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COMPACT SPACE-LIKE HYPERSURFACES
WITH CONSTANT SCALAR CURVATURE
IN LOCALLY SYMMETRIC LORENTZ SPACES

Yaning Wang and Ximin Liu

Abstract. A new class of \((n + 1)\)-dimensional Lorentz spaces of index 1 is introduced which satisfies some geometric conditions and can be regarded as a generalization of Lorentz space form. Then, the compact space-like hypersurface with constant scalar curvature of this spaces is investigated and a gap theorem for the hypersurface is obtained.

1. Introduction

Let \(N_{p}^{n+p}\) be an \((n + p)\)-dimensional connected semi-Riemannian manifold of index \(p\). It is called a semi-definite space of index \(p\). When we refer to index \(p\), we mean that there are only \(p\) negative eigenvalues of semi-Riemannian metric of \(N_{p}^{n+p}\) and the other eigenvalues are positive. In particular, \(N_{1}^{n+1}\) is called a Lorentz space when \(p = 1\). When the Lorentz space \(N_{1}^{n+1}\) is of constant curvature \(c\), we call it Lorentz space form, denote it by \(N_{1}^{n+1}(c)\), with de Sitter space \(S_{1}^{n+1}(1)\) and anti-de Sitter space \(H_{1}^{n+1}(-1)\) as its special cases. A hypersurface \(M\) of a Lorentz space is said to be space-like if the induced metric from that of the ambient space is positive definite.

The authors in \([3]\) introduced a class of Lorentz spaces \(\overline{M}\) of index 1. Let \(\nabla\), \(\overline{K}\) and \(\overline{R}\) denote the semi-Riemannian connection, sectional curvature and curvature tensor on \(\overline{M}\), respectively. For constant \(c_{1}\), \(c_{2}\) and \(c_{3}\), they considered Lorentz spaces which satisfy the following conditions:

(1) for any space-like vector \(u\) and any time-like vector \(v\), \(\overline{K}(u, v) = -\frac{c_{1}}{n}\),

(2) for any space-like vector \(u\) and \(v\), \(\overline{K}(u, v) \geq c_{2}\),

(3) \[|\nabla \overline{R}| \leq \frac{c_{3}}{n}.\]

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When $\overline{M}$ satisfies conditions (1) and (2), they say that $\overline{M}$ satisfies condition ($\ast$). When $\overline{M}$ satisfies conditions (1) $-$ (3), they say that $\overline{M}$ satisfies condition ($\ast\ast$).

Also they give some examples as following.

**Example 1.1.** The semi-Riemannian product manifold $H^k_1(-\frac{c_1}{n}) \times M^{n+1-k}(c_2)$, $c_1 > 0$. Its sectional curvature is given by

$$K(u_1, u_b) = K(u_a, u_b) = -\frac{c_1}{n}, \quad K(u_a, u_r) = 0, \quad K(u_r, u_s) = c_2,$$

where $a, b = 2, \ldots, k$; $r, s = k + 1, \ldots, n + 1$, $u_1$ and $u_a$, $u_r$ denote time-like and space-like vectors respectively.

**Example 1.2.** The semi-Riemannian product manifold $R^k_1 \times S^{n+1-k}(1)$. Its sectional curvature is given by

$$K(u_1, u_a) = K(u_a, u_b) = 0, \quad K(u_1, u_r) = 0, \quad K(u_r, u_s) = 1,$$

where $a, b = 2, \ldots, k$; $r, s = k + 1, \ldots, n + 1$. In particular, $R^1_1 \times S^n(1)$ is called Einstein Static Universe. Notice that it is not a Lorentz space form.


Now we consider Lorentz spaces which satisfy another condition:

(4) for any space-like vectors $u$ and $v$, $K(u, v) \leq c_2$.

When $\overline{M}$ satisfies conditions (1) and (4), we shall say that $\overline{M}$ satisfies conditions ($\overline{\ast}$). When $\overline{M}$ satisfies conditions (1), (3) and (4), we shall say that $\overline{M}$ satisfies condition ($\overline{\ast\ast}$). In this paper, we mainly discuss the compact space-like hypersurfaces with constant scalar curvature in a locally symmetric Lorentz spaces satisfying the condition ($\overline{\ast}$). It is worthy to point out that both Example 1.1 and 1.2 satisfy the condition ($\overline{\ast}$).

**Remark 1.3.** It is easy to see that a Lorentz space form $N^1_{n+1}(s)$ satisfies both conditions ($\ast\ast$) and ($\overline{\ast\ast}$), where $-\frac{c_1}{n} = c_2 = s$.

**Remark 1.4.** If a Lorentz space $\overline{M}$ is locally symmetric, then the condition (3) holds naturally, because $\nabla R = 0$ in this situation.

**Remark 1.5.** As discussed in section 4, our theorem extend the results in [6] under some geometric conditions.

2. Preliminaries

Let $(\overline{M}, \overline{g})$ be an $(n + 1)$-dimensional Lorentz space of index 1. Throughout the paper, manifolds are assumed to be connected and geometric objects are assumed
to be of class $C^\infty$. For any point $p \in M$, we choose a local field of semi-orthonormal frames $\{e_A\} = \{e_1, e_2, \ldots, e_{n+1}\}$ on a neighborhood of $p$, where $e_1, \ldots, e_n$ are space-like and $e_{n+1}$ is time-like. We use the following convention on the range of indices throughout the paper

$$A, B, \ldots = 1, \ldots, n + 1; \quad i, j, \ldots = 1, 2, \ldots, n.$$ 

Let $\{\omega_A\} = \{\omega_1, \omega_2, \ldots, \omega_{n+1}\}$ denote the dual frame fields of $\{e_A\}$ on $M$. The metric tensor $g$ of $M$ satisfies $g(e_A, e_B) = \delta_{AB}$, where $\epsilon_1 = \ldots = \epsilon_n = 1$ and $\epsilon_{n+1} = -1$. The canonical forms $\{\omega_A\}$ and the connection forms $\{\omega_{AB}\}$ satisfy the following structure equations

$$d\omega_A = -\sum_B \epsilon_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0,$$

(2.1)

$$d\omega_{AB} = -\sum_C \epsilon_C \omega_{AC} \wedge \omega_{CB} - \frac{1}{2} \sum_{C,D} \epsilon_C \epsilon_D R_{ABCD} \omega_C \wedge \omega_D.$$

(2.2)

The components $\overline{R}_{CD}$ of the Ricci tensor and the scalar curvature $\overline{R}$ are given respectively by

$$\overline{R}_{CD} = \sum_B \epsilon_B \overline{R}_{BCDB},$$

(2.3)

and

$$\overline{R} = \sum_A \epsilon_A \overline{R}_{AA}.$$  

(2.4)

The components $\overline{R}_{ABCD;E}$ of the covariant derivative of the Riemannian curvature tensor $\overline{R}$ are defined by

$$\sum_E \epsilon_E \overline{R}_{ABCD;E} = d\overline{R}_{ABCD} - \sum_E \epsilon_E (\overline{R}_{EBCD} \omega_{EA} + \overline{R}_{AECD} \omega_{EB} + \overline{R}_{ABED} \omega_{EC} + \overline{R}_{ABCE} \omega_{ED}).$$

(2.5)

Restricting the forms $\{\omega_A\}$ to a space-like hypersurface $M$ in $\overline{M}$, we have

$$\omega_{n+1} = 0,$$

(2.6)

and the induced metric $g$ of $M$ is given by $g = \sum_i \omega_i \otimes \omega_i$. It is well known that by Cartan’s Lemma we get

$$\omega_{(n+1)i} = \sum_j h_{ij} \omega_j, \quad h_{ij} = h_{ji},$$

(2.7)

where $h_{ij}$ are the coefficients of the second fundamental form of $M$. Then we denote by $H = \frac{1}{n} \sum_i h_{ii}$ and $S = \sum_{ij} h_{ij}^2$ the mean curvature and squared norm of the second fundamental form of $M$, respectively.
The structure equations of \( M \) are given by
\[
d\omega_i = -\sum_{j} \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0,
\]
\[
d\omega_{ij} = -\sum_{k} \omega_{ik} \wedge \omega_{kj} - \frac{1}{2} \sum_{k,l} R_{ijkl} \omega_k \wedge \omega_l.
\]
\[
R_{ijkl} = R_{ijlk} + (h_{ik} h_{jl} - h_{il} h_{jk}).
\]
The Gauss equation is given by
\[
R_{ijkl} = R_{ijlk} + \left(h_{ik} h_{jl} - h_{il} h_{jk}\right).
\]
The Ricci tensor and normalized scalar curvature of \( M \) are given respectively by
\[
R_{ij} = \sum_{k} R_{kijk} - nHh_{ij} + \sum_{k} h_{ik} h_{kj},
\]
and
\[
n(n - 1)R = \sum_{j,k} R_{kjjk} - n^2 H^2 + S.
\]
Let \( \overline{M} \) be a locally symmetric Lorentz space satisfying the condition \((\overline{\pi})\). We know that the scalar curvature \( \overline{R} \) of \( \overline{M} \) is a constant. By using the structure equations of \( \overline{M} \), we have
\[
\overline{R} = \sum_{A} \epsilon_{A} \overline{R}_{AA} = -2 \sum_{i} \overline{R}_{(n+1)ii(n+1)} + \sum_{i,j} \overline{R}_{iiji} = -2c_1 + \sum_{i,j} \overline{R}_{iiji},
\]
which means that \( \sum_{i,j} \overline{R}_{iiji} \) is a constant. We assume from now that the scalar curvature \( R \) of \( M \) is constant. Together with the above equation and \((2.12)\), we define a constant \( P \) by
\[
n(n - 1)P = \sum_{j,k} \overline{R}_{kjjk} - n^2 H^2 + S.
\]
By taking exterior differentiation of \((2.7)\) and defining \( h_{ijk} \) by
\[
\sum_{k} h_{ijk} \omega_{k} = dh_{ij} - \sum_{k} \left(h_{kj} \omega_{ki} + h_{ik} \omega_{kj}\right),
\]
we have the following Codazzi equation
\[
h_{ijk} - h_{ikj} = \overline{R}_{(n+1)ijk}.
\]
Similarly, we define \( h_{ijkl} \) by
\[
\sum_{l} h_{ijkl} \omega_{l} = dh_{ijk} - \sum_{l} \left(h_{ljk} \omega_{li} + h_{ilk} \omega_{lj} + h_{ijl} \omega_{lk}\right).
\]
By taking exterior differentiation of \((2.15)\), we have Ricci formula for the second fundamental form of \( M \)
\[
R_{ij} = -\sum_{k} R_{ik} R_{jkl} + h_{j} R_{rjkl}.
\]
Restricting (2.5) on $M$, $\overline{R}_{(n+1)ijk;l}$ is given by

$$
\overline{R}_{(n+1)ijk;l} = \overline{R}_{(n+1)ijkl} + \overline{R}_{(n+1)i(n+1)jk} + \sum_m \overline{R}_{mijk}h_{ml},
$$

(2.19)

where $\overline{R}_{(n+1)ijkl}$ denote the covariant derivative of $\overline{R}_{(n+1)ijk}$ as a tensor on $M$ by

$$
\sum_l \overline{R}_{(n+1)ijlk}\omega_l = d\overline{R}_{(n+1)ijk} - \sum_l \overline{R}_{(n+1)ljk}\omega_l + \sum_l \overline{R}_{(n+1)ilk}\omega_l - \sum_l \overline{R}_{(n+1)ijl}\omega_l.
$$

(2.20)

**Remark 2.1.** If $\overline{M}$ is a Lorentz space form of index 1, by a straightforward calculation we check that the sum of the last three terms of right-hand side of (2.19) goes to zero. Then we have $\overline{R}_{(n+1)ijk;l} = \overline{R}_{(n+1)ijkl}$, which is the same as in the case that the ambient space is a space form.

It is well known that the Laplacian $\Delta h_{ij}$ is defined by

$$
\Delta h_{ij} = \sum_k h_{ijkk}.
$$

(2.21)

By using Codazzi equation and Ricci formula, we get

$$
\Delta h_{ij} = \sum_k h_{ikjk} + \sum_k \overline{R}_{(n+1)ijkk} = \sum_k h_{kijk} + \sum_k \overline{R}_{(n+1)ijkk} = \sum_k \left( h_{kikj} - \sum_l (h_{kl}R_{lij} + h_{il}R_{lkj}) + \overline{R}_{(n+1)ijkk} \right).
$$

(2.22)

From the Codazzi equation $h_{ikjk} = h_{kkij} + \overline{R}_{(n+1)kijk}$, we have

$$
\Delta h_{ij} = \sum_k h_{kkij} + \sum_k \left( \overline{R}_{(n+1)ijkk} + \overline{R}_{(n+1)kijk} \right) - \sum_{k,l} (h_{kl}R_{lij} + h_{il}R_{lkj}).
$$

Together with Gauss equation and above equation and (2.19), we have

$$
\Delta h_{ij} = \sum_k h_{kkij} + \sum_k \left( \overline{R}_{(n+1)ijkk} + \overline{R}_{(n+1)kikk} \right)
$$

$$
- \sum_{k,l} \left( 2h_{kl} \overline{R}_{lij} + h_{jl} \overline{R}_{lkik} + h_{il} \overline{R}_{lkjk} \right) + Sh_{ij}
$$

$$
- \sum_k \left( h_{kk} \overline{R}_{(n+1)ij(n+1)} + h_{ij} \overline{R}_{(n+1)k(n+1)} \right) - nH \sum_l h_{il}h_{jl}.
$$

(2.23)
Thus
\[ \frac{1}{2} \Delta S = \sum_{i,j,k} h_{ijk}^2 + \sum_{i,j} h_{ij} \Delta h_{ij} \]
\[ = \sum_{i,j,k} h_{ijk}^2 + \sum_{i,j} (nH)_{ij} h_{ij} + \sum_{i,j,k} (R_{(n+1)ijk} + R_{(n+1)ikj}) h_{ij} \]
\[ - \left( nH \sum_{i,j} h_{ij} R_{(n+1)ij} + S \sum_{k} R_{(n+1)k(n+1)} \right) + S^2 \]
\[ - \sum_{i,j,k,l} (h_{kl} h_{ij} R_{lijk} + h_{il} h_{ij} R_{lkjk}) - nH \sum_{i,j,l} h_{il} h_{lj} h_{ij}. \]

(2.24)

3. Estimates of Laplacian and Key Lemmas

Let \( \hat{M} \) be a locally symmetric Lorentz space, i.e., \( \hat{R}_{ABCD;E} = 0 \). We also may choose a canonical bases \( \{e_1, e_2, \ldots, e_n\} \) such that \( h_{ij} = \lambda_i \delta_{ij} \), thus

(3.1)
\[ \hat{R}_{(n+1)ijk; k} + \hat{R}_{(n+1)ikj} = 0. \]

Noticing that \( \hat{M} \) satisfies condition \( (\mathfrak{r}) \), we have

(3.2)
\[ - \left( nH \sum_{i,j} h_{ij} \hat{R}_{(n+1)ij} + S \sum_{k} \hat{R}_{(n+1)k(n+1)} \right) \]
\[ = - \left( nH \sum_{i} \lambda_i \hat{R}_{(n+1)ii} + S \sum_{i} \hat{R}_{(n+1)i(n+1)} \right) \]
\[ = c_1 (S - nH^2). \]

Also we have

(3.3)
\[ - \sum_{i,j,k,l} 2(h_{kl} h_{ij} \hat{R}_{lijk} + h_{il} h_{ij} \hat{R}_{lkjk}) \]
\[ = 2 \sum_{j,k} \lambda_j \lambda_k \hat{R}_{kjjk} \leq c_2 \sum_{j,k} (\lambda_j - \lambda_k)^2 = 2c_2 (nS - n^2 H^2). \]

Substituting (3.1), (3.2) and (3.3) into (2.24), it yields that

(3.4) \[ \frac{1}{2} \Delta S \leq \sum_{i,k} h_{ii}^2 + \sum_{i} \lambda_i (nH)_{ii} + (2nc_2 + c_1)(S - nH^2) + (S^2 - nH \sum_{i} \lambda_i^2). \]

Lemma 3.1 ([7]). Let \( \{\mu_1, \mu_2, \ldots, \mu_n\} \) be real numbers satisfying \( \sum \mu_i = 0 \) and \( \sum \mu_i^2 = A \), where \( A \) is a constant no less than zero. Then we have

\[ \left| \sum_i \mu_i^2 \right| \leq \frac{n-2}{\sqrt{n(n-1)}} A^\frac{3}{2}, \]

and the equality holds if and only if at least \( n - 1 \) of the \( \mu_i \) are equal, i.e.,

\[ \mu_1 = \mu_2 = \ldots = \mu_{n-1} = -\sqrt{\frac{1}{n(n-1)}} A, \quad \mu_n = \sqrt{\frac{n-1}{n}} A. \]
Lemma 3.2. Let $M$ be a space-like hypersurface with constant normalized scalar curvature $R$ in locally symmetric $(n+1)$-dimensional Lorentz space satisfying the condition \( (\pi) \). If $h_{ijk} \geq 0$, then
\[
\sum_{i,j,k} h_{ijk}^2 \leq n^2 |\nabla H|^2.
\]

Proof. Notice that the following equation holds:
\[
n^2 |\nabla H|^2 = \sum_k \left( \sum_{i,j} h_{ijk} \right)^2 = \sum_{i,j,k,l,m} h_{ijk} h_{lmk} = \sum_{i,j,k} h_{ijk}^2 + \sum_{i\neq l,j,k,m} h_{ijk} h_{lmk} + \sum_{i,j\neq m,k} h_{ijm} h_{imk}.
\]

Then the proof follows from the above equation. \(\square\)

Next we will use the well known self-adjoint operator $\Box$ introduced in [1] to the function $nH$ and using (2.14), we have
\[
\Box(nH) := \sum_{i,j} (nH \delta_{ij} - h_{ij})(nH)_{ij}
\]
\[
= \frac{1}{2} \Delta(nH)^2 - \sum_i (nH)_i^2 - \sum_i \lambda_i(nH)_{ii}
\]
\[
= \frac{1}{2} \Delta(n(n-1)P) + \frac{1}{2} \Delta S - n^2 |\nabla H|^2 - \sum_i \lambda_i(nH)_{ii}.
\]

By (2.14), we know that $P$ is a constant, so we have $\frac{1}{2} \Delta(n(n-1)P) = 0$. Then substituting (3.4) to (3.5), we obtain
\[
\Box(nH) \leq \sum_{i,j,k} h_{ijk}^2 - n^2 |\nabla H|^2 + (2nc_2 + c_1)(S - nH^2) + (S^2 - nH \sum_i \lambda_i^3).
\]

Lemma 3.3. Let $M$ be a compact space-like hypersurface of dimension $n$ with constant scalar curvature in a locally symmetric Lorentz space which satisfies condition \( (\pi) \) and $h_{ijk} \geq 0$. Then we have the following inequality
\[
\Box(nH) \leq \frac{n-1}{n} (S - nP) \phi_P(S),
\]
where $\phi_P(S) = nc - 2(n-1)P + \frac{n-2}{n} S + \frac{n-2}{n} \sqrt{(n(n-1)P + S)(s - nP)}$ and $c = 2c_2 + \frac{c_1}{n}$.

Proof. We denote
\[
\mu_i = \lambda_i - H, \quad B = \sum_i \mu_i^2.
\]

It is obvious to see that
\[
\sum_i \mu_i = 0, \quad B = S - nH^2, \quad \sum_i \lambda_{ii}^3 = \sum_i \mu_{ii}^3 + 3HB + nH^3.
\]
By using Lemma 3.1, we have
\[ -nH \sum_i \lambda_i^3 = -n^2H^4 - 3nH^2B - nH \sum_i \mu_i^3 \]
(3.7)
\[ \leq 2n^2H^4 - 3nSH^2 + \frac{n(n-2)}{\sqrt{n(n-1)}}\|H\|B^2. \]

Substituting (3.7) to (3.6) and together with the Lemma 3.2, we get
(3.8) \[ \Box(nH) \leq B(2nc - nH^2 + B + \frac{n(n-2)}{\sqrt{n(n-1)}}\|H\|B^\frac{1}{2}). \]

It follows from (2.14) that
(3.9) \[ B = S - nH^2 = \frac{n-1}{n}(S - nP). \]

Putting the above equation into (3.8), we get
(3.10) \[ \Box(nH) \leq \frac{n-1}{n}(S - nP)\phi_H(S), \]
where
(3.11) \[ \phi_H(S) = nc - 2nH^2 + S + \frac{n(n-2)}{\sqrt{n(n-1)}}\|H\|\sqrt{S - nH^2}. \]

Putting (3.9) into (3.11), we have
\[ \phi_P(S) = nc - 2(n-1)P + \frac{n-2}{n}S + \frac{n-2}{n}\sqrt{(n-1)P + S}(S - nP). \]

Finally, (3.10) becomes
(3.12) \[ \Box(nH) \leq \frac{n-1}{n}(S - nP)\phi_P(S), \]
then we complete the proof.

4. Main theorems and proofs

**Theorem 4.1.** Let \( M \) be a compact space-like hypersurface of dimension \( n \) (where \( n > 2 \)) with constant scalar curvature in a locally symmetric Lorentz space of dimension \( n + 1 \) which satisfies condition (\( \overline{\pi} \)) and \( h_{ijk} \geq 0 \). If \( 0 \leq c \leq P \) or \( c \leq P < \frac{2}{n}c \) or \( P > \frac{1}{n-1}c, c < 0 \), then the norm square of the second fundamental form \( S \) satisfies
\[ S \geq \frac{n}{(n-2)(nP - 2c)}(n(n-1)P^2 - 4c(n-1)P + nc^2), \]
where \( P \) is given by (2.14) and \( c = 2c_2 + \frac{c_1}{n} \).

**Proof.** Since \( \Box \) is a self-adjoint operator and \( M \) is compact, then we have
(4.1) \[ \int_M \Box(nH) * 1 = 0. \]
We notice that $S - nP \geq 0$ holds naturally by (3.9) because $S \geq nH^2$. By taking integration on both sides of (3.12), we get $\phi_P(S) \geq 0$. By directly calculation we see that $\phi_P(S) \geq 0$ is equivalent to

$$S \geq \frac{n}{n-2}(2(n-1)P - nc)$$

or

$$\frac{n}{(n-2)(nP - 2c)}(n(n-1)P^2 - 4c(n-1)P + nc^2) \leq S < \frac{n}{(n-2)(nP - 2c)}(n(n-1)P^2 - 4c(n-1)P + nc^2).$$

By solving the above inequalities, we complete the proof.

**Theorem 4.2.** Let $M$ be a compact space-like hypersurface of dimension $n$ (where $n > 2$) with constant scalar curvature in a locally symmetric Lorentz space of dimension $n + 1$ which satisfies condition $(\bar{\pi})$ and $h_{ijk} \geq 0$ and $0 \leq c \leq P$ or $c \leq P < \frac{2}{n}c$ or $P > \frac{1}{n-1}c, c < 0$. If the norm square of the second fundamental form $S$ satisfies

$$nP \leq S \leq \frac{n}{(n-2)(nP - 2c)}(n(n-1)P^2 - 4c(n-1)P + nc^2),$$

then

(i) $S = nP$ and $M$ is totally umbilical, or

(ii) $S = \frac{n}{(n-2)(nP - 2c)}(n(n-1)P^2 - 4c(n-1)P + nc^2)$ and $M$ has two distinct principal curvatures.

**Proof.** Together with (4.2) and the definition of $P$, we see that the right-hand side term of (3.12) is non-positive. As in proof of Theorem 4.1, we take integration on both sides of (3.12) and notice (4.1), we have $(S - nP)\phi_P(S) = 0$. In particular, we notice that $\phi_P(S) = 0$ if and only if the equality holds in Lemma 3.1, thus we prove the theorem.

**Remark 4.3.** Let $\bar{M}$ in Theorem 4.2 be a Lorentz space form with constant sectional curvature $s$. In particular, we assume that $s = 1$ such that $\bar{M}$ is nothing but a de Sitter space. As seen in Remark 1.3, we have $-\frac{a_1}{n} = c_2 = 1$. Thus $c$ defined in Lemma 3.3 is 1. Then our theorem is just like Liu’s corollary in [6].

Finally, we discuss the compact space-like surface in a locally symmetric Lorentz spaces of dimension 3, i.e., the version of $n = 2$ of the Theorem 4.1. We using the convention of the ranges of the indexes as following

$$i, j, k = 1, 2, \quad A, B, C = 1, 2, 3.$$  

**Theorem 4.4.** Let $M$ be a compact space-like surface with constant scalar curvature in a locally symmetric 3-Minkowski space which satisfies condition $(\bar{\pi})$ and $h_{ijk} \geq 0$. Then

$$P \leq c,$$

where $P$ is given by (2.14) and $c = 2c_2 + \frac{a_1}{n}$ and $h_{ijk}$ is defined by (2.15).

**Proof.** We notice that when $n = 2$, (3.12) becomes $\Box(2H) \leq (S - 2P)(c - P)$. Taking integration on both sides of the inequality, then we complete the proof.
**Corollary 4.5.** Let $M$ be a compact space-like surface with constant scalar curvature in a locally symmetric 3-Minkowski space which satisfies condition $(\pi)$ and $h_{ijk} \geq 0$. If $P \geq c$, then

(i) $S = 2P$ and $M$ is totally umbilical, or

(ii) $P = c$.

The proof is the same as the proof of Theorem 4.2.

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