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A CONSTRUCTION OF LARGE GRAPHS
OF DIAMETER TWO AND GIVEN DEGREE
FROM ABELIAN LiftS OF DIPoLES

Dávid Mesežnikov

For any $d \geq 11$ we construct graphs of degree $d$, diameter 2, and order $\frac{8}{25} d^2 + O(d)$, obtained as lifts of dipoles with voltages in cyclic groups. For Cayley Abelian graphs of diameter two a slightly better result of $\frac{9}{25} d^2 + O(d)$ has been known [3] but it applies only to special values of degrees $d$ depending on prime powers.

Keywords: the degree-diameter problem, voltage assignment and lift, dipole

Classification: 05C12, 05C35

1. INTRODUCTION

Two types of restrictions that appear frequently in the design of large interconnection networks are limitations on the number of links emanating from a node and on the length of the shortest path between a pair of nodes. If networks are modeled by undirected graphs, the two requirements lead to design of large graphs of a given maximum degree and a given diameter. The search for largest such graphs is known as the degree-diameter problem. Since we will be interested only in the case of diameter 2, we just mention that by the Moore bound [5] the largest order (i.e., number of vertices) of a graph of diameter 2 and maximum degree $d$ is $d^2 + 1$ and that graphs of such an order exist only for degrees $d = 2, 3, 7$ and possibly 57.

In the past decades a number of techniques for constructing large graphs of a given degree and diameter have been developed. A fruitful method appears to be lifting graphs of a small order to comparatively large graphs by means of voltage assignments in finite groups; if the groups are Abelian one speaks about Abelian lifts. To avoid repetitiousness we refer to the basics of the method of lifting to [5] and references therein. In particular, Abelian lifts of dipoles (graphs of order 2) gave rise to the largest vertex-transitive and almost vertex-transitive graphs of diameter 2 and a given degree $d = (3q \pm 1)/2$, $q$ an odd prime power, whose order is $\frac{9}{25} d^2 + O(d)$, cf. [4, 5]. This led to interest in largest possible Abelian lifts of graphs of order 1 (equivalently, Cayley graphs of Abelian groups) and 2. From [7] it follows that the largest order of a graph of diameter 2 and degree $d$ obtained as an Abelian lift of a dipole is $\leq 0.932d^2 + O(d)$. In the other direction, constructions of [3] furnish Cayley graphs of degree $d$ and diameter 2 on Abelian groups.
of order \(\frac{1}{2}(d+1)^2\) if \(d = 3q - 1\) and \(\frac{3}{2}(d^2 - 4)\) if \(d = 4q - 2\), where in both cases \(q\) is an odd prime power. Moreover, in [3] the authors gave a construction of a Cayley graph of diameter 2 and degree \(d = 5p - 3\), where \(p\) is a prime congruent to 2 mod 3, on a cyclic group of order \(\frac{9}{25}d^2 + O(d)\).

In this note we offer a construction of graphs of degree \(d\), diameter 2, and order \(\frac{8}{25}d^2 + O(d)\), obtained as lifts of dipoles with voltages in cyclic groups. This is slightly worse than the aforementioned result of [3] but has the advantage that the construction works for general degrees \(d \geq 11\).

2. RESULTS

Our graphs will be always finite but may have loops and parallel (that is, multiple) edges. By \(D_{r,s}\) we denote a dipole, that is, a graph consisting of exactly two vertices joined by \(r\) parallel edges and having \(s\) loops at each vertex. Such a dipole is a regular graph of degree \(d = r + 2s\); with unspecified \(r\) and \(s\) we just speak about a dipole \(D\) of degree \(d\).

We are now ready to present and prove our results.

Theorem 2.1. For any \(d \geq 11\) there exists a graph of order \(\frac{8}{25}d^2 + O(d)\), degree \(d\), and diameter 2, arising as a lift of a dipole with voltages in a cyclic group.

Proof. Because of the nature of the statement it is sufficient to prove it for all sufficiently large \(d\) and we will do so for all \(d \geq 11\). We begin with degrees \(d \equiv 1 \mod 10\), that is, we let \(d = 10\ell + 1\) where \(\ell \geq 1\). For \(r = 8\ell + 1\) and \(s = \ell\), consider the dipole \(D = D_{r,s}\) as introduced before, of degree \(d = r + 2s = 10\ell + 1\) and with vertices \(u\) and \(v\). Further, let \(G = \mathbb{Z}_n\) be the cyclic group of order \(n = 16\ell^2 + 8\ell = \frac{4}{5}d^2 + O(d)\). On the dipole \(D\) we introduce a voltage assignment \(\alpha\) in \(G\) as follows. Letting \(k = 4\ell + 1\), the \(r = 2k - 1\) darts from \(u\) to \(v\) will be mapped bijectively by \(\alpha\) onto the set \(A = \{0, -1, -2, \ldots, -k + 1, k, 2k, \ldots, (k - 1)k\}\), and the set of all the \(2\ell = (k - 1)/2\) loops at both \(u\) and \(v\) are mapped bijectively by \(\alpha\) onto the set \(B = \{1, 2, 3, \ldots, (k - 1)/2\}\). The lift \(D^\alpha\) has \(2n = 2(k^2 - 1) = \frac{8}{25}d^2 + O(d)\) vertices and has degree \(d\).

We proceed by showing that the lift \(D^\alpha\) has diameter 2. It suffices to show that for any \(g \in \mathbb{Z}_n\) there exists a walk \(W\) in \(D\) of length at most two starting and ending at any of the two vertices \(u, v\) of \(D\) and such that \(\alpha(W) = g\). First we examine the \(u \rightarrow v\) walks. If \(g = kt \in A\) for some \(t\) such that \(0 \leq t \leq k - 1\), then \(W\) consists of the dart from \(u\) to \(v\) carrying the voltage \(kt \in A\). For \(g = ik + j\), where \(i \in \{0, 1, 2, \ldots, k - 1\}\) and \(j \in B \cup -B\), we can take \(W\) of length 2 composed of the dart from \(u\) to \(v\) with voltage \(ik\) and a suitable loop at \(u\) or at \(v\) carrying the voltage \(j\). Considering \(u \rightarrow u\) walks, for \(g \in A \cup -A\) the walk \(W\) consists of the dart from \(u\) to \(v\) with voltage \(g\) followed by the \(v\) to \(u\) dart with voltage 0. If \(g = ik + h\), where \(i, h \in \{1, 2, \ldots, k - 1\}\), then we choose \(W\) consisting of the \(u \rightarrow v\) dart with voltage \(ik\) and the \(v \rightarrow u\) dart with voltage \(h\). The cases of \(v \rightarrow v\) and \(v \rightarrow u\) walks can be dealt with in a similar way. This implies that the lift \(D^\alpha\) has diameter two.

We have thus proved the statement for all \(d \geq 11\) such that \(d \equiv 1 \mod 10\). For the remaining \(d = 10\ell + 1 + \delta\), where \(\ell \geq 1\) and \(1 \leq \delta \leq 9\) we modify the dipole \(D\) by
inserting extra $\lfloor \delta/2 \rfloor$ loops at both $u$ and $v$ that carry arbitrary distinct voltages in the set \{2$\ell + 1, \ldots, 2\ell + \lfloor \delta/2 \rfloor\} \subset Z_n$; if $\delta$ is odd we also insert an extra dart from $u$ to $v$ carrying the voltage 1 $\in Z_n$. By the above argument, the lift will have diameter 2, degree $d$, and order $\frac{8}{25}d^2 + O(d)$. \hfill \Box

The natural question of possible vertex-transitivity of the graphs constructed above is answered in the negative by our next result.

**Theorem 2.2.** The graphs constructed in the proof of Theorem 2.1 are not vertex-transitive if $d \geq 21$.

**Proof.** We keep to the notation introduced in the proof of Theorem 2.1. Let $F = \{v_i; \ i \in Z_n\}$ and $F_v = \{v_i; \ i \in Z_n\}$ be the fibres above $u$ and $v$, respectively, in the covering $D^\alpha \to D$ induced by the voltage assignment $\alpha$ in $Z_n$. Since $k$ is relatively prime to $n = k^2 - 1$, the element $k \in Z_n$ has order $n$. Let $k_0 = k(k-1)/2$ and $k_1 = k(k+1)/2$ be elements of $Z_n$. If $k \geq 9$, which is the case if $d \geq 21$, the dart of $D$ from $u$ to $v$ that carries the voltage $k_0$ is contained in no walk of length 3 of zero voltage, and the same is true for the dart from $u$ to $v$ of voltage $k_1$. (The condition $k \geq 9$ is needed because of the additional loops in the construction for $d \not\equiv 1 \mod 10$.) It follows that no edge of the form $u_i v_i m$ for $m \in \{k_0, k_1\}$ in the lift $D^\alpha$ lies in a triangle for any $i \in Z_n$. But as $k_1 - k_0 = k$, the cycle $C$ of the form

$$u_0 \to v_{k_1} \to u_k \to v_{k+k_1} \cdots \to u_{jk} \to v_{jk+k_1} \to u_{(j+1)k} \to v_{(j+1)k+k_1} \to \cdots$$

is a Hamilton cycle of $D^\alpha$ consisting of edges belonging to no triangle. Note also that every edge of $D^\alpha$ with both ends in $F_u$ lies in a triangle, with a similar conclusion for any edge with both ends in $F_v$.

Suppose now that $D^\alpha$ was a vertex-transitive graph and let $f$ be an automorphism that takes a vertex from $F_u$ onto a vertex from $F_v$. Since $f(C)$ is a Hamilton cycle again, with edges contained in no triangles, it follows that $f$ must interchange the sets $F_u$ and $F_v$. In other words, the fibres $F_u$ and $F_v$ form a block system for the automorphism group of $D^\alpha$. By the construction of $D^\alpha$ it is obvious that any edge of $D^\alpha$ that is a lift of a loop lies in a triangle containing vertices from both fibres, and such an edge lies in a largest number of such triangles if and only if the edge is a lift of the loop carrying the voltage 1. But such edges are either all in $F_u$ or all in $F_v$. Consequently, no automorphism $f$ as above exists, and we conclude that $D^\alpha$ is not a vertex-transitive graph. \hfill \Box

Let us remark that there is a lot of flexibility regarding the voltage assignment $\alpha$ in the proof of Theorem 2.1. It might be possible that a better choice of a voltage assignment could give vertex-transitive graphs but we have not been able to identify such assignments for general degrees $d$, and not even for small $d$ by computer [1].

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