{\theta}}.$$

The matrix $H(\hat{\alpha}, \hat{c}, \hat{\xi})$ is negative definite if the signs of the minor determinants Δ_1, Δ_2, and Δ_3 along the diagonal are $\Delta_1 < 0$, $\Delta_2 > 0$, $\Delta_3 < 0$. By elementary calculations we find that the first minor determinant $\Delta_1(\alpha, c, \xi) = -nc^2\alpha^2$ is always negative. The second minor determinant along the main diagonal is

$$\Delta_2(\alpha, c, \xi) = \frac{n^2}{a^2}\left(1 + \frac{1}{n}\sum_{i=1}^{n}\left(\frac{x_i - \xi}{a}\right)^c \ln^2\left(\frac{x_i - \xi}{a}\right)^c - \left(1 + \frac{1}{n}\sum_{i=1}^{n}\ln\left(\frac{x_i - \xi}{a}\right)^c\right)^2\right).$$

Easily we can see that

$$\lim \Delta_2(\alpha, c, \xi) = -\infty \quad \text{as} \quad \xi \to m-.$$

Thus for stationary values $(\hat{\alpha}, \hat{c}, \hat{\xi})$, where $\hat{\xi}$ is close to the asymptote $\xi = m$, the Hessian matrix is neither positive nor negative definite. Such a value corresponds to a saddle point.

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