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SOME PROPERTIES OF TANGENT DIRAC STRUCTURES
OF HIGHER ORDER

P. M. Kouotchop Wamba, A. Ntyam, and J. Wouafo Kamga

Abstract. Let $M$ be a smooth manifold. The tangent lift of Dirac structure on $M$ was originally studied by T. Courant in [3]. The tangent lift of higher order of Dirac structure $L$ on $M$ has been studied in [10], where tangent Dirac structure of higher order are described locally. In this paper we give an intrinsic construction of tangent Dirac structure of higher order denoted by $L^r$ and we study some properties of this Dirac structure. In particular, we study the Lie algebroid and the presymplectic foliation induced by $L^r$.

Introduction

Let $M$ be a differential manifold of dimension $m > 0$, in this paper, we denote by $\langle \cdot, \cdot \rangle_M : TM \times_M T^*M \to \mathbb{R}$ the usual canonical pairing. In [2], is defined the natural symmetric and skew-symmetric pairings on $TM \oplus T^*M$ by:

$\langle X \oplus \omega, Y \oplus \mu \rangle_+ = \frac{1}{2} (\omega(Y) + \mu(X))$

$\langle X \oplus \omega, Y \oplus \mu \rangle_- = \frac{1}{2} (\omega(Y) - \mu(X))$.

An almost-Dirac structure, or a Dirac bundle, on a manifold $M$ is a subbundle $L$ of vector bundle $TM \oplus T^*M$ which is maximally isotropic under the symmetric pairing $\langle \cdot, \cdot \rangle_+$. We denote by $\rho^M$ and $\rho^{*M}$ the natural projection of $TM \oplus T^*M$ onto $TM$ and $T^*M$ respectively. Clearly, $\rho^M(L)$ is a generalized distribution on $M$. We set

$\rho^M(L)^* = \bigcup_{x \in M} (\rho^M(L_x))^*$.

In [2], is defined a 2-form $\Omega_L : \rho^M(L) \to \rho^M(L)^*$ such that:

$\Omega_L(\rho^M(X, \omega))(\rho^M(Y, \mu)) = \langle X \oplus \omega, Y \oplus \mu \rangle_- = \omega(Y),$

and the bilinear bracket operation on the sections of $(TM \oplus T^*M \to M)$ by:

$[X \oplus \omega, Y \oplus \mu] = [X, Y] \oplus (L_X \mu - L_Y \omega + d(\langle X \oplus \omega, Y \oplus \mu \rangle_-)).$

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If $\Gamma(L)$ is closed under this bracket, the author of \cite{2} has said that the almost-Dirac structure $L$ is integrable or $L$ is a Dirac structure on $M$. This condition is equivalent to $T_L = 0$, where $T_L$ is the restriction on $L$ of 3-tensor $T$ defined on $TM \oplus T^*M$ by:

$$T(s_1, s_2, s_3) = \langle [s_1, s_2], s_3 \rangle_+.$$ 

Where $s_1, s_2, s_3 \in \Gamma(TM \oplus T^*M)$.

\textbf{Theorem 1.} An almost-Dirac structure $L$ is integrable if and only if $(L, [\cdot, \cdot], \rho_M|_L)$ is a Lie algebroid.

By this theorem, T. Courant in \cite{2} has shown that, if $L$ is an integrable Dirac structure, then the generalized distribution $\rho_M(L)$ generates a generalized foliation on $M$ and by the same way, we have:

\textbf{Theorem 2.} An integrable Dirac structure has a foliation by presymplectic leaves.

For the proof of these theorems, see \cite{2}.

In \cite{10}, we have defined the tangent lift of higher order $L^r$ ($r \geq 1$) of an almost-Dirac structure $L$ on a manifold $M$, and we have shown that this lifting is an almost-Dirac structure on a manifold $T^rM$. We have shown that, $L$ is integrable if and only if $L^r$ is integrable. In this paper we study some properties of $L^r$ namely the structures of Lie algebroid and generalized foliation induced by $L^r$. The main results of this paper are Theorems 3, 4, 5, 6 and Proposition 3.

All manifolds and maps are assumed to be infinitely differentiable. $r$ will be a natural integer ($r \geq 1$).

1. \textbf{Tangent lifts of higher order of some tensor fields revisited}

1.1. \textbf{Prolongations of sections of vector bundle.} For all $\alpha \in \{0, \ldots, r\}$, we denote by $\chi^{(\alpha)}: T^r \to T^r$ the natural transformation defined for all vector bundle $(E, M, \pi)$ and $\Psi \in C^{\infty}(\mathbb{R}, E)$ by:

$$\chi^{(\alpha)}_E(j^r_0 \Psi) = j^r_0(t^\alpha \Psi).$$

Where $t^\alpha \Psi$ is the smooth map defined for all $t \in \mathbb{R}$ by: $(t^\alpha \Psi)(t) = t^\alpha \Psi(t)$.

Let $S: M \to E$ be a smooth section on $E$, we define the section $\overline{S}^{(\alpha)}$ of $(T^rE, T^rM, T^r\pi)$ by:

$$\overline{S}^{(\alpha)} = \chi^{(\alpha)}_E \circ T^rS, \quad 0 \leq \alpha \leq r.$$

For the sake convenience we define $\overline{S}^{(\alpha)} = 0$ for all $\alpha > r$ or $\alpha < 0$.

\textbf{Definition 1.} This section $\overline{S}^{(\alpha)}$ of $T^rE$ is called $\alpha$-prolongation of order $r$ of $S$.

\textbf{Remark 1.} Let $(E, M, \pi)$ be a vector bundle and $\varphi: \pi^{-1}(U) \to U \times \mathbb{R}^n$ a local trivialization of $E$ over an open $U \subset M$. For $j = 1, \ldots, n$, we put:

$\varepsilon_j(x) = \varphi^{-1}(x, e_j)$ where $x \in U$ and $(e_j)_{j=1, \ldots, n}$ is the usual basis of $\mathbb{R}^n$. 


$(\varepsilon_j)_{j=1,\ldots,n}$ is a basis of sections of $E$ over $U$ associated to $\varphi$. Using the identification $T^r(U \times \mathbb{R}^n) = T^rU \times \mathbb{R}^{n(r+1)}$, we define a family of sections $(\varepsilon_j^\alpha)$, $1 \leq j \leq n$, $0 \leq \alpha \leq r$ of $T^rE$ over $T^rU$ by:

$$\varepsilon_j^\alpha(\tilde{x}) = T^r\varphi^{-1}(\tilde{x}, e_j^\alpha)$$

where $\tilde{x} \in T^rU$ and $(e_j^\alpha)$ the usual basis of $T^r\mathbb{R}^n = \mathbb{R}^{n(r+1)}$.

We have:

$$(1) \quad \varepsilon_j^\alpha = \tilde{\varepsilon}_j^{(\alpha)}, \quad \text{for all } j = 1, \ldots, n \text{ and } \alpha = 0, \ldots, r.$$

**Proposition 1.** Let $(E, M, \pi)$ be a vector bundle. If $\Psi$, $\Psi'$ are two tensor fields of type $(0, p)$ on the vector bundle $(T^rE, T^rM, T^r\pi)$ such that for all smooth sections $S_1, \ldots, S_p$ on $E$ and $\alpha_1, \ldots, \alpha_p \in \{0, 1, \ldots, r\}$ the equality

$$\Psi(S_1^{(\alpha_1)}, \ldots, S_p^{(\alpha_p)}) = \Psi'(S_1^{(\alpha_1)}, \ldots, S_p^{(\alpha_p)})$$

holds, then $\Psi = \Psi'$.

**Proof.** See [5]. □

For the prolongations of functions, vector fields and differential form of manifold $M$ to manifold $T^rM$ and related properties, see [5] or [11]. From now, we adopt the notations of [11].

1.2. **Prolongations of tensor fields of type $(0, p)$.** Let $(E, M, \pi)$ be a vector bundle and $\varphi$ a tensor field of type $(0, p)$ on $E$. We interpret a tensor $\varphi$ on $E$ as a $p$-linear mapping $\varphi: E \times_M \cdots \times_M E \to \mathbb{R}$ of the bundle product over $M$ of $p$-copies of $E$. For all $\alpha \in \{0, 1, \ldots, r\}$, we denote by $\tau_\alpha$ the linear form on $J^r_0(\mathbb{R}, \mathbb{R})$ defined by:

$$\tau_\alpha(f_0g) = \frac{1}{\alpha!} \frac{d^\alpha}{dt^\alpha}(g(t))|_{t=0}.$$  

We set:

$$(2) \quad \varphi^{(\alpha)} = \tau_\alpha \circ T^r\varphi;$$

$\varphi^{(\alpha)}$ is a tensor field of type $(0, p)$ on $(T^rE, T^rM, T^r\pi)$ called $\alpha$-prolongation of $\varphi$ from $E$ to $T^rE$. When $\alpha = r$, it is denoted by $\varphi^{(c)}$ called complete lift of $\varphi$ from $E$ to $T^rE$.

**Proposition 2.** $\varphi^{(\alpha)}$, $0 \leq \alpha \leq r$, is the only tensor field of type $(0, p)$ on $T^rE$ satisfying:

$$(3) \quad \varphi^{(\alpha)}(S_1^{(\alpha_1)}, \ldots, S_p^{(\alpha_p)}) = (\varphi(S_1, \ldots, S_p))^{(\alpha - \sum_{i=1}^p \alpha_i)}.$$  

for all $S_1, \ldots, S_p \in \Gamma(E)$ and $\alpha_1, \ldots, \alpha_p \in \{0, 1, \ldots, r\}$,
Theorem 3. \( (\cdot | \cdot) \) where
\[
\langle \cdot | \cdot \rangle = \langle \cdot | \cdot \rangle_{T^r M} = \tau_r \circ T^r \langle \cdot | \cdot \rangle_M.
\]
\[\text{Proof.} \] Let \( j_0^r \eta \in T^r M \), we have:
\[\varphi^{(\alpha)}(\widetilde{S}_1^{(\alpha_1)}, \ldots, \widetilde{S}_p^{(\alpha_p)})(j_0^r \eta) = \varphi^{(\alpha)}(\chi_{E}^{(\alpha_1)} \circ T^r S_1(j_0^r \eta), \ldots, \chi_{E}^{(\alpha_p)} \circ T^r S_p(j_0^r \eta)) = \varphi^{(\alpha)}(j_0^r (t^{\alpha_1} S_1 \circ \eta), \ldots, j_0^r (t^{\alpha_p} S_p \circ \eta)) = \tau_\alpha(j_0^r \varphi(t^{\alpha_1} S_1 \circ \eta, \ldots, t^{\alpha_p} S_p \circ \eta)) = \tau_\alpha(j_0^r t^{\alpha_1 + \cdots + \alpha_p} \varphi(S_1, \ldots, S_p) \circ \eta) = (t^{\alpha_1 + \cdots + \alpha_p} \varphi(S_1, \ldots, S_p))(\alpha)(j_0^r \eta) = (\varphi(S_1, \ldots, S_p))^{(\alpha-\sum_{i=1}^{p} \alpha_i)}(j_0^r \eta).
\]
The unicity comes from the equation [1] and Proposition [1]. \( \square \)

2. **Tangent Dirac structure of higher order**

2.1. Almost-Dirac structure of higher order. We denote by \( \alpha^r : T^r \circ T^r \to T^r \circ T^* \) and \( \kappa^r : T^r \circ T \to T \circ T^r \) the natural transformations defined in [1] and [2], such that for all manifold \( M \), we have:
\[\langle \kappa^r_{M}(u), v^* \rangle_{T^r M} = \langle u, \alpha^r_{M}(v^*) \rangle_{T^r M}, \quad (u, v^*) \in T^r TM \oplus T^* T^r M,\]
where \( \langle \cdot | \cdot \rangle_{T^r M} = \tau_r \circ T^r \langle \cdot | \cdot \rangle_M \). Let \( L \) be an almost-Dirac structure on \( m \)-dimensional manifold defined locally by the bundle morphisms \( a : U \times \mathbb{R}^m \to TM \) and \( b : U \times \mathbb{R}^m \to T^* M \). \( (e_j) \) denote the canonical basis of \( \mathbb{R}^m \). We set:
\[S_i : U \to L, \quad x \mapsto a(x, e_i) \oplus b(x, e_i),\]
\((S_i)_{1 \leq i \leq n} \) is a basis of sections of \( L \) over \( U \). In [10], we have showed that: the almost Dirac structure of order \( r \) \( L^r \) is determined by the maps \( a^r \) and \( b^r \) such that:
\[a^r = \kappa^r_{M} \circ T^r a \quad \text{and} \quad b^r = \varepsilon^r_{M} \circ T^r b;\]
where \( \varepsilon^r_{M} \) is the inverse map of \( \alpha^r_{M} \). The matrix form of \( a^r \) and \( b^r \) is given by:
\[a^r = \begin{pmatrix} a^r_{ij} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ (a^r_{ij}) & \cdots & a^r_{ij} \end{pmatrix} \quad \text{and} \quad b^r = \begin{pmatrix} (b_{ij})^r & \cdots & b_{ij} \\ \vdots & \ddots & \vdots \\ b_{ij} & \cdots & 0 \end{pmatrix}\]
So that,
\[L^r = (\kappa^r_{M} \oplus \varepsilon^r_{M})(T^r L) \subset TT^r M \oplus T^* T^r M.\]

**Theorem 3.** Let \( X \oplus \omega \in \Gamma(L) \), for all \( \alpha \in \{0, \ldots, r\} \), we have \( X^{(\alpha)} \oplus \omega^{(r-\alpha)} \in \Gamma(L^r) \).

**Proof.** If \((X, \omega) \in \Gamma(L)\) then, they are the maps \( \gamma_1, \ldots, \gamma_m \in C^\infty(U) \) such that:
\[X \oplus \omega = \sum_{i=1}^{m} \gamma^i S_i.\]
In this case,

\[
\begin{align*}
X|_U &= \gamma^i a^j_i \frac{\partial}{\partial x^j} \\
\omega|_U &= \gamma^i b_{ij} dx^j \\
X^{(\alpha)} &= (\gamma^i)^{(\nu)}(a^j_i)^{(\beta-\alpha-\nu)} \frac{\partial}{\partial x^j_{\beta}} .
\end{align*}
\]

We deduce that:

\[
X^{(\alpha)} = \left( \begin{array}{cccc}
a^j_i & 0 & \cdots & 0 \\
\dot{a}^j_i & a^j_i & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
(a^j_i)^{(r)} & (a^j_i)^{(r-1)} & \cdots & a^j_i \\
\end{array} \right) \left( \begin{array}{c}
0 \\
\vdots \\
\vdots \\
(\gamma^i)^{(r-\alpha)} \\
\end{array} \right).
\]

\[
\omega^{(r-\alpha)} = (\gamma^i b_{ij})^{(r-\alpha-\beta)} dx^j_{\beta} = (\gamma^i)^{(r-\nu)}(b_{ij})^{(\nu-\alpha-\beta)} dx^j_{\beta} .
\]

In the same way, we have:

\[
\omega^{(r-\alpha)} = \left( \begin{array}{cccc}
(b_{ij})^{(r)} & (b_{ij})^{(r-1)} & \cdots & b_{ij} \\
(b_{ij})^{(r-1)} & (b_{ij})^{(r-2)} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
(b_{ij}) & 0 & \cdots & 0 \\
\end{array} \right) \left( \begin{array}{c}
0 \\
\vdots \\
\vdots \\
(\gamma^i)^{(r-\alpha)} \\
\end{array} \right).
\]

Thus that \((X^{(\alpha)}, \omega^{(r-\alpha)}) \in \Gamma(L^r) \). □

For all \( X \oplus \omega \in \Gamma(TM \oplus T^*M) = \mathfrak{X}(M) \oplus \Omega^1(M) \), we set:

\[(X \oplus \omega)^{(\alpha)} = X^{(\alpha)} \oplus \omega^{(r-\alpha)} .\]

**Corollary 1.** Let \( L \) be an almost-Dirac structure on \( M \).

1. For all \( X \oplus \omega, Y \oplus \mu \in \mathfrak{X}(M) \oplus \Omega^1(M) \) and \( \alpha, \beta = 0, \ldots, r \), we have:

\[
[(X \oplus \omega)^{(\alpha)}, (Y \oplus \mu)^{(\beta)}] = [X \oplus \omega, Y \oplus \mu]^{(\alpha+\beta)} .
\]

2. For all \( f \in C^\infty(M) \) and \( X \oplus \omega \in \mathfrak{X}(M) \oplus \Omega^1(M) \), we have:

\[
(f \cdot (X \oplus \omega))^{(\alpha)} = \sum_{\beta=0}^{r-\alpha} f^{(\beta)} \cdot (X \oplus \omega)^{(\alpha+\beta)} .
\]

3. For all \( X \oplus \omega, Y \oplus \mu, Z \oplus \nu \in \Gamma(L) \), we have:

\[
\mathbb{T}_{L^r}((X \oplus \omega)^{(\alpha)}, (Y \oplus \mu)^{(\beta)}, (Z \oplus \nu)^{(\gamma)}) = \mathbb{T}_L(X \oplus \omega, Y \oplus \mu, Z \oplus \nu)^{(r-\alpha-\beta-\gamma)} ,
\]

for all \( \alpha, \beta, \gamma \in \{0, 1, \ldots, r\} \).

**Proof.** The proof comes of some properties of tangent lift of higher order of functions, vector fields and differential forms. □
For all $S \in \Gamma(L)$ and $\alpha \in \{0, 1, \ldots, r\}$, we have:

\[
(k_M^r \oplus \epsilon_M^r)(S^{(\alpha)}) = S^{(\alpha)}.
\]

**Theorem 4.** $\mathbb{T}_L^{(c)}$ is a complete lift of $\mathbb{T}_L$ from $L$ to $T^r L$. We denote by $\eta^r_M$ the inverse map of $k_M^r$. We have:

\[
(4) \quad \mathbb{T}_L^r = \mathbb{T}_L^{(c)} \circ \left( \bigoplus_{\alpha} (\eta^r_M \oplus \alpha^r_M) \right).
\]

**Proof.** $\mathbb{T}_L^r \circ \left( \bigoplus_{\alpha} (k_M^r \oplus \epsilon_M^r) \right)$ is a tensor field of type $(0, 3)$ on $T^r L$. Let $S_1, S_2, S_3 \in \Gamma(L)$ and $\alpha_1, \alpha_2, \alpha_3 \in \{0, 1, \ldots, r\}$, we have:

\[
\mathbb{T}_L^r \circ \left( \bigoplus_{\alpha} (k_M^r \oplus \epsilon_M^r) \right)(S_1^{(\alpha_1)}, S_2^{(\alpha_2)}, S_3^{(\alpha_3)}) = \mathbb{T}_L^r(S_1^{(\alpha_1)}, S_2^{(\alpha_2)}, S_3^{(\alpha_3)})
= \left( \mathbb{T}_L(S_1, S_2, S_3) \right)^{(r-\alpha_1-\alpha_2-\alpha_3)}
= \mathbb{T}_L^{(c)}(S_1^{(\alpha_1)}, S_2^{(\alpha_2)}, S_3^{(\alpha_3)}).
\]

We have the result by the Proposition 2.

**Remark 2.** The equation (4) shows that $L$ is integrable if and only if $L^r$ is integrable. Thus, we have given an intrinsic construction of tangent lift of higher order of an almost-Dirac structure, and we have shown independent of any local coordinates system that: this lifting is integrable if and only if the initial almost-Dirac structure is integrable.

Let $X \oplus \omega, Y \oplus \mu$ be sections of an almost-Dirac structure $L$. Define

\[
X \bullet (Y \oplus \mu) = [X, Y] \oplus \mathcal{L}_X \mu.
\]

**Definition 2.** $L$ is said invariance under $X \oplus \omega \in \Gamma(L)$ if and only if $X \bullet L \subseteq L$. When $L$ is integrable this is equivalent to say $d\omega|\rho_M(L) = 0$.

**Corollary 2.** If $L$ is an integrable Dirac structure invariant under $X \in \rho_M(\Gamma(L))$, then $L^r$ is invariant under $X^{(\alpha)}$ for all $\alpha = 0, \ldots, r$

**Proof.** Let $X \oplus \omega \in \Gamma(L)$, we have $X^{(\alpha)} \oplus \omega^{(r-\alpha)} \in \Gamma(L^r)$ by the equality

\[
d\omega^{(r-\alpha)} = (d\omega)^{(r-\alpha)} \quad \text{(see [11])},
\]

we deduce that $d\omega^{(r-\alpha)}|\rho(L^r) = 0$.

2.2. **Admissible functions of $L^r$.** Let $L$ be an integrable Dirac structure over $M$. A function $f$ is an admissible relatively to $L$, if there is vector field $X_f$ such that $(X_f, df) \in \Gamma(L)$. If $f$ and $g$ are two admissible functions, T. Courant defines in [2] their bracket by:

\[
\{f, g\} = X_f(g).
\]

**Proposition 3.** (1) If $f$ is an admissible function relatively to $L$, then $f^{(\alpha)}$ is an admissible function relatively to $L^r$ and we have:

\[
(5) \quad X_{f^{(\alpha)}} = (X_f)^{(r-\alpha)}.
\]
(2) For all \( f, g \) two admissible functions, \( \alpha, \beta = 0, \ldots, r \), we have:

\[
\{ f^{(\alpha)}, g^{(\beta)} \} = \{ f, g \}^{(\alpha + \beta - r)}.
\]

**Proof.** (1) If \( f \) is an admissible function, then \((X_f, df) \in \Gamma(L)\). For all \( \alpha \),

\[
((X_f)^{(r-\alpha)}, (df)^{(\alpha)}) \in \Gamma(L^r).
\]

Since \((df)^{(\alpha)} = df^{(\alpha)}\), it follows that \(((X_f)^{(r-\alpha)}, df^{(\alpha)}) \in \Gamma(L^r)\). Thus, \( f^{(\alpha)} \) is an admissible function relatively to \( L^r \) and \( X_f^{(\alpha)} = (X_f)^{(r-\alpha)} \).

(2) For \( \alpha, \beta = 0, \ldots, r \), we have:

\[
\{ f^{(\alpha)}, g^{(\beta)} \} = X_f^{(\alpha)}(g^{(\beta)}) = (X_f)^{(r-\alpha)}(g^{(\beta)}) = \{ f, g \}^{(\alpha + \beta - r)}
\]

\( \square \)

2.3. **The Lie algebroid** \((L^r, [\cdot, \cdot], \rho_{T^r M | L^r})\). For all \( \alpha \in \{0, 1, \ldots, r\} \), consider the map

\[
\chi^{(\alpha)}_{TM \oplus T^* M} : T^r (TM \oplus T^* M) \rightarrow T^r (TM \oplus T^* M),
\]

we have:

\[
\chi^{(\alpha)}_{TM \oplus T^* M} = \chi^{(\alpha)}_{TM} \oplus \chi^{(\alpha)}_{T^* M}.
\]

In this case, \( \chi^{(\alpha)}_L = \chi^{(\alpha)}_{TM} \oplus \chi^{(\alpha)}_{T^* M} |_{T^r L} \).

**Proposition 4.** Let \((E, [\cdot, \cdot], \rho)\) be a Lie algebroid. There is one and only one Lie algebroid structure on \( T^r E \) such that: For all \( S_1, S_2 \in \Gamma(E) \) and \( \alpha, \beta \in \{0, 1, \ldots, r\} \)

\[
[S_1^{(\alpha)}, S_2^{(\beta)}] = [S_1, S_2]^{(\alpha + \beta)}.
\]

The anchor map \( \rho^{(r)} \) is given by:

\[
\rho^{(r)} = \kappa^r_M \circ T^r \rho.
\]

This Lie algebroid structure is called tangent lift of order \( r \) of Lie algebroid \((E, [\cdot, \cdot], \rho)\).

**Proof.** See [9]. \( \square \)

**Theorem 5.** Let \( L \) be an integrable Dirac structure on \( M \). The tangent Lie algebroid of order \( r \) \( T^r L \), is isomorphic to the Lie algebroid \((L^r, [\cdot, \cdot], \rho_{T^r M | L^r})\) over \( T^r M \) induced by the integrable Dirac structure \( L^r \).

**Proof.** Let \((S_i)\) be a basis of sections of \( L \) over \( U \).

\[
S_i(x) = a(x, e_i) \oplus b(x, e_i), \quad \forall i = 1, \ldots, m
\]
we have $κ^r_M ⊕ ε^r_M(S_i^{(α)}) = S_i^{(α)}$.

The tangent Lie algebroid of order $r$ $T^rL$ is given by:

$$[S_i^{(α)}, S_j^{(β)}] = [S_i^{(α)}]^{(α+β)}$$

$$[κ^r_M ⊕ ε^r_M(S_i^{(α)}), κ^r_M ⊕ ε^r_M(S_j^{(β)})] = [S_i^{(α)}]^{(α+β)}$$

It follows that,

$$κ^r_M ⊕ ε^r_M|_{T^rL}: T^rL → L^r$$

is a Lie algebroids isomorphism.

2.4. Symplectic foliation induced by $L^r$. For the tangent lift of higher order of singular foliation of manifold $M$ to $T^rM$ we can see [9]. However, let $E$ be a smooth generalized distribution on $M$, we denote by $X_E$ the set of all local vector fields such that: for all $x ∈ M$, $X(x) ∈ E_x$. Let us notice that for a completely integrable distribution $E$, the family $X_E$ is a Lie subalgebra of the Lie algebra of vector fields on $M$.

**Proposition 5.** Let $E$ be a completely integrable generalized distribution on $M$. Then the distribution $E^r$ generated by the family $\{X^{(α)}, X ∈ X_E, 0 ≤ α ≤ r\}$ of vector fields on $T^rM$ is completely integrable.

**Proof.** See [9].

Let $F$ be a generalized foliation defined by $E$, the tangent lift of order $r$ of $F$ denoted by $T^rF$ is defined by $E^r$.

**Proposition 6.** If a submanifold $F ⊂ M$ is a leaf of generalized foliation $F$, then $T^rF$ is a leaf of generalized foliation $T^rF$.

**Proof.** See [9].

By the Propositions 5 and 6 we deduce this result.

**Theorem 6.** Let $L$ be an integrable Dirac structure, $F$ the generalized foliation induced by $L$ and $F$ a leaf of $F$.

1. The generalized foliation induced by $L^r$ is the tangent lift of order $r$ of generalized foliation $F$.

2. If $Ω_F$ is a presymplectic form on $F$ then $Ω_F^{(c)}$ is a presymplectic form on the leaf $T^rF$. Where $Ω_F^{(c)}$ is a complete lift of differential form $Ω_F$.

**Proof.** Let $X, Y ∈ ρ_M(Γ(L))$ tangent to $F$, we have:

$$Ω_{T^rF}(X^{(α)}, Y^{(β)}) = ω^{(r-α)}(Y^{(β)})$$

$$= (ω(Y))^{(r−α−β)}$$

$$= (Ω_F(X, Y))^{(r−α−β)}$$

$$= Ω_F^{(c)}(X^{(α)}, Y^{(β)})$$
Thus $\Omega_{T^rF} = \Omega_F^{(c)}$. □

These results generalize the properties of tangent lifting of higher order of Poisson manifold.

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