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BACKWARD DOUBLY STOCHASTIC DIFFERENTIAL EQUATIONS
WITH INFINITE TIME HORIZON*

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Abstract. We give a sufficient condition on the coefficients of a class of infinite horizon backward doubly stochastic differential equations (BDSDES), under which the infinite horizon BDSDES have a unique solution for any given square integrable terminal values. We also show continuous dependence theorem and convergence theorem for this kind of equations.

Keywords: infinite horizon backward doubly stochastic differential equations, filtration, backward stochastic integral

MSC 2010: 60H10

1. INTRODUCTION

Since the nonlinear backward stochastic differential equations (BSDEs in short) were introduced by Pardoux and Peng [9], the theory of BSDEs has been developed by many researchers in a series of papers (for example, Ma, Protter, and Yong [7], El Karoui, Peng, and Quenez [6] and the references therein). These papers basically study BSDEs for a fixed terminal time $T > 0$, i.e., in a finite time interval $[0, T]$. In order to investigate the case of infinite time interval, i.e., $T = \infty$, many researchers, for example, Peng [12], Darling and Pardoux [5], Peng and Shi [14] and so on, present many different assumptions. However, their results essentially require the terminal values to be decay in infinite horizon. Later Chen and Wang [4] were the first to show a kind of sufficient conditions on coefficients, under which for any square integrable random variables ξ as terminal values, BSDEs still have a unique pair of solutions

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for infinite horizon case. This result is pivotal for discussing the convergence of g -martingales which were introduced by Peng [13].

After Pardoux and Peng [9] introduced the theory of BSDEs, they [10] brought forward a new kind of BSDEs, that is a class of backward doubly stochastic differential equations (BDSDEs in short) with two different directions of stochastic integrals, i.e., the equations involve both a standard (forward) stochastic integral dW_t and a backward stochastic integral dB_t . They have proved the existence and uniqueness of solutions to BDSDEs under uniformly Lipschitz conditions on coefficients on a finite time interval $[0, T]$. That is, for a given terminal time $T > 0$, under the uniformly Lipschitz assumptions on coefficients f, g , given $\xi \in L^2(\Omega, \mathcal{F}_T, P; \mathbb{R}^k)$, the following BDSDE has a unique solution pair (y_t, z_t) in the interval $[0, T]$:

$$(1) \quad Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds + \int_t^T g(s, Y_s, Z_s) dB_s - \int_t^T Z_s dW_s, \quad t \in [0, T].$$

Pardoux and Peng also showed that BDSDEs can produce a probabilistic representation for certain quasilinear stochastic partial differential equations (SPDEs). Many researchers do their work in this area (for example, V. Bally and A. Matoussi [1], R. Buckdahn and J. Ma [2], [3], E. Pardoux [11], Peng and Shi [15], Zhu and Han [17] and the references therein). Infinite horizon BDSDEs are also very interesting, since they produce a probabilistic representation of certain quasilinear stochastic partial differential equations. Recently, Zhang and Zhao [16] got stationary solutions of SPDEs and infinite horizon BDSDEs, but under the assumption that the terminal value $\lim_{T \rightarrow \infty} e^{-KT} Y_T = 0$. This paper intends to study the existence and uniqueness of BDSDE (1) when $T = \infty$, our method being different from Zhang and Zhao. Due to sufficiently utilizing the properties of martingales, this method is essential for the theory of BSDEs. In this paper we give a sufficient condition on coefficients f, g under which for any square integrable random variable ξ , BDSDE (1) still has a unique solution pair when $T = \infty$. It is worth noting that in our argument, we have to restrict g to be independent of z . For the case of g being dependent on z , it is still an interesting open question. Our conditions are a special kind of Lipschitz conditions, which even include some cases of unbounded coefficients. At the end we will also give a continuous dependence theorem and a convergence theorem for this class of equations.

The paper is organized as follows: in Section 2 we present the setting of problems and the main assumptions; in Section 3 we prove the existence and uniqueness theorem for BDSDEs; at the end we discuss the continuous dependence theorem and the convergence theorem in Section 4.

2. SETTING OF INFINITE HORIZON BDSDEs

Notation. The Euclidean norm of a vector $x \in \mathbb{R}^k$ will be denoted by $|x|$, and for a $d \times k$ matrix A , we define $\|A\| = \sqrt{\text{Tr} AA^*}$, where A^* is the transpose of A .

Let (Ω, \mathcal{F}, P) be a complete probability space, let $\{W_t\}_{t \geq 0}$ and $\{B_t\}_{t \geq 0}$ be two mutually independent standard Brownian motions with values in \mathbb{R}^d and \mathbb{R}^l , respectively, defined on (Ω, \mathcal{F}, P) . Let \mathcal{N} denote the class of P -null sets of \mathcal{F} . For each $t \in [0, \infty)$, we define

$$\begin{aligned} \mathcal{F}_{0,t}^W &\doteq \sigma\{W_r; 0 \leq r \leq t\} \vee \mathcal{N}, & \mathcal{F}_{t,\infty}^B &\doteq \sigma\{B_r - B_t; t \leq r < \infty\} \vee \mathcal{N}, \\ \mathcal{F}_{0,\infty}^W &\doteq \bigvee_{0 \leq t < \infty} \mathcal{F}_{0,t}^W, & \mathcal{F}_{\infty,\infty}^B &\doteq \bigcap_{0 \leq t < \infty} \mathcal{F}_{t,\infty}^B, \end{aligned}$$

and

$$\mathcal{F}_t \doteq \mathcal{F}_{0,t}^W \vee \mathcal{F}_{t,\infty}^B, \quad t \in [0, \infty].$$

Note that $\{\mathcal{F}_{0,t}^W; t \in [0, \infty]\}$ is an increasing filtration and $\{\mathcal{F}_{t,\infty}^B; t \in [0, \infty]\}$ is a decreasing filtration, and the collection $\{\mathcal{F}_t, t \in [0, \infty]\}$ is neither increasing nor decreasing.

Suppose

$$\mathcal{F} = \mathcal{F}_\infty \doteq \mathcal{F}_{0,\infty}^W \vee \mathcal{F}_{\infty,\infty}^B.$$

For any $n \in \mathbb{N}$, let $S^2(\mathbb{R}^+; \mathbb{R}^n)$ denote the space of all $\{\mathcal{F}_t\}$ -measurable n -dimensional processes v with the norm $\|v\|_S \doteq \left[E \left(\sup_{s \geq 0} |v(s)| \right)^2 \right]^{1/2} < \infty$.

We denote similarly by $M^2(\mathbb{R}^+; \mathbb{R}^n)$ the space of all (classes of $dP \otimes dt$ a.e. equal) $\{\mathcal{F}_t\}$ -measurable n -dimensional processes v with the norm

$$\|v\|_M \doteq \left[E \int_0^\infty |v(s)|^2 ds \right]^{1/2} < \infty.$$

For any $t \in [0, \infty]$, let $L^2(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$ denote the space of all $\{\mathcal{F}_t\}$ -measurable n -valued random variables ξ satisfying $E|\xi|^2 < \infty$.

We also denote

$$B^2 \doteq \{(X, Y); X \in S^2(\mathbb{R}^+; \mathbb{R}^n), Y \in M^2(\mathbb{R}^+; \mathbb{R}^n)\}.$$

For each $(X, Y) \in B^2$, we define the norm of (X, Y) by

$$\|(X, Y)\|_B \doteq (\|X\|_S^2 + \|Y\|_M^2)^{1/2}.$$

Obviously B^2 is a Banach space.

Consider the infinite horizon backward doubly stochastic differential equation

$$(2) \quad y_t = \xi + \int_t^\infty f(s, y_s, z_s) ds + \int_t^\infty g(s, y_s) dB_s - \int_t^\infty z_s dW_s, \quad 0 \leq t \leq \infty,$$

where $\xi \in L^2(\Omega, \mathcal{F}, P; \mathbb{R}^k)$ is given. We note that the integral with respect to $\{B_t\}$ is a “backward Itô integral” and the integral with respect to $\{W_t\}$ is a standard forward Itô integral. These two types of integrals are particular cases of the Itô-Skorohod integral, see Nualart and Pardoux. Our aim is to find some conditions under which BDSDE (2) has a unique solution. Now we give the definition of a solution of BDSDE (2).

Definition 1. A pair of processes $(y, z): \Omega \times \mathbb{R}^+ \rightarrow \mathbb{R}^k \times \mathbb{R}^{k \times d}$ is called a solution of BDSDE (2), if $(y, z) \in B^2$ and satisfies BDSDE (2).

Let

$$\begin{aligned} f: \Omega \times \mathbb{R}^+ \times \mathbb{R}^k \times \mathbb{R}^{k \times d} &\rightarrow \mathbb{R}^k, \\ g: \Omega \times \mathbb{R}^+ \times \mathbb{R}^k &\rightarrow \mathbb{R}^{k \times l} \end{aligned}$$

satisfy the following assumptions:

(H1) For any $\omega \times t \in \Omega \times \mathbb{R}^+$, $(y, z) \in \mathbb{R}^k \times \mathbb{R}^{k \times d}$, $f(\cdot, y, z)$ and $g(\cdot, y)$ are $\{\mathcal{F}_t\}$ -progressively measurable processes such that

$$E \left(\int_0^\infty f(t, 0, 0) dt \right)^2 < \infty; \quad g(\cdot, 0) \in M^2(\mathbb{R}^+; \mathbb{R}^{k \times l}).$$

(H2) f and g satisfy the Lipschitz condition with Lipschitz coefficients $v := \{v(t)\}$ and $u := \{u(t)\}$, that is, there exist two positive non-random functions $\{v(t)\}$ and $\{u(t)\}$ such that

$$\begin{aligned} |f(t, y_1, z_1) - f(t, y_2, z_2)| &\leq v(t)|y_1 - y_2| + u(t)\|z_1 - z_2\|, \\ \|g(t, y_1) - g(t, y_2)\| &\leq u(t)|y_1 - y_2| \end{aligned}$$

for all $(t, y_i, z_i) \in \mathbb{R}^+ \times \mathbb{R}^k \times \mathbb{R}^{k \times d}$, $i = 1, 2$.

(H3) $\int_0^\infty v(t) dt < \infty$; $\int_0^\infty u^2(t) dt < \infty$.

3. EXISTENCE AND UNIQUENESS THEOREM

The following existence and uniqueness theorem is our main result.

Theorem 1. *Under the above conditions, in particular (H1), (H2), and (H3), Eq. (2) has a unique solution $(y, z) \in B^2$.*

In order to prove the existence and uniqueness theorem, we first give a priori estimate.

Lemma 2. Suppose (H1), (H2), and (H3) hold for f and g . For any $T \in [0, \infty]$, let $Y_T^i \in L^2(\Omega, \mathcal{F}_T, P; \mathbb{R}^k)$, (Y^i, Z^i) and $(y^i, z^i) \in B^2$ ($i = 1, 2$) satisfy the equation

$$(3) \quad \begin{aligned} Y_t^i &= Y_T^i + \int_t^T f(s, y_s^i, z_s^i) ds + \int_t^T g(s, y_s^i) dB_s \\ &\quad - \int_t^T Z_s^i dW_s, \quad 0 \leq t \leq T \leq \infty. \end{aligned}$$

Then there is a constant $C > 0$ such that, for any $\tau \in [0, T]$,

$$\begin{aligned} &\|((Y^1 - Y^2)I_{[\tau, T]}, (Z^1 - Z^2)I_{[\tau, T]})\|_B^2 \\ &\leq C[E|Y_T^1 - Y_T^2|^2 + l_{[\tau, T]} \|((y^1 - y^2)I_{[\tau, T]}, (z^1 - z^2)I_{[\tau, T]})\|_B^2] \end{aligned}$$

where $l_{[\tau, T]} = (\int_\tau^T v(s) ds)^2 + \int_\tau^T u^2(s) ds$ and $I_{[\tau, T]}(\cdot)$ is an indicator function.

Proof. Without loss of generality, we assume that $\tau = 0$, $T = \infty$, otherwise we can replace f by $fI_{[\tau, T]}$ and g by $gI_{[\tau, T]}$.

Set

$$\begin{aligned} \hat{Y}_t &= Y_t^1 - Y_t^2, \quad \hat{Z}_t = Z_t^1 - Z_t^2, \quad \hat{y}_t = y_t^1 - y_t^2, \quad \hat{z}_t = z_t^1 - z_t^2, \\ \hat{f}_t &= f(t, y_t^1, z_t^1) - f(t, y_t^2, z_t^2), \quad \hat{g}_t = g(t, y_t^1) - g(t, y_t^2). \end{aligned}$$

Then

$$(4) \quad \hat{Y}_t = \hat{Y}_T + \int_t^T \hat{f}_s ds + \int_t^T \hat{g}_s dB_s - \int_t^T \hat{Z}_s dW_s.$$

We define the filtration $\{\mathcal{G}_t\}_{0 \leq t \leq T}$ by

$$\mathcal{G}_t \doteq \mathcal{F}_{0,t}^W \vee \mathcal{F}_{0,\infty}^B.$$

Obviously \mathcal{G}_t is an increasing filtration. Since $(\hat{Y}, \hat{Z}) \in B^2$, $\{\int_0^t \hat{Z}_s dW_s\}$ is a \mathcal{G}_t -martingale. Thus from (4) it follows that

$$\hat{Y}_t = E^{\mathcal{G}_t} \left[\hat{Y}_T + \int_t^T \hat{f}_s ds + \int_t^T \hat{g}_s dB_s \right].$$

Note that

$$\begin{aligned}
E\left(\int_0^\infty |\hat{f}_s| \, ds\right)^2 &\leq E\left(\int_0^\infty (v(s)|\hat{y}_s| + u(s)\|\hat{z}_s\|) \, ds\right)^2 \\
&\leq 2E\left(\sup_{t \geq 0} |\hat{y}_t| \cdot \int_0^\infty v(s) \, ds\right)^2 + 2E\left(\int_0^\infty u^2(s) \, ds \cdot \int_0^\infty \|\hat{z}_s\|^2 \, ds\right) \\
&\leq 2\left[\left(\int_0^\infty v(s) \, ds\right)^2 + \int_0^\infty u^2(s) \, ds\right] \cdot \|(\hat{y}, \hat{z})\|_B^2 < \infty,
\end{aligned}$$

and

$$E\int_0^\infty \|\hat{g}_s\|^2 \, ds \leq E\int_0^\infty u^2(s)|\hat{y}_s|^2 \, ds \leq \left(\int_0^\infty u^2(s) \, ds\right) \cdot \|\hat{y}\|_S^2 < \infty.$$

Applying the Doob inequality and the B-D-G inequality, we can deduce

$$\begin{aligned}
(6) \quad \|\hat{Y}\|_S^2 &= E(\sup_{t \geq 0} |\hat{Y}_t|)^2 \\
&\leq 2E\left(\sup_{t \geq 0} E^{\mathcal{G}_t} \left[|\hat{Y}_T| + \int_t^T |\hat{f}_s| \, ds\right]\right)^2 + 2E\left(\sup_{t \geq 0} E^{\mathcal{G}_t} \left[\left|\int_t^T \hat{g}_s \, dB_s\right|\right]\right)^2 \\
&\leq 8E\left(|\hat{Y}_T| + \int_0^\infty |\hat{f}_s| \, ds\right)^2 + 2c_0 E\int_0^\infty \|\hat{g}_s\|^2 \, ds \\
&\leq 16\left(E|\hat{Y}_T|^2 + E\left(\int_0^\infty |\hat{f}_s| \, ds\right)^2\right) + 2c_0 E\int_0^\infty \|\hat{g}_s\|^2 \, ds \\
&\leq (16 + 2c_0)\left(E|\hat{Y}_T|^2 + E\left(\int_0^\infty |\hat{f}_s| \, ds\right)^2 + E\int_0^\infty \|\hat{g}_s\|^2 \, ds\right),
\end{aligned}$$

where $c_0 > 0$ is a constant.

On the other hand, from (4) it follows that

$$\begin{aligned}
(7) \quad \|\hat{Z}\|_M^2 &= E\left\langle \int_0^\cdot \hat{Z}_s \, dW_s \right\rangle_\infty \\
&= E\left(\hat{Y}_T + \int_0^\infty \hat{f}_s \, ds + \int_0^\infty \hat{g}_s \, dB_s\right)^2 \\
&\quad - E\left|E^{\mathcal{G}_0} \left[\hat{Y}_T + \int_0^\infty \hat{f}_s \, ds + \int_0^\infty \hat{g}_s \, dB_s\right]\right|^2 \\
&\leq E\left(\hat{Y}_T + \int_0^\infty \hat{f}_s \, ds + \int_0^\infty \hat{g}_s \, dB_s\right)^2 \\
&\leq 3E\left(|\hat{Y}_T|^2 + \left(\int_0^\infty |\hat{f}_s| \, ds\right)^2 + \int_0^\infty \|\hat{g}_s\|^2 \, ds\right),
\end{aligned}$$

where $\langle M \rangle$ is the variation process generated by the martingale M .

Consequently, (6) and (7) imply that

$$\begin{aligned}
\|(\hat{Y}, \hat{Z})\|_B^2 &= \|\hat{Y}\|_S^2 + \|\hat{Z}\|_M^2 \\
&\leq (19 + 2c_0) \left(E|\hat{Y}_T|^2 + E \left(\int_0^\infty |\hat{f}_s| \, ds \right)^2 + E \int_0^\infty |\hat{g}_s|^2 \, ds \right) \\
&\leq (57 + 6c_0) (E|\hat{Y}_T|^2 + l_{[0, \infty]} \|(\hat{y}, \hat{z})\|_B^2) \\
&= C (E|\hat{Y}_T|^2 + l_{[0, \infty]} \|(\hat{y}, \hat{z})\|_B^2),
\end{aligned}$$

where $C = (57 + 6c_0)$ is a constant and $l_{[0, \infty]} = (\int_0^\infty v(s) \, ds)^2 + \int_0^\infty u^2(s) \, ds$.

For any $\tau, T \in [0, \infty]$, we set $f_1(t, y_t, z_t) = f(t, y_t, z_t)I_{[\tau, T]}$, and $g_1(t, y_t) = g(t, y_t)I_{[\tau, T]}$. Then f_1 , and g_1 satisfy the assumptions (H1), (H2), and (H3), and their Lipschitz constants are $vI_{[\tau, T]}$, $uI_{[\tau, T]}$. Repeating the above process, we can obtain the desired result. \square

Now we give the proof of Theorem 1.

P r o o f. The proof of Theorem 1 is divided into two steps.

Step 1. We assume $l_{[0, \infty]} = (\int_0^\infty v(s) \, ds)^2 + \int_0^\infty u^2(s) \, ds \leq 1/2C$. For any $(y, z) \in B^2$, let

$$M_t \doteq E^{\mathcal{G}_t} \left[\xi + \int_0^\infty f(s, y_s, z_s) \, ds + \int_0^\infty g(s, y_s) \, dB_s \right], \quad 0 \leq t \leq \infty.$$

We will prove $\{M_t\}$ is a square integrable \mathcal{G}_t -martingale. From (H1)–(H3), it follows that

$$\begin{aligned}
&E \left(\left| \xi + \int_0^\infty f(s, y_s, z_s) \, ds + \int_0^\infty g(s, y_s) \, dB_s \right|^2 \right) \\
&\leq E \left(|\xi| + \int_0^\infty |f(s, y_s, z_s)| \, ds + \left| \int_0^\infty g(s, y_s) \, dB_s \right| \right)^2 \\
&\leq 3E|\xi|^2 + 9E \left(\int_0^\infty |f(s, 0, 0)| \, ds \right)^2 + 9E \left(\int_0^\infty v(s)|y_s| \, ds \right)^2 \\
&\quad + 9E \left(\int_0^\infty u(s)\|z_s\| \, ds \right)^2 + 6E \int_0^\infty (\|g(s, 0)\|^2 + u^2(s)|y_s|^2) \, ds \\
&\leq 3E|\xi|^2 + 9E \left(\int_0^\infty |f(s, 0, 0)| \, ds \right)^2 + 9 \left(\int_0^\infty v(s) \, ds \right)^2 \cdot \|y\|_S^2 \\
&\quad + 9 \int_0^\infty u^2(s) \, ds \cdot \|z\|_M^2 + 6E \int_0^\infty \|g(s, 0)\|^2 \, ds \\
&\quad + 6 \int_0^\infty u^2(s) \, ds \cdot \|y\|_S^2 < \infty,
\end{aligned}$$

which means $\{M_t\}$ is a square integrable \mathcal{G}_t -martingale. According to an obvious extension of Itô's martingale representation theorem, there exists a unique \mathcal{G}_t -progressively measurable process Z_t with values in $\mathbb{R}^{k \times d}$ such that

$$(8) \quad E \int_0^\infty |Z_t|^2 dt < \infty,$$

$$M_t = E^{\mathcal{G}_0} \left[\xi + \int_0^\infty f(s, y_s, z_s) ds + \int_0^\infty g(s, y_s) dB_s \right] + \int_0^t Z_s dW_s, \quad 0 \leq t \leq \infty.$$

Let

$$(9) \quad Y_t \doteq E^{\mathcal{G}_t} \left[\xi + \int_t^\infty f(s, y_s, z_s) ds + \int_t^\infty g(s, y_s) dB_s \right], \quad 0 \leq t \leq \infty.$$

Then it is not difficult to check that (8) and (9) are equivalent to

$$(10) \quad Y_t = \xi + \int_t^\infty f(s, y_s, z_s) ds + \int_t^\infty g(s, y_s) dB_s - \int_t^\infty Z_s dW_s, \quad 0 \leq t \leq \infty.$$

We show that $\{Y_t\}$ and $\{Z_t\}$ are in fact \mathcal{F}_t -measurable. For Y_t , this is obvious since for each t ,

$$Y_t = E^{\mathcal{G}_t} \left[\xi + \int_t^\infty f(s, y_s, z_s) ds + \int_t^\infty g(s, y_s) dB_s \right] = E(\Theta / \mathcal{F}_t \vee \mathcal{F}_{0,t}^B),$$

where $\Theta = \xi + \int_t^\infty f(s, y_s, z_s) ds + \int_t^\infty g(s, y_s) dB_s$ is indeed $\mathcal{F}_{0,\infty}^W \vee \mathcal{F}_{t,\infty}^B$ -measurable. Hence, $\mathcal{F}_{0,t}^B$ is independent of $\mathcal{F}_t \vee \sigma(\Theta)$, and

$$Y_t = E(\Theta / \mathcal{F}_t).$$

Now

$$\int_t^\infty Z_s dW_s = \xi + \int_t^\infty f(s, y_s, z_s) ds + \int_t^\infty g(s, y_s) dB_s - Y_t$$

and the right-hand side is $\mathcal{F}_{0,\infty}^W \vee \mathcal{F}_{t,\infty}^B$ -measurable. Thus, from Itô's martingale representation theorem, $\{Z_s, s > t\}$ is $\mathcal{F}_{0,s}^W \vee \mathcal{F}_{t,\infty}^B$ -adapted. Consequently, Z_s is $\mathcal{F}_{0,s}^W \vee \mathcal{F}_{t,\infty}^B$ -measurable for any $t < s$, and, thus, Z_t is \mathcal{F}_t -measurable. So $(Y, Z) \in B^2$. Therefore, equation (10) yields a mapping from B^2 to B^2 , and we denote it by φ , that is,

$$\varphi: (y, z) \rightarrow (Y, Z).$$

If φ is a contractive mapping with respect to the norm $\|\cdot\|_B$, then by the fixed-point theorem there exists a unique $(y, z) \in B^2$ satisfying (10), which is just the unique solution to BDSDE (2).

Now we are in the position to prove that φ is a contractive mapping. Suppose $(y^i, z^i) \in B^2$, let (Y^i, Z^i) be the map φ of (y^i, z^i) ($i = 1, 2$), that is

$$\varphi(y^i, z^i) = (Y^i, Z^i), \quad i = 1, 2.$$

We denote

$$\begin{aligned} \hat{Y} &= Y^1 - Y^2, & \hat{Z} &= Z^1 - Z^2, & \hat{y} &= y^1 - y^2, & \hat{z} &= z^1 - z^2, \\ \hat{f}_t &= f(t, y^1, z^1) - f(t, y^2, z^2), & \hat{g}_t &= g(t, y^1) - g(t, y^2). \end{aligned}$$

By Lemma 2, we have

$$\|\varphi(y^1, z^1) - \varphi(y^2, z^2)\|_B^2 = \|(\hat{Y}, \hat{Z})\|_B^2 \leq Cl_{[0, \infty)} \|(\hat{y}, \hat{z})\|_B^2.$$

Due to $l_{[0, \infty)} \leq 1/2C$, it follows that φ is a contractive mapping from B^2 to B^2 .

Step 2. Since $\int_0^\infty v(t) dt < \infty$; $\int_0^\infty u^2(t) dt < \infty$, there exists a sufficiently large constant T such that

$$\left(\int_T^\infty v(s) ds \right)^2 + \int_T^\infty u^2(s) ds \leq \frac{1}{2C}.$$

Let

$$f_1(t, y, z) \doteq I_{[T, \infty)}(t)f(t, y, z), \quad g_1(t, y) \doteq I_{[T, \infty)}(t)g(t, y),$$

then (H1)–(H3) hold for f_1 and g_1 , whose Lipschitz coefficients are $\bar{v}(t) = I_{[T, \infty)}v(t)$ and $\bar{u}(t) = I_{[T, \infty)}u(t)$. Obviously,

$$\left(\int_0^\infty \bar{v}(s) ds \right)^2 + \int_0^\infty \bar{u}^2(s) ds \leq \frac{1}{2C}.$$

By Step 1, there exists a unique $(\tilde{y}, \tilde{z}) \in B^2$ satisfying

$$\tilde{y}_t = \xi + \int_t^\infty f_1(s, \tilde{y}_s, \tilde{z}_s) ds + \int_t^\infty g_1(s, \tilde{y}_s) dB_s - \int_t^\infty \tilde{z}_s dW_s, \quad 0 \leq t \leq \infty.$$

For $(\tilde{y}_t, \tilde{z}_t)$ given as above, let us consider the infinite BDSDE

$$\begin{cases} \bar{y}_t = \int_t^T f(s, \bar{y}_s + \tilde{y}_s, \bar{z}_s + \tilde{z}_s) ds \\ \quad + \int_t^T g(s, \bar{y}_s + \tilde{y}_s) dB_s - \int_t^T \bar{z}_s dW_s, \quad 0 \leq t \leq T; \\ \bar{y}_t \equiv 0, \quad \bar{z}_t \equiv 0, \quad t > T. \end{cases}$$

According to the results of Pardoux and Peng [10], the above BDSDE has a unique solution (\bar{y}, \bar{z}) in $[0, T]$, thus the above BDSDE has a unique solution such that $(\bar{y}, \bar{z}) \equiv (0, 0)$ for every $t > T$. Let

$$y \doteq \bar{y} + \tilde{y}, \quad z \doteq \bar{z} + \tilde{z}.$$

It is easy to check that (y_t, z_t) is the unique solution of (2). □

4. CONTINUOUS DEPENDENCE THEOREM

In this section we will discuss the convergence of solutions of infinite horizon BDSDEs. First we give the following continuous dependence theorem.

Theorem 3. *Suppose $\xi_i \in L^2(\Omega, \mathcal{F}, P; \mathbb{R}^k)$ ($i = 1, 2$), and (H1)–(H3). Let (y^i, z^i) be the solutions of BDSDE (2) corresponding to the terminal data $\xi = \xi_1$, $\xi = \xi_2$, respectively. Then there exists a constant $\bar{C} > 0$ such that*

$$\|(y^1 - y^2, z^1 - z^2)\|_B^2 \leq \bar{C}E|\xi_1 - \xi_2|^2.$$

Proof. Set $\hat{y} := y^1 - y^2$, $\hat{z} := z^1 - z^2$. Since $(\int_0^\infty v(s) ds)^2 + \int_0^\infty u^2(s) ds < \infty$, we can choose a strictly increasing sequence $0 = t_0 < t_1 < \dots < t_n < t_{n+1} = \infty$ such that

$$I_{[t_i, t_{i+1}]} = \left(\int_{t_i}^{t_{i+1}} v(s) ds \right)^2 + \int_{t_i}^{t_{i+1}} u^2(s) ds \leq \frac{1}{2C}, \quad i = 0, 1, \dots, n.$$

Applying Lemma 2, we have

$$\begin{aligned} \|(\hat{y}, \hat{z})I_{[t_i, t_{i+1}]} \|_B^2 &\leq CE|\hat{y}_{t_{i+1}}|^2 + CI_{[t_i, t_{i+1}]} \|(\hat{y}, \hat{z})I_{[t_i, t_{i+1}]} \|_B^2 \\ &\leq CE|\hat{y}_{t_{i+1}}|^2 + \frac{1}{2} \|(\hat{y}, \hat{z})I_{[t_i, t_{i+1}]} \|_B^2. \end{aligned}$$

Thus,

$$\begin{aligned} (11) \quad \|(\hat{y}, \hat{z})I_{[t_i, t_{i+1}]} \|_B^2 &\leq 2CE|\hat{y}_{t_{i+1}}|^2 \\ &\leq 2CE \left(\left(\sup_{t_{i+1} \leq s \leq t_{i+2}} |\hat{y}_s| \right)^2 + \int_{t_{i+1}}^{t_{i+2}} \|\hat{z}_s\|^2 ds \right) \\ &= 2C \|(\hat{y}, \hat{z})I_{[t_{i+1}, t_{i+2}]} \|_B^2, \quad i = 0, 1, \dots, n-1. \end{aligned}$$

In particular, we have

$$(12) \quad \|(\hat{y}, \hat{z})I_{[t_n, \infty]} \|_B^2 \leq 2CE|\xi_1 - \xi_2|^2.$$

From (11) and (12), it follows that

$$\begin{aligned} \|(\hat{y}, \hat{z}) \|_B^2 &\leq \sum_{i=0}^n \|(\hat{y}, \hat{z})I_{[t_i, t_{i+1}]} \|_B^2 \\ &\leq (2C + (2C)^2 + \dots + (2C)^{n+1})E|\xi_1 - \xi_2|^2 \\ &= \frac{2C((2C)^{n+1} - 1)}{2C - 1} E|\xi_1 - \xi_2|^2 = \bar{C}E|\xi_1 - \xi_2|^2. \end{aligned}$$

Thus the desired result is obtained. □

Now we can assert the following convergence theorem for infinite horizon BDSDEs.

Theorem 4. Suppose $\xi, \xi_i \in L^2(\Omega, \mathcal{F}, P; \mathbb{R}^k)$ ($i = 1, 2, \dots$), let (H1)–(H3) hold for f and g . Let (y^i, z^i) be the solutions of the following BDSDE:

$$(13) \quad y_t^i = \xi_i + \int_t^\infty f(s, y_s^i, z_s^i) ds + \int_t^\infty g(s, y_s^i) dB_s - \int_t^\infty z_s^i dW_s, \quad 0 \leq t < \infty.$$

If $E|\xi_i - \xi|^2 \rightarrow 0$ as $i \rightarrow \infty$, then there exists a pair $(y, z) \in B^2$ such that $\|(y^i - y, z^i - z)\|_B \rightarrow 0$ as $i \rightarrow \infty$. Furthermore, (y, z) is the solution of the following BDSDE:

$$(14) \quad y_t = \xi + \int_t^\infty f(s, y_s, z_s) ds + \int_t^\infty g(s, y_s) dB_s - \int_t^\infty z_s dW_s, \quad 0 \leq t < \infty.$$

Proof. For any $n, m \geq 1$, let (y^n, z^n) and (y^m, z^m) be the solutions of (13) corresponding to ξ_n and ξ_m , respectively. Due to Theorem 3, there exists a constant $\bar{C} > 0$ such that

$$\begin{aligned} \|(y^n - y^m, z^n - z^m)\|_B^2 &\leq \bar{C}E|\xi_n - \xi_m|^2 \\ &\leq 2\bar{C}(E|\xi_n - \xi|^2 + E|\xi_m - \xi|^2) \rightarrow 0 \quad \text{as } n, m \rightarrow \infty, \end{aligned}$$

which means that $\{(y^i, z^i), i = 1, 2, \dots\}$ is a Cauchy sequence in B^2 . Thus, there exists a pair $(y, z) \in B^2$ such that $\|(y^i - y, z^i - z)\|_B \rightarrow 0$ as $i \rightarrow \infty$. Hence,

$$\begin{aligned} &E \left| \int_t^\infty (f(s, y_s^i, z_s^i) - f(s, y_s, z_s)) ds \right|^2 \\ &\leq E \left(\int_0^\infty (v(s)|y_s^i - y_s| + u(s)\|z_s^i - z_s\|) ds \right)^2 \\ &\leq 2 \left[\left(\int_0^\infty v(s) ds \right)^2 + \int_0^\infty u^2(s) ds \right] \cdot \|(y^i - y, z^i - z)\|_B^2 \rightarrow 0 \quad \text{as } i \rightarrow \infty, \end{aligned}$$

and

$$\begin{aligned} &E \left| \int_t^\infty (g(s, y_s^i) - g(s, y_s)) dB_s \right|^2 \\ &\leq E \int_0^\infty u^2(s) |y_s^i - y_s|^2 ds \\ &\leq \left(\int_0^\infty u^2(s) ds \right) \cdot \|(y^i - y)\|_S^2 \rightarrow 0, \quad \text{as } i \rightarrow \infty. \end{aligned}$$

Thus, for any $t \in \mathbb{R}^+$, $\int_t^\infty f(s, y_s^i, z_s^i) ds \rightarrow \int_t^\infty f(s, y_s, z_s) ds$ and $\int_t^\infty g(s, y_s^i) dB_s \rightarrow \int_t^\infty g(s, y_s) dB_s$ in $L^2(\Omega, \mathcal{F}, P)$. Taking the limit on both sides of (13), we deduce that (y, z) is the solution to BDSDE (14). The desired result is obtained. \square

The following corollary shows the relation between the solution of infinite horizon BDSDE (2) and of the finite time BDSDE

$$(15) \quad \begin{aligned} y_t = & E^{\mathcal{F}_T}[\xi] + \int_t^T f(s, y_s, z_s) ds + \int_t^T g(s, y_s) dB_s \\ & - \int_t^T z_s dW_s, \quad 0 \leq t \leq T < \infty. \end{aligned}$$

Corollary 5. Assume $\xi \in L^2(\Omega, \mathcal{F}, P; \mathbb{R}^k)$, let (H1)–(H3) hold for f and g . Let (y, z) be the solution of BDSDE (2). For any $T > 0$, let (y^T, z^T) be the solutions of the finite time interval BDSDE (15). Then $(y^T, z^T) \rightarrow (y, z)$ in B^2 as $T \rightarrow \infty$.

Proof. Note that $E^{\mathcal{F}_T}[\xi] \rightarrow \xi$ in $L^2(\Omega, \mathcal{F}, P; \mathbb{R}^k)$ as $T \rightarrow \infty$. The proof is straightforward from Theorem 4. \square

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