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*Archivum Mathematicum*, Vol. 48 (2012), No. 3, 183--196

Persistent URL: [http://dml.cz/dmlcz/142988](http://dml.cz/dmlcz/142988)

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ON $\mu$-SINGULAR AND $\mu$-EXTENDING MODULES

YAHYA TALEBI AND ALI REZA MONIRI HAMZEKOLAEE

Abstract. Let $M$ be a module and $\mu$ be a class of modules in Mod$^{-R}$ which is closed under isomorphisms and submodules. As a generalization of essential submodules Özcan in [8] defines a $\mu$-essential submodule provided it has a non-zero intersection with any non-zero submodule in $\mu$. We define and investigate $\mu$-singular modules. We also introduce $\mu$-extending and weakly $\mu$-extending modules and mainly study weakly $\mu$-extending modules. We give some characterizations of $\mu$-co-H-rings by weakly $\mu$-extending modules.

Let $R$ be a right non-$\mu$-singular ring such that all injective modules are non-$\mu$-singular, then $R$ is right $\mu$-co-H-ring if and only if $R$ is a QF-ring.

1. Introduction

Let $R$ be a ring with identity. All modules we consider are unitary right $R$-modules and we denote the category of all such modules by Mod$^{-R}$.

Let $\mu$ be a class of modules. For any module $M$ the trace of $\mu$ in $M$ is denoted by $\text{Tr}(\mu, M) = \sum \{\text{Im} f : f \in \text{Hom}(C, M), C \in \mu\}$. Dually the reject of $\mu$ in $M$ is denoted by $\text{Rej}(M, \mu) = \bigcap \{\text{Ker} f : f \in \text{Hom}(M, C), C \in \mu\}$.

Let $N$ be a submodule of $M$ ($N \leq M$). The notations $N \ll M$, $N \leq_{e} M$ and $N \leq_{d} M$ is used for a small submodule, an essential submodule and a direct summand of $M$, respectively. $\text{Soc}(M)$ will denote the socle of $M$. An $R$-module $M$ is said to be small, if $M \cong L \ll K$ for some $R$-modules $L$ and $K$. Dually, $M$ is called singular if $M \cong N/K$ such that $K \leq_{e} N$. Every module $M$ contains a largest singular submodule which is denoted by $Z(M)$. Then $Z(M) = \text{Tr}(U, M)$ where $U$ denotes the class of all singular modules.

Simple modules split into four disjoint classes by combining the exclusive choices [injective or small] and [projective or singular]. Also note that if a module $M$ is singular and projective, then it is zero.

Talebi and Vanaja in [10], define cosingular modules as a dual of singular modules. Let $M$ be a module and $\mathcal{M}$ denotes the class of all small modules. Then $\overline{Z}(M) = \bigcap \{\ker g : g \in \text{Hom}(M, L), L \in \mathcal{M}\}$ is a submodule of $M$. Then $M$ is called cosingular (non-cosingular) if $\overline{Z}(M) = 0$ ($\overline{Z}(M) = M$). Every small module

2010 Mathematics Subject Classification: primary 16S90; secondary 16D10, 16D70, 16D99.

Key words and phrases: $\mu$-essential submodule, $\mu$-singular module, $\mu$-extending module, weakly $\mu$-extending module.

Received April 6, 2011, revised June 25, 2012. Editor J. Trlifaj.

DOI: 10.5817/AM2012-3-183
is cosingular. The class of all cosingular modules is closed under submodules, direct
sums and direct products ([10, Corollary 2.2]).

In Section 2 we give the definition of \( \mu \)-singular modules and discuss some
properties of such modules. It is proved that \( R \) is a GCO-ring (i.e. every simple
singular module is injective) if and only if for every \( M \)-singular module \( N, Z(N) = N \)
if and only if for every \( \delta \)-singular module \( N, Z(N) = N \) (Corollary 2.14). When
we consider the class of all finitely cogenerated modules \( FC \) we prove
that every finitely cogenerated \( R \)-module is projective if and only if for every
\( FC \)-singular \( R \)-module \( N, \operatorname{Rej}(N, FC) = N \) if and only if \( R \) is semisimple Artinian
(Corollary 2.16).

In Section 3, we define \( \mu \)-extending and weakly \( \mu \)-extending modules and show
that any direct summand of a weakly \( \mu \)-extending module and any homomorphic
image of a weakly \( \mu \)-extending module are weakly \( \mu \)-extending modules (Proposi-
tion 3.12 and Corollary 3.13).

In Section 4, we discuss when a direct sum of weakly \( \mu \)-extending modules is a
weakly \( \mu \)-extending module. We show that a direct sum of a \( \mu \)-singular module
and a semisimple module is weakly \( \mu \)-extending (Theorem 4.2).

In Section 5, we study rings in which every projective module is \( \mu \)-extending. We
call such rings \( \mu \)-co-H-ring. We show that a ring \( R \) is \( \mu \)-co-H-ring if and only if every
\( R \)-module is weakly \( \mu \)-extending (Theorem 5.3). Let \( R \) be a right non-
\( \mu \)-singular ring such that all injective modules are non-\( \mu \)-singular, then \( R \) is right
\( \mu \)-co-H-ring if and only if \( R \) is a QF-ring (Corollary 5.6).

In this paper \( \mu \) will be a class in \( \text{Mod} - R \) which is closed under isomorphisms and
submodules, unless otherwise stated. We shall call any member of \( \mu \), a \( \mu \)-module.

In this article we denote the following classes:
\[ S = \{ M \in \text{Mod} - R, M \text{ is simple} \}, \]
\[ M = \{ M \in \text{Mod} - R, M \text{ is small} \}, \]
\[ \delta = \{ M \in \text{Mod} - R, Z(M) = 0 \}, \]
\[ \mu - \text{Sing} = \{ M \in \text{Mod} - R, M \text{ is } \mu \text{-singular} \}, \]
\[ FC = \{ M \in \text{Mod} - R, M \text{ is finitely cogenerated} \}. \]

2. \( \mu \)-SINGULAR MODULES

\( \text{Özcan} \) in [8], investigate some properties of \( \mu \)-essential submodules. Let \( M \) be
a module and \( N \leq M \). Then \( N \) is called a \( \mu \)-essential submodule of \( M \), denoted
by \( N \leq_{\mu} M \), if \( N \cap K \neq 0 \) for any nonzero submodule \( K \) of \( M \) such that \( K \in \mu \).
Now we list the properties of \( \mu \)-essential submodules. We omit the proofs because
they are similar to those for essential submodules (see, [4]).

**Lemma 2.1.** Let \( M \) be a module.

a) Let \( N \leq L \leq M \). Then \( N \leq_{\mu} M \) if and only if \( N \leq_{\mu} L \leq_{\mu} M \).
b) If \( K_1 \leq_{\mu} L_1 \leq M, K_2 \leq_{\mu} L_2 \leq M \), then \( K_1 \cap K_2 \leq_{\mu} L_1 \cap L_2 \).
c) If \( f : N \to M \) is a homomorphism and \( K \leq_{\mu} M \), then \( f^{-1}(K) \leq_{\mu} N \).
d) If \( N/L \leq_{\mu} K/L \leq M/L \), then \( N \leq_{\mu} K \).
e) Let \( K_i \ (i \in I) \) be an independent family of submodules of \( M \). If \( K_i \leq_{\mu} L_i \leq M \)
for all \( i \in I \), then \( \bigoplus_{i \in I} K_i \leq_{\mu} \bigoplus_{i \in I} L_i \).
Definition 2.2. Let $M$ be a module. $M$ is called $\mu$-singular if $M \cong K/L$ such that $L \leq_{\mu} K$.

Every module $M$ contains a largest $\mu$-singular submodule which we denote by $Z_{\mu}(M) = \text{Tr}(\mu - \text{Sing}, M)$ where $\mu - \text{Sing}$ is the class of all $\mu$-singular modules. Then $Z(M) \leq Z_{\mu}(M)$. If $M$ is a $\mu$-singular module (i.e. $Z_{\mu}(M) = M$) and a $\mu$-module, then $M$ is singular. For, let $M \in \mu$ and $M \cong K/L$ where $L \leq_{\mu} K$. We claim that $L \leq_{e} K$. Let $0 \neq X \leq K$ and assume that $L \cap X = 0$. Then $X \cong (L \oplus X)/L \leq K/L$ and so $X \in \mu$. Since $L \leq_{\mu} K$ we have a contradiction. This proves that $M$ is singular. If $Z_{\mu}(M) = 0$, then $M$ is called non-$\mu$-singular.

Proposition 2.3. Let $M$ be a $\mu$-singular module and $f \in \text{Hom}_R(R, M)$. Then $\ker f \leq_{\mu} R$.

Proof. By assumption, $f(R) \cong L/K$ where $K \leq_{\mu} L$. Since $R$ is projective, there exists a homomorphism $g: R \to L$ such that $\pi g = f$ where $\pi$ is the natural epimorphism $L \to L/K$. Then $\ker f = g^{-1}(K) \leq_{\mu} R$ by Lemma 2.1

Proposition 2.4. Let $P$ be a projective module and $X \leq P$. Then $P/X$ is $\mu$-singular if and only if $X \leq_{\mu} P$.

Proof. If $I \leq R$ and $R/I$ is $\mu$-singular, then $I \leq_{\mu} R$ by Proposition 2.3. Let $P/X$ be $\mu$-singular and assume $X \nleq_{\mu} P$. Let $F$ be a free module such that $F = P \oplus P'$, $P' \leq P$. Then $F/(X \oplus P') \cong P/X$ is $\mu$-singular and $X \oplus P' \nleq_{\mu} F$. So we may assume that $P$ is free i.e. $P = \bigoplus R_i$, each $R_i$ is a copy of $R$. Then $R_i/(R_i \cap X) \cong (R_i + X)/X \leq P/X$ is $\mu$-singular. So $R_i \cap X \leq_{\mu} R_i$. This implies that $(\bigoplus R_i) \cap X \leq_{\mu} \bigoplus R_i = P$, i.e. $X \leq_{\mu} P$.

Lemma 2.5. Let $M$ be a module. Then $Z_{\mu}(M) = \{x \in M \mid xI = 0, I \leq_{\mu} R\}$.

Proof. Let $xI = 0$ for some $I \leq_{\mu} R$. Then $R/I$ is $\mu$-singular. Define $f: R/I \to xR$ by $r + I \mapsto xr$. Hence, $x \in \text{Tr}(\mu - \text{Sing}, M)$. Conversely assume that $x = x_1 + \cdots + x_n = f_1(l_1) + \cdots + f_n(l_n)$ and $x_i \in \text{Im} f_i$ where $f_i: L_i \to M$ such that $L_i$ is $\mu$-singular. For each $i$ we have $l_i R \cong R/\text{ann}(l_i)$ which implies that $I_i = \text{ann}(l_i) \leq_{\mu} R$ by Proposition 2.3. Take $I = \bigcap_{i=1}^n I_i$. Then $I \leq_{\mu} R$ by Lemma 2.1 and $xI = 0$. This completes the proof.

Proposition 2.6. A module $M$ is non-$\mu$-singular if and only if $\text{Hom}_R(N, M) = 0$ for all $\mu$-singular modules $N$.

Proof. See [4, Proposition 1.20].

Proposition 2.7. Let $M$ be a non-$\mu$-singular module and $N \leq M$. Then $M/N$ is $\mu$-singular if and only if $N \leq_{\mu} M$.

Proof. If $M/N$ is $\mu$-singular and $x$ is a nonzero element of $M$. Then $\exists I = 0$ for some $I \leq_{\mu} R$. So, $xI \leq N$. Since $M$ is non-$\mu$-singular, we have $xI \neq 0$ and thus $xR \cap N \neq 0$. Therefore, $N \leq_{\mu} M$.

Proposition 2.8. (1) The class of all non-$\mu$-singular modules is closed under submodules, direct products, $\mu$-essential extension and module extension.

(2) The class of all $\mu$-singular modules is closed under submodules, factor modules and direct sums.
Proof. It follows from Lemma 2.5 and [4, Proposition 1.22]. □

Proposition 2.9. Assume that $R$ is a right non-$\mu$-singular ring, then:

1. $Z_\mu(M/Z_\mu(M)) = 0$ for any $R$-module $M$.
2. An $R$-module $M$ is $\mu$-singular if and only if $\text{Hom}_R(M, N) = 0$ for all non-$\mu$-singular modules $N$.
3. The class of all $\mu$-singular modules is closed under module extension and $\mu$-essential extension.
4. The set of all $\mu$-essential right ideals of $R$ denoted by $\mathcal{P}(R)$ is closed under finite products.

Proof. The proof is easy by [4, Proposition 1.23] and Lemma 2.5. □

Proposition 2.10. Let $M$ be a simple module. Then $M$ is either $\mu$-singular or projective, but not both.

Proof. See [4, Proposition 1.24]. □

It is easy to see that a ring $R$ is right non-$\mu$-singular if and only if all projective right $R$-modules are non-$\mu$-singular.

From the properties of $\mu$-singular modules and Proposition 2.4 the following can be seen easily.

Proposition 2.11. For an $R$-module $M$ the following are equivalent:

1. $M$ is $\mu$-singular:
2. $M \cong F/K$ with $F$ a projective (free) module and $K \leq_{\mu e} F$;
3. For every $m \in M$, the right annihilator $\text{ann}_R(m)$ is $\mu$-essential in $R$.

Lemma 2.12. Let $M$ be a module. If $Z_\mu(M) = 0$ and $K \leq_{c} M$, then $Z_\mu(M/K) = 0$.

Theorem 2.13. Let $\mu$ be closed under factor modules. Then the following are equivalent:

1. Every $\mu$-module is projective;
2. For every singular module $N$, $\text{Rej}(N, \mu) = N$;
3. For every $\mu$-singular module $N$, $\text{Rej}(N, \mu) = N$;
4. For every simple singular module $N$, $\text{Rej}(N, \mu) = N$.

Proof. (1) $\Rightarrow$ (2) Let $N$ be a singular module and $g: N \to L$ where $L \in \mu$. Then $N/\ker g \in \mu$. By (1), $N/\ker g$ is projective. Since $N$ is singular, we have that $N = \ker g$. Hence $\text{Rej}(N, \mu) = N$.

(2) $\Rightarrow$ (3) Let $N$ be a $\mu$-singular module and $g: N \to L$ a homomorphism where $L \in \mu$. Then $N/\ker g \in \mu$. This implies that $\text{Rej}(N/\ker g, \mu) = 0$. Since $N/\ker g$ is $\mu$-singular and a $\mu$-module, it is singular. Then by (2), $N = \ker g$. Hence $\text{Rej}(N, \mu) = N$.

(3) $\Rightarrow$ (2) and (2) $\Rightarrow$ (4) are clear.

(4) $\Rightarrow$ (1) Let $N$ be a $\mu$-module. We claim that $N$ is semisimple. Let $x \in N$ and $K$ a maximal submodule of $xR$. Then $xR/K$ is a simple $\mu$-module. By (4), it cannot be singular. Hence $xR/K$ is projective. This implies that $K$ is a direct summand of $xR$. Hence $N$ is semisimple. By above process every simple submodule of $N$ is projective. It follows that $N$ is projective. □
If we consider the class $\mathcal{M}$ of all small modules, we have a characterization of GCO-rings. A ring $R$ is called a GCO-ring if every simple singular module is injective.

**Corollary 2.14.** The following are equivalent for a ring $R$:

1. Every small module is projective;
2. Every singular module is non-cosingular;
3. Every $\mathcal{M}$-singular module is non-cosingular;
4. $R$ is a GCO-ring;
5. Every $\delta$-singular module is non-cosingular.

**Proof.** (1) $\Leftrightarrow$ (4) is by [7, Theorem 1.5]. (2) $\Leftrightarrow$ (4) is by [9, Theorem 4.1].

Simple modules are either injective or small. Hence (1)–(4) are equivalent by Theorem 2.13.

(5) $\Rightarrow$ (2) is clear.

(3) $\Rightarrow$ (5) It is clear since $\mathcal{M} \subseteq \delta$, every $\delta$-singular module is $\mathcal{M}$-singular. □

For the class $\delta$ of all cosingular modules we have the following corollary.

**Corollary 2.15.** If the class $\delta$ is closed under the factor modules the following are equivalent:

1. Every cosingular module is projective;
2. For every singular module $N$, $\text{Rej}(N, \delta) = N$;
3. For every $\delta$-singular module $N$, $\text{Rej}(N, \delta) = N$;
4. $R$ is a GCO-ring.

**Proof.** See [9, Theorems 4.1 and 4.2] and Theorem 2.13. □

A module $M$ is called *finitely cogenerated* if $\text{Soc}(M)$ is finitely generated and essential submodule of $M$. Let $\mathcal{FC}$ be the class of all finitely cogenerated $R$-modules. Note that $\mathcal{FC}$ is closed under submodules. We next give a characterization of semisimple Artinian rings which is taken from [8]. We give the proof for completeness.

**Corollary 2.16.** The following statements are equivalent for a ring $R$:

1. Every finitely cogenerated $R$-module is projective;
2. For every singular module $N$, $\text{Rej}(N, \mathcal{FC}) = N$;
3. For every $\mathcal{FC}$-singular module $N$, $\text{Rej}(N, \mathcal{FC}) = N$;
4. $R$ is semisimple Artinian.

**Proof.** (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3) By Theorem 2.13. □

(4) $\Rightarrow$ (1) is clear.

(2) $\Rightarrow$ (4) Let $E$ be an essential right ideal of $R$. Suppose that $a$ is an element of $R$ but $a \notin E$. Let $F$ be a right ideal of $R$ maximal with respect to the properties that $E$ is contained in $F$ and $a \notin F$. Then $(aR + F)/F$ is simple singular. By (2), we have a contradiction. Hence $R$ is semisimple Artinian. □

A ring $R$ is a *quasi-Frobenius* ring (briefly QF-ring) if and only if every right $R$-module is a direct sum of an injective module and a singular module. In this result we may take $\mu$-singular modules instead of singular as the following result shows.
Theorem 2.17. The following are equivalent for a ring $R$:

1. $R$ is a QF-ring;
2. Every right $R$-module is a direct sum of an injective module and a $\mu$-singular module.

Proof. (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (1) Let $M$ be a projective $R$-module. Then $M$ is a direct sum of an injective module and a $\mu$-singular module. Since projective $\mu$-singular modules are zero, $M$ is injective. Then $R$ is a QF-ring. □

3. $\mu$-EXTENDING MODULES

In this section $\mu$-extending modules will be introduced. Then we define and study weakly $\mu$-extending modules. It is proved that any factor module, any direct summand and any fully invariant submodule of a weakly $\mu$-extending module are weakly $\mu$-extending.

Definition 3.1. Let $M$ be a module. Then $M$ is called an $\mu$-extending module if for every submodule $N$ of $M$ there exists a direct summand $D$ of $M$ such that $N \leq_{\mu e} D$.

Clearly every essential submodule is $\mu$-essential. So $\mu$-extending modules are a generalization of extending modules.

Note that by [5, Proposition 2.4], a module $M$ is extending if and only if every closed submodule is direct summand. This may not be true for a $\mu$-extending module.

Let $M$ be a module and $K$ a submodule of $M$. Then $K$ is called a $\mu$-closed submodule, denoted by $K \leq_{\mu c} M$, provided $K \leq_{\mu e} L \leq M$ implies $K = L$, i.e. $K$ doesn’t have any proper $\mu$-essential extension. A $\mu$-closed submodule is closed but the converse is true when $M$ is a $\mu$-module (see [8, Corollary 1.1]).

Proposition 3.2. The following statements hold for a module $M$.

1. If $K \leq_{\mu c} M$, then whenever $Q \leq_{\mu e} M$ such that $K \subseteq Q$, then $Q/K \leq_{\mu e} M/K$.
2. If $L \leq_{\mu e} M$, then $L/K \leq_{\mu e} M/K$.

Proof. (1) Suppose $K \leq_{\mu c} M$. Let $Q \leq_{\mu e} M$ such that $K \subseteq Q$. Let $P/K \leq M/K$ be a $\mu$-module such that $(Q/K) \cap (P/K) = 0$. By Lemma 2.1(b), $K = Q \cap P \leq_{\mu e} P$ and hence $K = P$. Thus $Q/K \leq_{\mu e} M/K$.

(2) It is clear by Lemma 2.1(d). □

The following proposition is clear by definitions.

Proposition 3.3. Let $M$ be a $\mu$-extending module. Then every $\mu$-closed submodule is a direct summand.

We next give an equivalent condition for a $\mu$-extending module.

Proposition 3.4. Let $M$ be a module. Then $M$ is $\mu$-extending if and only if for each submodule $A$ of $M$ there exists a decomposition $M = M_1 \oplus M_2$ such that $A \leq M_1$ and $A + M_2 \leq_{\mu e} M_2$. 
Proof. Let $M$ be $\mu$-extending and $A \leq M$. Then there exists a decomposition $M = M_1 \oplus M_2$ such that $A \leq \mu e M_1$. Since $\{A, M_2\}$ is an independent family of submodules of $M$ the result follows from Lemma 2.1.

The converse is clear. □

A module $M$ is called $\mu$-uniform if every proper nonzero submodule is $\mu$-essential in $M$.

**Proposition 3.5.** An indecomposable module $M$ is $\mu$-extending if and only if $M$ is $\mu$-uniform.

**Definition 3.6.** Let $M$ be a module. Then $M$ is called weakly $\mu$-extending if for every submodule $N$ of $M$ there exists a direct summand $K$ of $M$ such that $N \leq K$ and $K/N$ is $\mu$-singular.

The definition shows that every $\mu$-extending module is weakly $\mu$-extending. Also any $\mu$-singular module is weakly $\mu$-extending.

Let $M$ be a $\mu$-singular module with unique composition series $M \supset U \supset V \supset 0$. By [2], $N = M \oplus (U/V)$ is not extending. But $N$ is weakly $\mu$-extending.

We next give some equivalent conditions for weakly $\mu$-extending modules.

**Proposition 3.7.** The following are equivalent for a module $M$:

1. $M$ is weakly $\mu$-extending;
2. For every $N \leq M$ there exists a decomposition $M = K \oplus K'$ such that $N \leq K$ and $M/(K' + N)$ is $\mu$-singular;
3. For every $N \leq M$ there exists a decomposition $M/N = K/N \oplus K'/N$ such that $K \leq_a M$ and $M/K'$ is $\mu$-singular;
4. For every $N \leq M$, there exists a direct summand $K$ of $M$ such that $N \leq K$ and for any $x \in K$ there is a right ideal $I$ with $I \leq_\mu R$ such that $xI \leq N$.

**Proposition 3.8.** Let $M$ be a non-$\mu$-singular or projective module. Then, $M$ is $\mu$-extending if and only if $M$ is weakly $\mu$-extending.

Proof. It is easy by Propositions 2.7 and 2.4. □

Some special submodules of a weakly $\mu$-extending module are weakly $\mu$-extending. Recall that a submodule $N$ of $M$ is called fully invariant if $f(N) \subseteq N$ for each $f \in \text{End}(M)$. A module $M$ is called a duo module, if every submodule of $M$ is fully invariant.

**Proposition 3.9.** Let $N \leq M$ be fully invariant and $M$ a weakly $\mu$-extending module. Then $N$ is weakly $\mu$-extending.

Proof. Let $L \leq N \leq M$. By assumption, there exists a decomposition $M = K \oplus K'$ such that $L \leq K$ and $K/L$ is $\mu$-singular. Since $N$ is fully invariant, we have $N = (N \cap K) \oplus (N \cap K')$. Obviously, $L \leq N \cap K$ and $(N \cap K)/L \leq K/L$ is $\mu$-singular. Hence $N$ is weakly $\mu$-extending. □
Proposition 3.10. Let $M$ be a module and $N$ a submodule of $M$.

(1) If $M$ is weakly $\mu$-extending and the intersection of $N$ with any direct summand of $M$ is a direct summand of $N$, then $N$ is weakly $\mu$-extending.

(2) If $N$ is weakly $\mu$-extending and $D$ a direct summand of $M$ such that $(D + N)/D$ is non-$\mu$-singular, then $D \cap N$ is a direct summand of $N$.

(3) If $M$ is weakly $\mu$-extending and $(D + N)/D$ is non-$\mu$-singular for any direct summand $D$ of $M$, then $N$ is weakly $\mu$-extending if and only if $D \cap N$ is a direct summand of $N$ for any direct summand $D$ of $M$.

Proof. (1) It is similar to the proof of Proposition 3.9.

(2) Let $Y = D \cap N$. Since $N$ is weakly $\mu$-extending, there is a direct summand $K$ of $N$ such that $K/Y$ is $\mu$-singular. By assumption, $N/Y \cong (D + N)/D$ is non-$\mu$-singular. Hence, $K/Y \leq N/Y$ is both $\mu$-singular and non-$\mu$-singular. It follows that $K = Y$ is a direct summand of $N$.

(3) It is a consequence of (1) and (2). □

The following proposition shows the equivalent condition of a cyclic submodule of a module to be weakly $\mu$-extending over a right weakly $\mu$-extending ring.

Proposition 3.11. Let $R$ be a right weakly $\mu$-extending ring and $M$ a cyclic right $R$-module such that every nonzero direct summand of $M$ contains a nonzero $\mu$-module. Then the following are equivalent:

(1) $M$ is non-$\mu$-singular;

(2) Every cyclic submodule of $M$ is projective and weakly $\mu$-extending;

(3) Every cyclic submodule of $M$ is projective.

Proof. (1) $\Rightarrow$ (2) Suppose that $M$ is non-$\mu$-singular and $N$ a cyclic submodule of $M$. Then there is a right ideal $I$ of $R$ such that $N \cong R/I$. Since $R$ is $\mu$-extending and $N$ is non-$\mu$-singular, $I$ is a $\mu$-closed submodule of $R_R$, hence $I$ is a direct summand of $R_R$. Thus $N$ is isomorphic to a direct summand of $R_R$. Therefore, $N$ is projective and weakly $\mu$-extending.

(2) $\Rightarrow$ (3) It is clear.

(3) $\Rightarrow$ (1) For any $m \in Z_\mu(M)$, $mR$ is projective and is isomorphic to $R/\text{ann}_r(m)$, where $\text{ann}_r(m)$ is the right annihilator of $m$. Since $R$ is right weakly $\mu$-extending and $mR$ is $\mu$-singular, then $\text{ann}_r(m) \leq_\mu R$ is a direct summand of $R$. Then, $R = \text{ann}_r(m) \oplus L$. By assumption, if $L \neq 0$ then it contains a nonzero $\mu$-module. Hence, $\text{ann}_r(m) = R$ and $m = 0$. Hence, $Z_\mu(M) = 0$. □

Any factor module of a $\mu$-singular module is $\mu$-singular and we show that any image of a weakly $\mu$-extending module is weakly $\mu$-extending. The direct summand of a $\mu$-extending module may not be $\mu$-extending. For weakly $\mu$-extending modules, we first show the following proposition and then show that any direct summand of a weakly $\mu$-extending module is weakly $\mu$-extending.

Proposition 3.12. Let $M$ be a weakly $\mu$-extending module. Then any homomorphic image of $M$ is weakly $\mu$-extending.

Proof. Let $f : M \to N$ be an epimorphism and $L$ a submodule of $N$. Then there is a submodule $H$ of $M$ such that $L \cong H/\text{Ker} f$. Since $M$ is weakly $\mu$-extending,
there are direct summands $K, K'$ of $M$ such that $M = K \oplus K'$, $H \leq K$ and that $K/H$ is $\mu$-singular. So $N \cong M/\text{Ker } f = (K/\text{Ker } f) \oplus (K' + \text{Ker } f)/\text{Ker } f$ and $L \cong H/\text{Ker } f \leq K/\text{Ker } f$. Since $(K/\text{Ker } f)/(H/\text{Ker } f) \cong K/H$ is $\mu$-singular, $N$ is weakly $\mu$-extending. \hfill \Box

**Corollary 3.13.** (1) Let $M$ be a weakly $\mu$-extending module. Then any direct summand of $M$ is weakly $\mu$-extending.

(2) Let $M$ be a $\mu$-extending module. Then any non-$\mu$-singular homomorphic image of $M$ is $\mu$-extending.

**Corollary 3.14.** The following are equivalent:

(1) Every (resp., finitely generated) module is weakly $\mu$-extending;

(2) Every (resp., finitely generated) projective module is weakly $\mu$-extending.

**Proposition 3.15.** Let $R$ be a right non-$\mu$-singular ring and $f : M \to M'$ an epimorphism. Suppose that $M'$ is weakly $\mu$-extending and $\text{Ker } f$ is $\mu$-singular, then $M$ is weakly $\mu$-extending.

**Proof.** Let $N$ be a submodule of $M$. First, we assume that $\text{Ker } f \subseteq N \leq M$, then $f(N) \leq N'$. Since $M'$ is weakly $\mu$-extending, there is a decomposition, $M' = K \oplus H$ such that $K/f(N)$ is $\mu$-singular. So $M = f^{-1}(K) + f^{-1}(H)$. Since $\text{Ker } f \leq f^{-1}(H)$ and $\text{Ker } f$ is injective, then $f^{-1}(H) = T \oplus \text{Ker } f$ for some submodule $T$ of $f^{-1}(H)$. Thus $M = f^{-1}(K) + T$. Since $f^{-1}(K) \cap T \leq f^{-1}(K) \cap f^{-1}(H) = \text{Ker } f$ and $f^{-1}(K) \cap T \leq \text{Ker } f \cap T = 0$, we have $M = f^{-1}(K) \oplus T$ and $N \leq f^{-1}(K)$.

For any $x \in f^{-1}(K)$, $f(x) \in K$ and there is an $\mu$-essential right ideal $I$ of $R$ such that $f(x)I \leq f(N)$. It is easy to see that $xI \leq N$ and that $f^{-1}(K)/N$ is $\mu$-singular.

Now we assume that $N$ does not contain $\text{Ker } f$. Set $L = N + \text{Ker } f$, then $f(L) = f(N)$. As the case above, there is a decomposition $M = f^{-1}(K) \oplus T$ such that $f^{-1}(K)/L$ is $\mu$-singular. Since $\text{Ker } f$ is $\mu$-singular, we have that $(N + \text{Ker } f)/N \cong \text{Ker } f/(N \cap \text{Ker } f)$ is $\mu$-singular. Since $R$ is right non-$\mu$-singular, by Proposition 2.9, we have that $f^{-1}(K)/N$ is $\mu$-singular. In either case, $M$ is weakly $\mu$-extending. \hfill \Box

**Proposition 3.16.** Let $R$ be a right non-$\mu$-singular ring and $M$ a weakly $\mu$-extending module. Then $M = Z_\mu(M) \oplus T$ for some $\mu$-extending submodule $T$ of $M$ and $T$ is $Z_\mu(M)$-injective.

**Proof.** If $Z_\mu(M) = 0$ or $Z_\mu(M) = M$, it is clear.

Suppose that $0 < Z_\mu(M) < M$. Since $M$ is weakly $\mu$-extending, there are direct summands $K, T$ of $M$ such that $M = K \oplus T$, $Z_\mu(M) \leq K$ and that $K/Z_\mu(M)$ is $\mu$-singular. So $K$ is $\mu$-singular. Since $Z_\mu(M) = Z_\mu(K) \oplus Z_\mu(T) = K \oplus Z_\mu(T)$, so $Z_\mu(M) = K$ and $T$ is non-$\mu$-singular. By Proposition 3.12, $T$ is $\mu$-extending.

Since for any submodule $N$ of $Z_\mu(M)$, $\text{Hom}_R(N, T) = 0$, so $T$ is $Z_\mu(M)$-injective, as required. \hfill \Box

**Corollary 3.17.** Let $R$ be a right non-$\mu$-singular ring and $M$ an injective module. Then $Z_\mu(M)$ is injective.
Corollary 3.18. Let $R$ be a right non-$\mu$-singular ring and $M$ an indecomposable weakly $\mu$-extending module. Then $M$ is either a $\mu$-singular module or a non-$\mu$-singular $\mu$-uniform module.

Proposition 3.19. Let $M$ be a weakly $\mu$-extending module which contains maximal submodules. Then for any maximal submodule $N$ of $M$, either $M/N$ is $\mu$-singular or $M = N \oplus S$ for some simple submodule $S$ of $M$.

Proof. Let $N$ be a maximal submodule of $M$ and suppose that $M/N$ is not $\mu$-singular. Then $N$ is a direct summand of $M$, i.e., $M = N \oplus S$ for some submodule $S$ of $M$. Since $S \cong M/N$, so $S$ is simple. □

A module $M$ is called local if it has a largest submodule, i.e., a proper submodule which contains all other proper submodules. For a local module $M$, $\text{Rad}(M)$, the Jacobson radical of $M$ is small in $M$.

Corollary 3.20. Let $M$ be a local weakly $\mu$-extending module. Then $M/\text{Rad}(M)$ is $\mu$-singular.

Proposition 3.21. Let $R$ be a right hereditary ring and $M$ an injective module. Then any factor module of $M$ is a direct sum of an injective module and a $\mu$-singular injective module.

Proof. Let $L$ be any factor module of $M$, then there is a submodule $N$ of $M$ such that $L \cong M/N$. Since any injective module is weakly $\mu$-extending, there are direct summands $K, K'$ of $M$ such that $M = K \oplus K'$, $N \subseteq K$ and that $K/N$ is $\mu$-singular. So $L \cong M/N = K/N \oplus (K' + N)/N$. Since $R$ is hereditary and $M$ is injective, so $M/N$ is injective. Thus $K/N$ is a $\mu$-singular injective module and $(K' + N)/N$ is injective. □

4. Direct sum of weakly $\mu$-extending modules

A direct sum of $\mu$-singular modules is also $\mu$-singular. But a direct sum of $\mu$-extending modules may not be $\mu$-extending. Also a direct sum of weakly $\mu$-extending modules need not be weakly $\mu$-extending (see Example 2.4).

It may be interesting to see when a direct sum of weakly $\mu$-extending modules is weakly $\mu$-extending.

Proposition 4.1. Let $M = \bigoplus_{i \in I} M_i$ be a distributive module. Then $M$ is weakly $\mu$-extending if and only if each $M_i$ is weakly $\mu$-extending for $i \in I$.

Proof. Let $N$ be any submodule of $M$, then $N = \bigoplus_{i \in I} (N \cap M_i)$. Since $M_i$ is weakly $\mu$-extending, there is direct summand $H_i \leq_d M_i$, such that $M_i = H_i \oplus H_i'$ and $(N \cap M_i) \leq H_i$ and that $H_i/(N \cap M_i)$ is $\mu$-singular for $i \in I$. Hence $M = (\bigoplus_{i \in I} H_i) \oplus (\bigoplus_{i \in I} H_i')$ and $(N = \bigoplus_{i \in I} (N \cap M_i)) \leq (H = \bigoplus_{i \in I} H_i)$. Since 

$H = \bigoplus_{i \in I} \frac{H_i}{i \in I, \#(N \cap M_i)} \cong \bigoplus_{i \in I} \frac{H_i}{N \cap M_i}$ is $\mu$-singular, so $M$ is weakly $\mu$-extending. □

Theorem 4.2. Let $M = M_1 \oplus M_2$ with $M_1$ being $\mu$-singular ($\mu$-uniform) and $M_2$ semisimple. Then $M$ is weakly $\mu$-extending.
Proposition 4.5. Let $N$ be any submodule of $M$. Then $N + M_1 = M_1 \oplus [(N + M_1) \cap M_2]$. Since $M_2$ is semisimple, then $(N + M_1) \cap M_2$ is a direct summand of $M_2$ and therefore $N + M_1$ is a direct summand of $M$. Note that $(N + M_1)/N \cong M_1/(N \cap M_1)$ is $\mu$-singular, since $M_1$ is $\mu$-singular ($\mu$-uniform). So $M$ is weakly $\mu$-extending. □

Proposition 4.3. Let $M = M_1 \oplus M_2$ with $M_1$ being weakly $\mu$-extending and $M_2$ semisimple. Suppose that for any submodule $N$ of $M$, $N \cap M_1$ is a direct summand of $N$. Then $M$ is weakly $\mu$-extending.

Proof. Let $N$ be any submodule of $M$. As in Theorem 4.2, $N + M_1$ is a direct summand of $M$. By the hypothesis, $N = (N \cap M_1) \oplus K$ for some submodule $K$ of $N$. Since $M_1$ is weakly $\mu$-extending, there is a direct summand $T$ of $M_1$ such that $T/(N \cap M_1)$ is $\mu$-singular. But $N + M_1 = (N \cap M_1) + K + M_1 = M_1 \oplus K$, so $(T \oplus K)/N = (T \oplus K)/[(N \cap M_1) \oplus K] \cong T/(N \cap M_1) \oplus K/K$ is $\mu$-singular. Since $T \oplus K$ is a direct summand of $N + M_1$ and hence a direct summand of $M$, then $M$ is weakly $\mu$-extending.

Proposition 4.4. Let $M = M_1 \oplus M_2$ with $M_1$ being weakly $\mu$-extending and $M_2$ injective. Suppose that for any submodule $N$ of $M$, we have $N \cap M_2$ is a direct summand of $N$, then $M$ is weakly $\mu$-extending.

Proof. Let $N \leq M$. By the hypothesis, there is a submodule $N'$ of $N$ such that $N = (N \cap M_2) \oplus N'$. Note that $N' \cap M_2 = 0$ and hence $(M_2 + N')/N' \cong M_2$ is an injective module, so there is a submodule $M'$ of $M$ containing $N'$ such that $M/N' = [(M_2 + N')/N'] \oplus (M'/N')$. Thus it is easy to see that $M = M_2 \oplus M'$ and that $M' \cong M/M_2 \cong M_1$. Hence $M'$ is weakly $\mu$-extending. There are direct summands $K, K'$ of $M'$ such that $M' = K \oplus K'$ and that $K/N'$ is $\mu$-singular. Since $N \cap M_2$ is a submodule of an injective module $M_2$, so there is a direct summand $H$ of $M_2$ such that $H/(N \cap M_2)$ is $\mu$-singular. Following from the fact that $(H \oplus K)/[(N \cap M_2) \oplus N'] \cong [H/(N \cap M_2)] \oplus (K/N')$ and that $H \oplus K \leq M$, then $M$ is weakly $\mu$-extending.

Proposition 4.5. Let $M = M_1 \oplus M_2$ such that $M_1$ is weakly $\mu$-extending and $M_2$ is an injective module. Then $M$ is weakly $\mu$-extending if and only if for every submodule $N$ of $M$ such that $N \cap M_2 \neq 0$, there is a direct summand $K$ of $M$ such that $K/N$ is $\mu$-singular.

Proof. Suppose that for every submodule $N$ of $M$ such that $N \cap M_2 \neq 0$, there is a direct summand $K$ of $M$ such that $K/N$ is $\mu$-singular. Let $N$ be a submodule of $M$ such that $N \cap M_2 = 0$. Then, since $(M_2 + N)/N \cong M_2$ is an injective module, there is a submodule $M'$ of $M$ containing $N$ such that $M/N = (M'/N) \oplus ((M_2 + N)/N)$. It is easy to see that $M = M' \oplus M_2$. Since $M'/N \cong M/M_2 \cong M_1$ is weakly $\mu$-extending, there is a direct summand $K$ of $M'$, hence of $M$, such that $K/N$ is $\mu$-singular. So $M$ is weakly $\mu$-extending. The converse is obvious.

5. Rings whose projective modules are $\mu$-extending

In [6], a ring $R$ is called a right co-$H$-ring if every projective right $R$-module is extending. It is known that a ring $R$ is a right co-$H$-ring if and only if $R$ is right
-extending (i.e., any direct sum of $R_R$ is extending). In this section we introduce rings in which all projective right modules are $\mu$-extending. We call such rings $\mu$-co-H-rings. It is easy to check that a ring $R$ is $\mu$-co-H-ring if and only if any direct sum of $R_R$ is $\mu$-extending.

**Lemma 5.1.** Let $R$ be a ring. A projective $R$-module $M$ is weakly $\mu$-extending if and only if every factor module of $M$ is a direct sum of a $\mu$-singular module and a projective module.

**Proof.** Suppose that $M$ is weakly $\mu$-extending. Let $M'$ be any factor module of $M$, then there is a submodule $N$ of $M$ such that $M/N \cong M'$. Since $M$ is weakly $\mu$-extending, then there are direct summands $K, K'$ of $M$ such that $M = K \oplus K'$ and $K/N$ is $\mu$-singular. Thus $M/N = (K/N) \oplus ((K' + N)/N)$. As $M$ is projective, $K' \cong (K' + N)/N$ is projective. Conversely, let $N$ be any submodule of $M$, then $M/N$ is a direct sum of a $\mu$-singular module and a projective module. We may assume that $M/N = S/N \oplus T/N$, where $S/N$ is $\mu$-singular and $T/N$ is projective. Then $M = S + T$ and as $M/S \cong T/N$ is projective, $S$ is a direct summand of $M$. Thus $M$ is weakly $\mu$-extending. □

**Lemma 5.2.** Let $R$ be any right non-$\mu$-singular ring. Then the following are equivalent:

1. All modules are weakly $\mu$-extending;
2. All projective modules are weakly $\mu$-extending;
3. All non-$\mu$-singular modules are $\mu$-extending.

**Proof.** (1) $\iff$ (2) By Corollary 3.14

(1) $\iff$ (3) This is a consequence of Propositions 3.8 and 3.12 and the fact that over a right non-$\mu$-singular ring all projective modules are non-$\mu$-singular. □

As an immediate consequence of Lemmas 5.1, 5.2 and Proposition 3.8 we have:

**Theorem 5.3.** Let $R$ be any ring, then the following are equivalent:

1. $R$ is a right $\mu$-co-H-ring;
2. All right $R$-modules are weakly $\mu$-extending;
3. All projective right $R$-modules are $\mu$-extending;
4. All projective right $R$-modules are weakly $\mu$-extending;
5. Every factor module of any projective module is a direct sum of a $\mu$-singular module and a projective module.

**Theorem 5.4.** Let $R$ be a right non-$\mu$-singular ring, consider the following:

1. $R$ is a right $\mu$-co-H-ring;
2. Every non-$\mu$-singular module is projective;
3. Every module is weakly $\mu$-extending;
4. Every non-$\mu$-singular module is $\mu$-extending.

Then (1) $\iff$ (3) $\iff$ (4) and (1) $\Rightarrow$ (2).

**Proof.** (1) $\Rightarrow$ (2) Suppose that $R$ is a right $\mu$-co-H-ring and $M$ a non-$\mu$-singular module. Then there is a projective module $P$ and an epimorphism $f: P \to M$. Set $K = Ker f$, then $K$ is a $\mu$-closed submodule of $P$. Since $P$ is $\mu$-extending, then $K$
is a direct summand of \( P \) and hence \( M \) is isomorphic to a direct summand of \( P \). Thus \( M \) is projective.

(1) \( \iff \) (3) By Theorem 5.3

(1) \( \implies \) (4) It is clear by Propositions 3.8 and 3.12

(4) \( \implies \) (1) Since a ring \( R \) is right non-\( \mu \)-singular if and only if all projective modules are non-\( \mu \)-singular, by (4), all projective modules are \( \mu \)-extending and \( R \) is a right \( \mu \)-co-H-ring.

\[ \square \]

**Corollary 5.5.** Let \( R \) be a ring such that all \( \mu \)-singular modules are projective, then \( R \) is a right \( \mu \)-co-H-ring if and only if \( R \) is semisimple.

**Proof.** Suppose that \( R \) is a right \( \mu \)-co-H-ring. Let \( M \) be an \( R \)-module module and \( N \) a submodule of \( M \), then by Theorem 5.3 \( M \) is weakly \( \mu \)-extending, i.e., there is a direct summand \( K \) of \( M \) such that \( N \leq K \) and \( K/N \) is \( \mu \)-singular. By hypothesis, \( K/N \) is projective, so \( N \) is a direct summand of \( K \) and hence a direct summand of \( M \). Thus \( M \) is semisimple and \( R \) is semisimple.

The converse is obvious. \[ \square \]

It is known from [3, Theorem 24.20] that a ring \( R \) is a QF-ring if and only if all projective modules are injective if and only if all injective modules are projective. Obviously every QF-ring \( R \) is a left and right \( \mu \)-co-H-ring. As an immediate consequence of Theorem 5.4 we have:

**Corollary 5.6.** Let \( R \) be a right non-\( \mu \)-singular ring such that all injective modules are non-\( \mu \)-singular. Then \( R \) is a right \( \mu \)-co-H-ring if and only if \( R \) is a QF-ring.

**References**


