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ON THE $H^p$-$L^q$ BOUNDEDNESS OF SOME FRACTIONAL INTEGRAL OPERATORS

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Abstract. Let $A_1, \ldots, A_m$ be $n \times n$ real matrices such that for each $1 \leq i \leq m$, $A_i$ is invertible and $A_i - A_j$ is invertible for $i \neq j$. In this paper we study integral operators of the form

$$Tf(x) = \int k_1(x - A_1y)k_2(x - A_2y) \cdots k_m(x - A_my)f(y)\,dy,$$

$$k_i(y) = \sum_{j \in \mathbb{Z}} 2^{jn/q_i} \varphi_{i,j}(2^j y), 1 \leq q_i < \infty, 1/q_1 + 1/q_2 + \ldots + 1/q_m = 1 - r, 0 \leq r < 1,$$

and $\varphi_{i,j}$ satisfying suitable regularity conditions. We obtain the boundedness of $T$: $H^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ for $0 < p < 1/r$ and $1/q = 1/p - r$. We also show that we can not expect the $H^p$-$H^q$ boundedness of this kind of operators.

Keywords: integral operator, Hardy space

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1. Introduction

In [4] the authors obtain the $L^p$ boundedness, $p > 1$, for a class of maximal operators on the three dimensional Heisenberg group. The operators they consider have relevance in the analysis on $\text{SL}(\mathbb{R}^3)$. Some of them actually arise in the study of the boundary behavior of Poisson integrals on the symmetric space $\text{SL}(\mathbb{R}^3)/\text{SO}(3)$. To obtain the principal results, they analyze the $L^2(\mathbb{R})$ boundedness of integral operators of the form

$$Tf(x) = \int |x - y|^{-\alpha}|x + y|^{\alpha - 1}f(y)\,dy,$$

$0 < \alpha < 1$.

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A natural question is if these operators are also bounded from $L^p(\mathbb{R})$ into $L^q(\mathbb{R})$ for certain $1 < p, q < \infty$, and if this kind of results still hold for larger dimensions or for more general kernels. In this context, in [3] the authors study integral operators on $\mathbb{R}^n$ with kernels of the form

$$k(x, y) = k_1(x - a_1 y)k_2(x - a_2 y)\ldots k_m(x - a_m y),$$

with $a_j \in \mathbb{R} \setminus \{0\}$, $a_i \neq a_j$ for $i \neq j$, $1 \leq i, j \leq m$ and

$$k_i(y) = \sum_{j \in \mathbb{Z}} 2^{jn/q_i}\varphi_{i,j}(2^j y),$$

for certain functions $\varphi_{i,j}$ satisfying some regularity properties. They obtain that this operator is bounded from $L^p(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$ for $1 < p < 1/r$ and $1/q = 1/p - r$.

Now we consider the following natural generalization of these operators. For $n, m \in \mathbb{N}$, let $A_1, \ldots, A_m$ be real $n \times n$ matrices such that for each $1 \leq i \leq m$, $A_i$ is invertible and $A_i - A_j$ is invertible if $i \neq j$. Let $m > 1$, $q_1, \ldots, q_m$ be real numbers, $1 < q_i < \infty$ such that

$$\frac{1}{q_1} + \frac{1}{q_2} + \ldots + \frac{1}{q_m} = 1 - r$$

for some $0 \leq r < 1$. If $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multiindex, we denote $|\alpha| = \alpha_1 + \ldots + \alpha_n$, and $D^\alpha = \partial^{|\alpha|}/\partial y_1^{\alpha_1} \ldots \partial y_n^{\alpha_n}$. For $1 \leq i \leq m$ let $\{\varphi_{i,j}\}_{j \in \mathbb{Z}}$ be a family of smooth and non negative real functions defined on $\mathbb{R}^n$, such that

$$\text{supp}(\varphi_{i,j}) \subset \{y \in \mathbb{R}^n : 2^{-1} \leq |y| \leq 2\}$$

and such that for each multiindex $\alpha = (\alpha_1, \ldots, \alpha_n)$ there exists $M_\alpha$ such that

$$\sup_{j \in \mathbb{Z}} \|D^\alpha \varphi_{i,j}\|_\infty \leq M_\alpha.$$

Let

$$k(x, y) = k_1(x - A_1 y)k_2(x - A_2 y)\ldots k_m(x - A_m y),$$

with

$$k_i(y) = \sum_{j \in \mathbb{Z}} 2^{jn/q_i}\varphi_{i,j}(2^j y),$$

and let $T$ be the integral operator with kernel $k(x, y)$, i.e.

$$T f(x) = \int k(x, y) f(y) \, dy.$$

We observe that if $\varphi_{i,j} = \varphi_{i,k}$ for all $j, k \in \mathbb{Z}$ then $k_i(2^s y) = 2^{-sn/q_i} k_i(y)$. So $k_i$ is “homogeneous” of degree $-n/q_i$ and then the “homogeneity degree” of $k$ is $-n(1-r)$. 626
The Hardy-Littlewood-Sobolev theorem shows that the Riesz potential operator $I_{nr}$, with kernel $1/|y|^{n(1-r)}$, is bounded from $L^p(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$, for $0 < r < 1$, $1 < p < 1/r$ and $1/q = 1/p - r$. Also for the endpoint cases, it is known that $I_{nr}$ is not bounded from $L^1$ into $L^{1/(1-r)}$ and neither from $L^{1/r}(\mathbb{R}^n)$ into $L^{\infty}(\mathbb{R}^n)$ (See [6], p. 119). In 1960 E. Stein and G. Weiss [8] used the theory of harmonic functions of several variables to prove that these operators are bounded from $H^1(\mathbb{R}^n)$ to $L^{1/(1-r)}(\mathbb{R}^n)$ and in 1980 M. Taibleson and G. Weiss, using the molecular characterization of the real Hardy spaces, obtained the boundedness of these operators from $H^p(\mathbb{R}^n)$ into $H^q(\mathbb{R}^n)$, where $0 < p < 1$ and $1/q = 1/p - r$ (see [9]).

Also in [1] the authors obtain the $H^p(\mathbb{R}^n) - L^q(\mathbb{R}^n)$ boundedness, $n/(n + \alpha) \leq p < 1$, $1/q = 1/p - \alpha/n$, for the homogeneous fractional convolution operators $T_{\Omega, \alpha}$ given by

$$T_{\Omega, \alpha} f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x - y)}{|x - y|^{n - \alpha}} f(y) dy,$$

where $0 < \alpha < n$, $\Omega$ is homogeneous of degree zero on $\mathbb{R}^n$ with $\Omega \in L^s(S^{n-1})$, $s > 1$.

In [5] we obtain the $H^p(\mathbb{R}^n) - L^p(\mathbb{R}^n)$ boundedness, $0 < p \leq 1$, of integral operators with kernels of the form

$$k(x, y) = |x - a_1 y|^{-\alpha_1} \ldots |x - a_m y|^{-\alpha_m},$$

where $a_i \neq a_j$ for $i \neq j$, $m > 1$ and $\alpha_1 + \ldots + \alpha_m = n$ and we also show that we can not expect the $H^p(\mathbb{R}^n)$ boundedness of them. These kernels can be expressed as in (1), with $r = 0$.

In this paper we obtain the $H^p(\mathbb{R}^n) - L^q(\mathbb{R}^n)$ boundedness of the operator $T$ defined by (2), for $0 < p < 1/r$ and $1/q = 1/p - r$. By duality we obtain the corresponding $L^{1/r}(\mathbb{R}^n) \to \text{BMO}(\mathbb{R}^n)$ boundedness. Also, in the last section, for each $0 < r < 1$ we give an example of an operator $T_r$ on $H^p(\mathbb{R})$, having a kernel of the form (3) with $m = 2$ and $\alpha_1 + \alpha_2 = 1 - r$, that is not bounded from $H^p(\mathbb{R})$ into $H^q(\mathbb{R})$ for $0 < p \leq 1/(1 + r)$ and $1/q = 1/p - r$.

Throughout this paper, $c$ will denote a positive constant not necessarily the same at each occurrence.

2. Preliminary results

We note that the condition $1/q = 1/p - r$, $1 < p < 1/r$ is necessary for the boundedness from $L^p(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$ of certain subfamily of operators of the form (2).

Remark 1. A standard homogeneity argument shows that if an operator with general kernel $k$ with “homogeneity degree” $-n(1 - r)$ is bounded from $L^p(\mathbb{R}^n)$ into
for some \(1 < p, q < \infty\), then \(1/q = 1/p - r\). Now for \(l \in \mathbb{Z}\), let \(T^l\) be the integral operator with kernel \(k^l = k_1^l(x - A_1y) \ldots k_m^l(x - A_my)\), where \(k_i^l(y) = \sum_{j \in \mathbb{Z}} 2^{jn/q_i} \varphi_{i,j}(2^j y)\). If for each \(1 \leq i \leq m\), \(\varphi_{i,j} = \varphi_{i,k}\) for all \(j, k \in \mathbb{Z}\) then \(T^l = T\).

Also, if all the operators \(T^l\) are bounded from \(L^p(\mathbb{R}^n)\) into \(L^q(\mathbb{R}^n)\) for some \(1 < p, q < \infty\), and \(0 < \sup \|T^l\|_{p,q} \leq C < \infty\), then \(1/q = 1/p - r\). Indeed for \(l \in \mathbb{Z}\) we denote \(f_i(x) = 2^{-ln} f(2^{-l} x)\) then

\[
T(f_i)(x) = 2^{-ln(1-r)} T^l f(2^{-l} x),
\]

so

\[
\|T f\|_q = \|T((f-\mathbb{1})_i)\|_q \leq 2^{-ln(1-r)+nl/q} \|T^l (f-\mathbb{1})\|_q \\
\leq C 2^{-ln(1-r)+nl/q} \|f-\mathbb{1}\|_p = C 2^{-ln(1/q-1/p+r)} \|f\|_p
\]

and then \(1/q - 1/p + r = 0\).

With respect to the endpoint \((p,q) = (1,1/(1-r))\) and \((p,q) = (1/r,0)\), as in the case of the Riesz potentials, we can not expect \(L^p(\mathbb{R}^n) - L^q(\mathbb{R}^n)\) boundedness. For the first one we take \(f = \chi_B\) the characteristic function of the unit ball of \(\mathbb{R}^n\) and \(k(x,y) = 1/|x - A_1y|^{n/q_1} \ldots 1/|x - A_my|^{n/q_m}\). A simple computation shows that for \(|x| \gg 1\), \(T f(x) \geq c/|x|^{n(1-r)}\) and then \(T f \notin L^{1/(1-r)}\). The second case follows by duality.

**Lemma 1.** If \(k(x,y)\) is the kernel defined by (1) and \(\alpha = (\alpha_1, \ldots, \alpha_n)\) is a multi-index then

\[
\left| \frac{\partial^{|\alpha|}}{\partial y_1^{\alpha_1} \ldots \partial y_n^{\alpha_n}} k(x,y) \right| \leq c \left( \prod_{i=1}^{m} |x - A_iy|^{-\frac{\alpha_i}{q_i}} \right) \left( \sum_{i=1}^{m} |x - A_iy|^{-\alpha_i} \right)^{|\alpha|}
\]

with \(c\) independent of \(x, y\).

**Proof.** We denote \(D^\alpha_y = \partial^{|\alpha|}/\partial y_1^{\alpha_1} \ldots \partial y_n^{\alpha_n}\). By the Leibniz formula,

\[
D^\alpha_y k(x,y) = D^\alpha_y \left( \prod_{1 \leq i \leq m} k_i(x - A_i y) \right) \\
= \sum_{\Gamma_1 + \ldots + \Gamma_m = \alpha} c_{\Gamma_1, \ldots, \Gamma_m} D_{y_1}^{\Gamma_1} (k_1(x - A_1 y)) \ldots D_{y_m}^{\Gamma_m} (k_m(x - A_m y)),
\]

now

\[
k_i(x - A_i y) = \sum_{j \in \mathbb{Z}} 2^{jn/q_i} \varphi_{i,j}(2^j(x - A_i y)).
\]
For each fixed \( x \) only a finite number of \( j \)'s (independent of \( x \)) are involved in the above sum, also \( 2^j \ll 2|x - A_i y|^{-1} \) for \( 2^j (x - A_i y) \in \text{supp} \varphi_{i,j} \), also sup \( \|D^\alpha \varphi_{i,j}\|_\infty \ll \infty \), so

\[
|D_y^{\Gamma_i}(k_i(x - A_i y))| = \left\| \sum_{j \in \mathbb{Z}} 2^{jn/q_i} D_y^{\Gamma_i}(\varphi_{i,j}(2^j(x - A_i y))) \right\| \leq c|x - A_i y|^{-n/q_i - |\Gamma_i|}
\]

thus

\[
|D_y^\alpha k(x, y)| \leq c \sum_{\Gamma_1 + \ldots + \Gamma_m = \alpha} c_{\Gamma_1, \ldots, \Gamma_m} \prod_{1 \leq i \leq m} |x - A_i y|^{-n/q_i - |\Gamma_i|}
\]

\[
= c \left( \prod_{1 \leq i \leq m} |x - A_i y|^{-n/q_i} \right) \left( \sum_{\Gamma_1 + \ldots + \Gamma_m = \alpha} c_{\Gamma_1, \ldots, \Gamma_m} \prod_{1 \leq i \leq m} |x - A_i y|^{-|\Gamma_i|} \right)
\]

\[
\leq c \left( \prod_{1 \leq i \leq m} |x - A_i y|^{-n/q_i} \right) \left( \sum_{1 \leq l \leq m} |x - A_l y|^{-1} \right)^{|\alpha|}.
\]

\[
\Box
\]

3. The main results

As we have said in the introduction, in the case that \( A_i \) is a multiple of the identity, in [3] the authors obtain that \( T \) is well defined on \( L^p(\mathbb{R}^n) \) and that it is bounded from \( L^p(\mathbb{R}^n) \) into \( L^q(\mathbb{R}^n) \) for \( 1 < p < 1/r \) and \( 1/q = 1/p - r \). We will show that with slight modifications on the proofs, this result still holds for \( A_i \) satisfying the above stated hypothesis.

**Proposition 2.** Let \( T \) be the operator defined by (2). If \( 1 < p < 1/r \), \( 0 \leq r < 1 \) and \( 1/q = 1/p - r \), then \( T \) is a well defined and bounded operator from \( L^p(\mathbb{R}^n) \) into \( L^q(\mathbb{R}^n) \).

**Proof.** As in the proof of Lemma 2.1 in [3] we obtain that for \( l \in \mathbb{Z}, 1/(1-r) < p \leq \min_{1 \leq i \leq m} p_i/q_i(1-r) \)

\[
\left\| \sum_{s_1, \ldots, s_m \leq -l} \prod_{1 \leq i \leq m} 2^{s_i n/q_i} \varphi_{i,s_i}(2^{s_i}(x - A_i y)) \right\|_{L^p(dy)} \leq c 2^{nl/p},
\]

and also as in the proof of Lemma 2.2 in the same paper,

\[
\left\| \sum_{s_i \geq -l} 2^{s_i n/q_i} \varphi_{i,s_i}(2^{s_i}(x - A_i y)) \prod_{j \neq i} 2^{-l n/q_j} \varphi_{j,-l}(2^{-l}(x - A_j y)) \right\|_{L^p(dy)} \leq c,
\]

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with $c$ independent of $x$ and $l$. Now we follow the proof of Theorem 3.1 in [3] with the following changes. We take

$$d = \min_{1 \leq i \leq m} \left( \min_{|y|=1} \frac{|A_i(y)|}{2}, \min_{|y|=1, j \neq i} \frac{|A_i(y) - A_j(y)|}{2} \right)$$

and

$$D = \max_{1 \leq i \leq m, |y|=1} |A_i(y)|,$$

for $x \in \mathbb{R}^n \setminus \{0\}$ we define $l = l(x)$ such that $2^l \leq |x| \leq 2^{l+1}$ and we set, for $1 \leq i \leq m$,

$$R_i = R_i(x) = \{ y \in \mathbb{R}^n : |y - A_i(x)| \leq 2^l d \},$$

we also set

$$R_{m+1} = \{ y \in \mathbb{R}^n : |y| \leq 2^l D \} \cap \left( \bigcup_{1 \leq i \leq m} R_i \right)^c \quad \text{and} \quad R_{m+2} = \left( \bigcup_{1 \leq i \leq m+1} R_i \right)^c.$$

\[
\square
\]

Let $0 < p \leq 1$. We recall that a $p$-atom is a measurable function $a$ supported on a ball $B$ of $\mathbb{R}^n$ satisfying

a) $\|a\|_\infty \leq \|B\|^{-1/p}$,

b) $\int y^\beta a(y) \, dy = 0$ for every multiindex $\beta$ with $|\beta| \leq np^{-1} - 1$.

It is well known that for $0 < p \leq 1$ the distributions of $H^p(\mathbb{R}^n)$ can be approximated by adequate linear combinations of $p$-atoms. (See Theorem 2, p. 107 in [7].)

**Theorem 3.1.** Let $T$ be the operator defined by (2). If $0 \leq r < 1$, $0 < p \leq 1$ and $1/q = 1/p - r$, then $T$ is a bounded operator from $H^p(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$.

**Proof.** If $0 \leq r < 1$, $0 < p \leq 1$, $1/q = 1/p - r$ and $f \in H^p(\mathbb{R}^n)$ we write $f = \sum_{j \in \mathbb{N}} \lambda_j a_j$, where $a_j$ is a $p$-atom and $\sum_{j \in \mathbb{N}} |\lambda_j|^p \leq c \|f\|_{H^p}^p$. So the theorem will be proved if we obtain that there exists $c > 0$ such that $\|T a\|_{L^q} \leq c$ with $c$ independent of the $p$-atom $a$, since this estimate and the inequality $\left( \sum_{j \in \mathbb{N}} |\lambda_j|^q \right)^{1/q} \leq \left( \sum_{j \in \mathbb{N}} |\lambda_j|^p \right)^{1/p}$ give $\|T f\|_q \leq c \|f\|_{H^p}$. We denote by $B(y_0, \delta)$ the closed ball centered at $y_0$ with radius $\delta$. Let $a$ be supported on a ball $B = B(y_0, \delta)$, and for each $1 \leq i \leq m$ let $B^*_i = B(A_i y_0, 4D \delta)$ with $D$ defined as in the proof of Proposition 2. We decompose $\mathbb{R}^n = \bigcup_{1 \leq i \leq m} B^*_i \cup R$, where $R = \left( \bigcup_{1 \leq i \leq m} B^*_i \right)^c$. Proposition 2 gives that $T$ is bounded
from $L^{p_0}(\mathbb{R}^n)$ into $L^{q_0}(\mathbb{R}^n)$ for $1/q_0 = 1/p_0 - r$, $1 < p_0 < 1/r$. Since $q < q_0$ we use the Hölder inequality with $q_0/q$ and $q_0/(q_0 - q)$ to obtain

$$\int \bigcup_{1 \leq i \leq m} B_i^* |Ta(x)|^q \, dx \leq \sum_{1 \leq i \leq m} \int_{B_i^*} |Ta(x)|^q \, dx$$

$$\leq c \sum_{1 \leq i \leq m} |B_i^*|^{1 - q/q_0} \|Ta\|_{q_0}^q \leq c\delta^{n - nq/q_0} \|a\|_{p_0}^q$$

$$\leq c\delta^{n - nq/q_0} \left( \int_{B} |a|^{p_0} \right)^{q/p_0} \leq c\delta^{n - nq/q_0} \delta^{-nq/r} \delta^{nq/p_0} = c.$$ 

To study the integral on

$$R = \{x \in \mathbb{R}^n : |x - A_i y_0| > 4\delta, \text{ for all } 1 \leq i \leq m\},$$

we suppose $n/(n + N) < p \leq n/(n + N - 1)$ for some $N \in \mathbb{N}$. Let $k(x, y)$ be defined by (1). The moment condition $b)$ satisfied by the $p$-atom $a$ allows us to write

(4) $$\int_R \left| \int_B k(x, y)a(y) \, dy \right|^q \, dx = \int_R \left| \int_B (k(x, y) - q_N(x, y))a(y) \, dy \right|^q \, dx$$

where $q_N(x, y)$ is the degree $N - 1$ Taylor polynomial of the function $y \to k(x, y)$ expanded around $y_0$. By the standard estimate of the remainder term in the Taylor expansion, there exists $\xi$ between $y$ and $y_0$ such that

$$|k(x, y) - q_N(x, y)| \leq c|y - y_0|^N \sum_{k_1 + \ldots + k_n = N} \left| \frac{\partial^N}{\partial y_1^{k_1} \ldots \partial y_n^{k_n}} k(x, \xi) \right|$$

$$\leq c|y - y_0|^N \left( \prod_{i=1}^m |x - A_i \xi|^{-n/q_i} \right) \left( \sum_{i=1}^m |x - A_i \xi|^{-1} \right)^N,$$

where the last inequality follows from Lemma 1. Since $x \in R$ and $y \in B$, it follows that $|x - A_i \xi| \geq c|x - A_i y_0|$ for $1 \leq i \leq m$. So

(5) $$|k(x, y) - q_N(x, y)| \leq c|y - y_0|^N \left( \prod_{i=1}^m |x - A_i y_0|^{-n/q_i} \right) \left( \sum_{i=1}^m |x - A_i y_0|^{-1} \right)^N.$$ 

For $1 \leq k \leq m$, let

$$R_k = \{x \in R : |x - A_k y_0| \leq |x - A_j y_0| \text{ for all } j \neq k\}.$$
We note that \( R = \bigcup_{k=1}^{m} R_k \) and that \( R_k \subseteq (B_k^*)^c \). So, from (4) and (5), we have

\[
\int_R \left| \int_B k(x, y) a(y) \, dy \right|^q \, dx 
\leq c \int_R \left( \int_B \left( \prod_{i=1}^{m} |x - A_i y_0|^{-n/q_i} \right) \left( \sum_{l=1}^{m} |x - A_l y_0|^{-1} \right)^N |y - y_0|^{N} |a(y)| \, dy \right)^q \, dx 
\leq c \sum_{1 \leq k \leq m} \int_{R_k} \left( \prod_{i=1}^{m} |x - A_i y_0|^{-q_{n/q_i}} \left( \sum_{l=1}^{m} |x - A_l y_0|^{-1} \right)^q N \right) \left( \int_B |y - y_0|^{N} |a(y)| \, dy \right)^q \, dx 
\leq c \sum_{1 \leq k \leq m} \int_{(B_k^*)^c} \left( \int_B |y - y_0|^{N} |a(y)| \, dy \right)^q |x - A_k y_0|^{-q(1-r)(m|x - A_k y_0|^{-1})} \, dx 
\leq c \sum_{1 \leq k \leq m} \delta^q N^{-nq/p+nq} \int_{4D\delta}^\infty t^{-q(n(1-r)+N)+n-1} \, dt = c,
\]

with \( c \) independent of the \( p \)-atom \( a \), since \(-q(n(1-r)+N) + n < 0\). \( \square \)

We recall that a locally integrable function \( f \) belongs to BMO(\( \mathbb{R}^n \)) if the inequality

\[
\frac{1}{|B|} \int_B |f(x) - f_B| \, dx \leq A
\]

holds for all balls \( B \subset \mathbb{R}^n \); here \( f_B = |B|^{-1} \int_B f \, dx \). The dual result to the previous theorem, corresponding to the case \( p = 1 \), is the following.

**Corollary 3.** Let \( T \) be the operator defined by (2). Then \( T \) is bounded from \( L^{1/r}(\mathbb{R}^n) \) into BMO(\( \mathbb{R}^n \)) for \( 0 \leq r < 1 \).

**Proof.** It is well known that the dual space of \( H^1(\mathbb{R}^n) \) is the space BMO(\( \mathbb{R}^n \)). Let \( \widetilde{T} \) be the integral operator with kernel \( \widetilde{k}(x, y) = \widetilde{k}_1(x - A_1^{-1} y) \ldots \widetilde{k}_m(x - A_m^{-1} y) \), with \( \widetilde{k}_i(x) = k_i(A_i x) \). Since for each \( 1 \leq i \leq m \), it can be checked that \( A_i^{-1} \) is invertible and \( A_i^{-1} - A_j^{-1} \) is invertible if \( i \neq j \), the previous theorem gives us the boundedness of \( \widetilde{T} \) from \( H^1(\mathbb{R}^n) \) into \( L^{1/(1-r)} \). Now it is easy to check that \( T \) is the adjoint operator of \( \widetilde{T} \), so the corollary follows. \( \square \)
4. A COUNTEREXAMPLE

In this section we show that we can not expect that operators of the form (2) be bounded from $H^p(\mathbb{R})$ into $H^q(\mathbb{R})$ with $0 < p \leq 1/(1 + r)$ and $1/q = 1/p - r$.

For $n = 1$ and $0 < r < 1$ we consider the integral operator

$$
T_r f(x) = \int \frac{f(y) \, dy}{|x - y|^{(1-r)/2} |x + y|^{(1-r)/2}},
$$

we will show that for a given 1-atom $a$, $\int T_r a(x) \, dx \neq 0$.

We observe that $T_r a \in L^1(\mathbb{R})$ and that $\int T_r a(x) \, dx = (T_r a)(0)$, where the Fourier transform of a integrable function $f$ is given by $\widehat{f}(\xi) = \int_\mathbb{R} e^{-ix\xi} f(x) \, dx$. Thus it is enough to show that $(T_r a)(0) \neq 0$. Let $\varphi \in S(\mathbb{R})$ be an even function such that $\varphi(0) = 1$ and for $\varepsilon > 0$ let $\varphi_\varepsilon(x) = \varphi(\varepsilon x)$. Now $(T_r a)(0) = \lim_{\varepsilon \to 0} (\varphi_\varepsilon T_r a)(0)$ so we will compute

$$
(\varphi_\varepsilon T_r a)(0) = \int \varphi(\varepsilon x) \left( \int |x^2 - y^2|^{(r-1)/2} a(y) \, dy \right) \, dx
= \int a(y) \left( \int |x^2 - y^2|^{(r-1)/2} \varphi(\varepsilon x) \, dx \right) \, dy
= \int a(y) |y|^r \left( \int |z^2 - 1|^{(r-1)/2} \varphi(\varepsilon |z|) \, dz \right) \, dy
= \int a(y) |y|^r \left( \int (|z^2 - 1|^{(r-1)/2}) (\varphi_{\varepsilon |y|}(|\sigma|) \, d\sigma \right) \, dy.
$$

Since $-\frac{1}{2} < -\frac{1}{2}r < 0$, the Fourier transform of the function $|z^2 - 1|^{(r-1)/2}$ is

$$
\Gamma \left( \frac{r + 1}{2} \right) \sqrt{\pi} \left( \frac{\sigma}{2} \right)^{-r/2} J_{r/2}(\sigma) + \left| \frac{\sigma}{2} \right|^{-r/2} \left( \frac{\cos(\pi r/2) J_{-r/2}(|\sigma|) - J_{r/2}(|\sigma|)}{\sin(\pi r/2)} \right),
$$

where

$$
J_p(s) = \frac{2(s/2)^p}{\Gamma(p + \frac{1}{2}) \sqrt{\pi}} \int_0^1 (1 - t^2)^{p - \frac{1}{2}} \cos(st) \, dt
$$

is the Bessel function of order $p > -\frac{1}{2}$ (see p. 185–188 in [2]). So

$$
(\varphi_\varepsilon T_r a)(0)
= c_r \int a(y) \int |\varepsilon \sigma|^{-r} \left( \int_0^1 (1 - t^2)^{(r-1)/2} \cos(\varepsilon |y||\sigma|t) \, dt \right) \widehat{\varphi}(\sigma) \, d\sigma \, dy
+ 2 \left( 1 - \frac{1}{\sin(\pi r/2)} \right) \int a(y) |y|^r \int_0^1 (1 - t^2)^{(r-1)/2} \cos(\varepsilon |y||\sigma|t) \, dt \widehat{\varphi}(\sigma) \, d\sigma \, dy,
$$

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thus it is easy to check that
\[
\lim_{\varepsilon \to 0} (\varphi_\varepsilon T_r a)(0) = 2 \left( 1 - \frac{1}{\sin(\pi r/2)} \right) \int_0^1 (1 - t^2)^{(r-1)/2} dt \int a(y)|y|^r dy.
\]
We take the 1-atom
\[
a_\delta(y) = \begin{cases} 2\delta & \text{for } -\frac{1}{2} \leq y \leq 0, \\ -\delta & \text{for } 0 < y \leq 1 \end{cases}
\]
with \(0 < \delta \leq \frac{1}{3}\). A computation shows that \(\int a_\delta(y)|y|^r dy = \delta(2^{-r} - 1)/(r + 1)\), so
\[
\int T_r a_\delta(x) \, dx = (T_r a_\delta)(0) = 2\delta \frac{2^{-r} - 1}{r + 1} \left( 1 - \frac{1}{\sin(\pi r/2)} \right) \int_0^1 (1 - t^2)^{(r-1)/2} dt \neq 0.
\]
We note that
\[
\lim_{r \to 0} \int T_r a_\delta(x) \, dx = 2\delta \ln(2) = \int T_0 a_\delta(x) \, dx,
\]
where the last equality is computed in [5]. Also \(a_\delta \in H^p(\mathbb{R})\) for \(\frac{1}{2} < p \leq 1/(1 + r)\), and \(T_r a_\delta\) does not belong to \(H^q(\mathbb{R})\) for \(1/q = 1/p - r\) since \(\int T_r a_\delta \neq 0\). For \(0 < p \leq \frac{1}{2}\) we take \(N\) any fixed integer with \(N > p^{-1} - 1\), then the set of all bounded, compactly supported functions for which \(\int_\mathbb{R} x^\alpha f(x) \, dx = 0\) for all \(\alpha\) with \(0 \leq \alpha < N\), is dense in \(H^p(\mathbb{R})\) (see 5.2b), p. 128 in [7]). In particular, there exists \(b \in H^p(\mathbb{R})\) such that \(\|a_\delta - b\|_{H^{1/(1+r)}(\mathbb{R})} < |(T_r a_\delta)(0)|/2c\). Then
\[
\left| \int T_r b(x) \, dx \right| \geq \left| \int T_r a_\delta(x) \, dx \right| - \int |T_r b(x) - T_r a_\delta(x)| \, dx \\
\geq |(T_r a_\delta)(0)| - c\|a_\delta - b\|_{H^{1/(1+r)}(\mathbb{R})} \geq \frac{|(T_r a_\delta)(0)|}{2},
\]
where the second inequality follows from Theorem 3.1 with \(p = 1/(1 + r)\). But then \(T_r\) is not bounded on \(H^p(\mathbb{R})\) into \(H^q(\mathbb{R})\) for \(1/q = 1/p - r\), since \(\int T_r b(x) \, dx \neq 0\).

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