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ON GENERALIZED MOSER-TRUDINGER INEQUALITIES WITHOUT BOUNDARY CONDITION

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Abstract. We give a version of the Moser-Trudinger inequality without boundary condition for Orlicz-Sobolev spaces embedded into exponential and multiple exponential spaces. We also derive the Concentration-Compactness Alternative for this inequality. As an application of our Concentration-Compactness Alternative we prove that a functional with the sub-critical growth attains its maximum.

Keywords: Orlicz space, Orlicz-Sobolev space, embedding theorem, sharp constant, Moser-Trudinger inequality, concentration-compactness principle

MSC 2010: 46E35, 46E30, 26D10, 49J99

1. Introduction

Throughout the paper $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$, is an open bounded connected domain from the class $C^{1,\theta}$ for some $\theta \in (0,1]$, ω_{n-1} denotes the surface of the unit sphere and $\mathcal{L}_n$ is the n-dimensional Lebesgue measure.

If $\Omega \subset \mathbb{R}^n$ is an open bounded set and $W_0^{1,p}(\Omega)$ denotes the usual completion of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$, then it is well known that

$$\begin{aligned} &W_0^{1,p}(\Omega) \subset L^{np/(n-p)}(\Omega) \quad \text{if } 1 \leqslant p < n,\\ &W_0^{1,p}(\Omega) \subset L^\infty(\Omega) \quad \text{if } n < p. \end{aligned}$$

In the borderline case $p = n$ we have

$$\begin{aligned} &W_0^{1,n}(\Omega) \subset L^q(\Omega) \quad \text{for every } q \in [1,\infty),\\ &W_0^{1,n}(\Omega) \not\subset L^\infty(\Omega). \end{aligned}$$

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This case is studied more precisely by Trudinger [22] who showed that

$$W_0^{1,n}(\Omega) \subset L^{\Phi}(\Omega),$$

where $L^{\Phi}(\Omega)$ is the Orlicz space corresponding to the Young function $\Phi(t) = \exp(t^{n/(n-1)}) - 1$. Moreover, for the functions from $W_0^{1,n}(\Omega)$ there is also the famous Moser-Trudinger inequality [19]

(1.1)
$$\sup_{\|\nabla u\|_{L^n(\Omega)}\leqslant 1} \int_{\Omega} \exp(K|u|^{n/(n-1)})\,\mathrm{d}x \begin{cases} \leqslant C(n,K,\mathcal{L}_n(\Omega)) & \text{when } K \leqslant n\omega_{n-1}^{1/(n-1)}, \\ = \infty & \text{when } K > n\omega_{n-1}^{1/(n-1)}. \end{cases}$$

An important extension of inequality (1.1) is its version for the space $W^{1,n}(\Omega)$ where $\Omega \subset \mathbb{R}^n$ is a bounded connected domain from the class $C^{1,\theta}$, $\theta \in (0,1]$, given in [8] (see also [6]). In such a version the borderline parameter $n\omega_{n-1}^{1/(n-1)}$ in (1.1) turns to $n(\frac{1}{2}(\omega_{n-1}))^{1/(n-1)}$.

The aim of this paper is to obtain an analogue of the above result for Orlicz-Sobolev spaces embedded into exponential and multiple exponential spaces.

If $l \in \mathbb{N}$ and $\alpha < n-1$, we set

(1.2)
$$\gamma = \frac{n}{n-1-\alpha} > 0, \qquad B = 1 - \frac{\alpha}{n-1} = \frac{n}{(n-1)\gamma} > 0$$
$$\text{and} \quad K_{l,n,\alpha} = \begin{cases} B^{1/B} n\omega_{n-1}^{\gamma/n} & \text{for } l = 1 \\ B^{1/B} \omega_{n-1}^{\gamma/n} & \text{for } l \geqslant 2. \end{cases}$$

The following is known, if Ω is an open bounded set. The space $W_0 L^n \log^{\alpha} L(\Omega)$ of the Sobolev type, modeled on the Zygmund space $L^n \log^{\alpha} L(\Omega)$, is continuously embedded into the Orlicz space with the Young function that behaves like $\exp(t^{\gamma})$ for large t (see [16] and [10]). Moreover, it is shown in [10] (see also [7] and [11]) that in the limiting case $\alpha = n-1$ we have the embedding into a double exponential space, i.e. the space $W_0 L^n \log^{n-1} L \log^{\alpha} \log L(\Omega)$, $\alpha < n-1$, is continuously embedded into the Orlicz space with the Young function that behaves like $\exp(\exp(t^{\gamma}))$ for large t. Further, in the limiting case $\alpha = n-1$ we have the embedding into a triple exponential space and so on. The borderline case is always $\alpha = n-1$ and for $\alpha > n-1$ we have the embedding into $L^{\infty}(\Omega)$. It is well known that the Zygmund space $L^n \log^{\alpha} L(\Omega)$ coincides with the Orlicz space $L^{\Phi}(\Omega)$, where

$$\lim_{t\to\infty} \frac{\Phi(t)}{t^n \log^{\alpha}(t)} = 1,$$

the space $L^n \log^{n-1} L \log^{\alpha} \log L(\Omega)$ coincides with $L^{\Phi}(\Omega)$ where

$$\lim_{t\to\infty} \frac{\Phi(t)}{t^n \log^{n-1}(t) \log^{\alpha}(\log(t))} = 1,$$

and so on (see for example [20, Lemma 8.1]). For other results concerning these spaces we refer the reader to [11], [12], [13], [14], [15] and [20].

The following notation enables us to deal with the multiple exponential spaces comfortably. For $j \in \mathbb{N}$, let us write

$$\log_{[j]}(t) = \log(\log_{[j-1]}(t)), \quad \text{where } \log_{[1]}(t) = \log(t)$$

and

$$\exp_{[j]}(t) = \exp(\exp_{[j-1]}(t)), \quad \text{where } \exp_{[1]}(t) = \exp(t).$$

Let $l \in \mathbb{N}$ and $\alpha < n-1$. Then we have the above mentioned embedding results for any Young function Φ satisfying

$$\lim_{t\to\infty} \frac{\Phi(t)}{t^n \Big(\prod_{j=1}^{l-1} \log_{[j]}^{n-1}(t)\Big) \log_{[l]}^{\alpha}(t)} = 1 \tag{1.3}$$

(for $l = 1$ we read (1.3) as $\lim_{t\to\infty} \Phi(t)/(t^n \log_{[1]}^{\alpha}(t)) = 1$). As Ω is bounded, all Young functions satisfying (1.3) give the same Orlicz-Sobolev space.

Moser-type results. The following theorem summarizes the known versions of (1.1) for embedding into single and multiple exponential spaces in the case of functions vanishing on the boundary (and without any assumption concerning the regularity of the boundary).

Theorem 1.1. *Let $K \geqslant 0$, $l \in \mathbb{N}$, $n \geqslant 2$ and $\alpha < n-1$. Suppose that $\Omega \subset \mathbb{R}^n$ is an open bounded set. Let Φ be a Young function satisfying* (1.3).

(i) *If $u \in W_0L^{\Phi}(\Omega)$, then*

$$\int_\Omega \exp_{[l]}(K|u(x)|^{\gamma})\,\mathrm{d}x < \infty.$$

(ii) *If $K < K_{l,n,\alpha}$ and $u \in W_0L^{\Phi}(\Omega)$ with $\|\Phi(\nabla u)\|_{L^1(\Omega)} \leqslant 1$, then*

$$\int_\Omega \exp_{[l]}(K|u(x)|^{\gamma})\,\mathrm{d}x \leqslant C(l,n,\alpha,\Phi,\mathcal{L}_n(\Omega),K).$$

(iii) *If $K > K_{l,n,\alpha}$, then*

$$\sup_{u\in W_0L^{\Phi}(\Omega),\|\Phi(\nabla u)\|_{L^1(\Omega)}\leqslant 1} \int_\Omega \exp_{[l]}(K|u(x)|^{\gamma})\,\mathrm{d}x = \infty.$$

(iv) *Suppose that* $K = K_{l,n,\alpha}$ *and there are* $a \in (0, \min\{1, 1/\gamma\})$ *and* $t_0 \geqslant \exp_{[l]}(1)$ *such that* Φ *satisfies*

$$\Phi(t) \geqslant t^n \left(\prod_{j=1}^{l-1} \log_{[j]}^{n-1}(t)\right) \log_{[l]}^{\alpha}(t)(1 + \log_{[l]}^{-a}(t)) \quad \textit{for } t \in [t_0, \infty) \tag{1.4}$$

and $u \in W_0 L^{\Phi}(\Omega)$ *with* $\|\Phi(\nabla u)\|_{L^1(\Omega)} \leqslant 1$. *Then*

$$\int_{\Omega} \exp_{[l]}(K|u(x)|^{\gamma})\, \mathrm{d}x \leqslant C(l, n, \alpha, \Phi, \mathcal{L}_n(\Omega)).$$

(v) *Suppose that* $K = K_{l,n,\alpha}$ *and there are* $t_0 \geqslant \exp_{[l]}(1)$, $a \in (0, \min\{1, B\})$ *and* $C > 0$ *such that*

$$\Phi(t) \leqslant \begin{cases} Ct^n & \textit{for } t \in [0, t_0], \\ t^n \Big(\prod_{j=1}^{l-1} \log_{[j]}^{n-1}(t)\Big) \log_{[l]}^{\alpha}(t)(1 - \log_{[l]}^{-a}(t)) & \textit{for } t \in [t_0, \infty). \end{cases} \tag{1.5}$$

Then

$$\sup_{u \in W_0 L^{\Phi}(\Omega), \|\Phi(\nabla u)\|_{L^1(\Omega)} \leqslant 1} \int_{\Omega} \exp_{[l]}(K|u(x)|^{\gamma})\, \mathrm{d}x = \infty.$$

The first assertion follows from [10, Remarks 3.11 (iv)]. In the case $l \geqslant 2$, all four remaining assertions of Theorem 1.1 can be found in [5, Theorem 1.1, Theorem 1.2, Theorem 4.2 and Theorem 4.1]. In the case $l = 1$, assertions (ii), (iii), (iv) follow from [17, Theorem 1.1, Theorem 1.2 and Theorem 4.2] while assertion (v) is given in [1, Example 5.1].

Notice that even though all Young functions satisfying (1.3) with fixed $l \in \mathbb{N}$ and $\alpha < n-1$ give the same Orlicz-Sobolev space, they give different Moser-type results in the critical case $K = K_{l,n,\alpha}$.

Next we state the main result of this paper. First, we define the median of a given measurable function $u\colon \Omega \to \mathbb{R}$ by

$$\operatorname{med}(u) = \sup\Big\{t \in \mathbb{R}\colon \mathcal{L}_n(\{x \in \Omega\colon u(x) > t\}) > \frac{\mathcal{L}_n(\Omega)}{2}\Big\}.$$

Theorem 1.2. *Let* $K \geqslant 0$, $l \in \mathbb{N}$, $n \geqslant 2$ *and* $\alpha < n-1$. *Suppose that* $\Omega \subset \mathbb{R}^n$ *is a bounded connected domain from the class* $C^{1,\theta}$ *for some* $\theta \in (0,1]$. *Let* Φ *be a Young function satisfying* (1.3).

(i) *If* $u \in WL^{\Phi}(\Omega)$, *then*

$$\int_{\Omega} \exp_{[l]}(K|u(x)|^{\gamma})\, \mathrm{d}x < \infty.$$

(ii) *If* $K < K_{l,n,\alpha}(\frac{1}{2})^{\gamma/n}$, $M \geqslant 0$, $u \in WL^{\Phi}(\Omega)$ *with* $\|\Phi(\nabla u)\|_{L^1(\Omega)} \leqslant 1$ *and* $|\mathrm{med}(u)| \leqslant M$, *then*

$$\int_{\Omega} \exp_{[l]}(K|u(x)|^{\gamma})\,\mathrm{d}x \leqslant C(l,n,\alpha,\Phi,\mathcal{L}_n(\Omega),K,M).$$

(iii) *If* $K > K_{l,n,\alpha}(\frac{1}{2})^{\gamma/n}$ *and* $M > 0$, *then there is a smooth domain* $\tilde{\Omega} \subset \mathbb{R}^n$ *such that*

$$\sup_{u\in WL^{\Phi}(\tilde{\Omega}),\|\Phi(\nabla u)\|_{L^1(\tilde{\Omega})}\leqslant 1,|\mathrm{med}(u)|\leqslant M} \int_{\tilde{\Omega}} \exp_{[l]}(K|u(x)|^{\gamma})\,\mathrm{d}x = \infty.$$

(iv) *Suppose that* $K = K_{l,n,\alpha}(\frac{1}{2})^{\gamma/n}$, $M \geqslant 0$, Φ *satisfies* (1.4), $u \in WL^{\Phi}(\Omega)$ *with* $\|\Phi(\nabla u)\|_{L^1(\Omega)} \leqslant 1$ *and* $|\mathrm{med}(u)| \leqslant M$. *Then*

$$\int_{\Omega} \exp_{[l]}(K|u(x)|^{\gamma})\,\mathrm{d}x \leqslant C(l,n,\alpha,\Phi,\mathcal{L}_n(\Omega),M).$$

(v) *Suppose that* $K = K_{l,n,\alpha}(\frac{1}{2})^{\gamma/n}$, $M > 0$ *and* Φ *satisfies* (1.5). *Then there is a smooth domain* $\tilde{\Omega} \subset \mathbb{R}^n$ *such that*

$$\sup_{u\in WL^{\Phi}(\tilde{\Omega}),\|\Phi(\nabla u)\|_{L^1(\tilde{\Omega})}\leqslant 1,|\mathrm{med}(u)|\leqslant M} \int_{\tilde{\Omega}} \exp_{[l]}(K|u(x)|^{\gamma})\,\mathrm{d}x = \infty.$$

The basic strategy of the proof of Theorem 1.2 is similar to the one concerning the proof of Theorem 1.1 given in [17] and [5]. However, due to the fact that we are dealing with the space $WL^{\Phi}(\Omega)$ instead of $W_0L^{\Phi}(\Omega)$, similarly to [8], we have to employ the signed non-increasing rearrangement instead of the radially symmetric rearrangement. Therefore, we use some results concerning the isoperimetric function from [8]. Moreover, we also need to derive some new estimates concerning the norm of the isoperimetric function with respect to the associated Young function to Φ.

Concentration-Compactness Principle. Our next result is a version of the Concentration-Compactness Alternative by Lions [18, Theorem I6 and Remark I.18] (see also [3]), which states that for non-concentrating sequences we can take K slightly exceeding the number $K_{l,n,\alpha}(\frac{1}{2})^{\gamma/n}$ in Theorem 1.2 (iii).

Theorem 1.3. *Let* $l \in \mathbb{N}$, $n \geqslant 2$ *and* $\alpha < n-1$. *Suppose that* $\Omega \subset \mathbb{R}^n$ *is a bounded connected domain of the class* $C^{1,\theta}$ *for some* $\theta \in (0,1]$. *Let* Φ *be a Young function satisfying* (1.3). *Let* $\{u_k\}_{k=1}^{\infty} \subset WL^{\Phi}(\Omega)$ *satisfy* $\|\Phi(|\nabla u_k|)\|_{L^1(\Omega)} \leqslant 1$ *and*

$$(1.6)\qquad u_k \rightharpoonup u \text{ in } WL^{\Phi}(\Omega), \quad u_k \to u \text{ a.e. in } \Omega \quad \text{and} \quad \Phi(|\nabla u_k|) \overset{*}{\rightharpoonup} \mu \text{ in } \mathcal{M}(\bar{\Omega})$$

for some $u \in WL^{\Phi}(\Omega)$ *and* $\mu \in \mathcal{M}(\bar{\Omega})$.

(i) *If u is a constant function (i.e. $u \equiv u_0$, where $u_0 \in \mathbb{R}$) and $\mu = \delta_{x_0}$ for some $x_0 \in \bar{\Omega}$, and*

$$\int_\Omega \exp_{[l]}\Big(K_{l,n,\alpha}\Big(\frac{1}{2}\Big)^{\gamma/n}|u_k|^\gamma\Big)\,\mathrm{d}x \overset{k\to\infty}{\longrightarrow} c + \exp_{[l]}\Big(K_{l,n,\alpha}\Big(\frac{1}{2}\Big)^{\gamma/n}|u_0|^\gamma\Big)\mathcal{L}_n(\Omega) \tag{1.7}$$

for some $c \in [0, \infty)$, then

$$\exp_{[l]}\Big(K_{l,n,\alpha}\Big(\frac{1}{2}\Big)^{\gamma/n}|u_k|^\gamma\Big) \overset{*}{\rightharpoonup} c\delta_{x_0} + \exp_{[l]}\Big(K_{l,n,\alpha}\Big(\frac{1}{2}\Big)^{\gamma/n}|u_0|^\gamma\Big)\mathcal{L}_n|_\Omega \quad \text{in } \mathcal{M}(\bar{\Omega}).$$

(ii) *If u is a constant function and μ is not a Dirac mass concentrated at one point, then there is $p > 1$ such that*

$$\exp_{[l]}\Big(K_{l,n,\alpha}\Big(\frac{1}{2}\Big)^{\gamma/n}p|u_k|^\gamma\Big) \quad \text{is bounded in } L^1(\Omega).$$

(iii) *If u is not a constant function, then*

$$\exp_{[l]}\Big(K_{l,n,\alpha}\Big(\frac{1}{2}\Big)^{\gamma/n}p|u_k|^\gamma\Big) \quad \text{is bounded in } L^1(\Omega)$$

for every

$$p < P := \begin{cases} \big(1 - \int_\Omega \Phi(|\nabla u|)\,\mathrm{d}x\big)^{-\gamma/n} & \text{if } \int_\Omega \Phi(|\nabla u|)\,\mathrm{d}x < 1, \\ \infty & \text{if } \int_\Omega \Phi(|\nabla u|)\,\mathrm{d}x = 1. \end{cases}$$

Moreover, in both cases (ii) *and* (iii) *we have*

$$\exp_{[l]}\Big(K_{l,n,\alpha}\Big(\frac{1}{2}\Big)^{\gamma/n}p|u_k|^\gamma\Big) \overset{k\to\infty}{\longrightarrow} \exp_{[l]}\Big(K_{l,n,\alpha}\Big(\frac{1}{2}\Big)^{\gamma/n}p|u|^\gamma\Big) \quad \text{in } L^1(\Omega). \tag{1.8}$$

The case (i) in the above theorem is called the Concentration. In this case, the assumption (1.7) is not satisfied automatically, because it may happen for a concentrating sequence that the integrals on the left-hand side of (1.7) tend to infinity. In fact, the proof of Theorem 1.2 (v) is based on the construction of such a concentrating sequence. The second case is when we have (ii) or (iii) and hence (1.8) is satisfied. This case is called the Compactness.

Let us note that a version of Theorem 1.3 for the space $W_0L^\Phi(\Omega)$ can be found in [4], [1] and [2]. Our proof of Theorem 1.3 is inspired by [4] and [3].

As an application of our Concentration-Compactness Alternative we prove that a functional with the sub-critical growth attains its maximum.

Theorem 1.4. *Let $l \in \mathbb{N}$, $n \geqslant 2$, $\alpha < n-1$ and $\lambda > 0$. Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded connected domain of the class $C^{1,\theta}$ for some $\theta \in (0,1]$, and let Φ be a Young function satisfying* (1.3). *Suppose that the function $F\colon \mathbb{R} \to \mathbb{R}$ is continuous. Further suppose that either*

$$\lim_{|t|\to\infty} \frac{F(t)}{\exp_{[l]}(K|t|^\gamma)} = 0 \quad \text{for some } K < K_{l,n,\alpha}\Big(\frac{1}{2}\Big)^{\gamma/n} \tag{1.9}$$

or Φ satisfies the additional condition (1.4) *and*

$$\lim_{|t|\to\infty} \frac{F(t)}{\exp_{[l]}(K_{l,n,\alpha}(\frac{1}{2})^{\gamma/n}|t|^\gamma)} = 0. \tag{1.10}$$

Then the functional

$$\Lambda_F(u) = \int_\Omega F(u(x))\,\mathrm{d}x$$

attains its maximum on the set

$$\{u \in WL^\Phi(\Omega)\colon\ \|\Phi(|\nabla u|)\|_{L^1(\Omega)} + \lambda\|\Phi(|u|)\|_{L^1(\Omega)} \leqslant 1\}.$$

Our proof of Theorem 1.4 demonstrates a standard application of the Concentration-Compactness Alternative in the situations when the Concentration phenomenon is harmless. It is based on showing that the maximizing sequence $\{u_k\}$ has a subsequence weakly convergent to a function u such that $\Lambda_F(u)$ is maximal. We use the fact that the sequence $\{u_k\}$ is bounded in $WL^\Phi(\Omega)$, hence passing to a subsequence we can guarantee that (1.6) is satisfied. If we have cases (ii) or (iii) from Theorem 1.3, then we use a version of (1.8) to show that $\Lambda_F(u) = \lim\limits_{k\to\infty} \Lambda_F(u_k)$. If (i) occurs, the result is obtained using (1.9) and (1.10), respectively.

The paper is organized as follows. After Preliminaries we prove some technical estimates which enable us to use generalized Hölder's inequality in the proof of Theorem 1.2 (i), (ii) and (iv). In the fourth section we recall some properties of the concentrating sequences from the proof of Theorem 1.1 (iii) and (v). These sequences are later used in the proof of Theorem 1.2 (iii) and (v). In the fifth section we prove the generalized Moser-Trudinger inequality (Theorem 1.2). The sixth section is devoted to the Concentration-Compactness Alternative (Theorem 1.3). The sixth section also contains the proof of the result concerning the functional with the subcritical growth (Theorem 1.4). We also discuss the sharpness of the condition $p < P$ in Theorem 1.3 (iii).

Notation. For the measure μ on $\mathbb{R}^n$ let $\mu|_\Omega$ be its restriction to Ω, i.e. $\mu|_\Omega(A) = \mu(A \cap \Omega)$ for every measurable set $A \subset \mathbb{R}^n$.

By $\mathcal{M}(A)$ we denote the set of all Radon measures on a ompact set A. We write that $\mu_j \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}(A)$ if $\int_A \psi \, \mathrm{d}\mu_j \to \int_A \psi \, \mathrm{d}\mu$ for every $\psi \in C(A)$. It is well known that each sequence bounded in $L^1(A)$ contains a subsequence converging weakly* in $\mathcal{M}(A)$.

When we integrate with respect to the n-dimensional Lebesgue measure, we often write $\int_\Omega f$ instead of $\int_\Omega f(x)\,\mathrm{d}x$.

By $B(x_0, R)$ we denote an open Euclidean ball in $\mathbb{R}^n$ centered at x_0 with the radius $R > 0$. If $x_0 = 0$ we simply write $B(R)$.

By C we denote a generic positive constant which may depend on l, n, α, $\mathcal{L}_n(\Omega)$, Φ, K and $\operatorname{med}(u)$. This constant may vary from expression to expression as usual. Sometimes we say that for every $\varepsilon > 0$ something is true. Then the constants C in such a case may depend also on a fixed $\varepsilon > 0$.

Young functions and Orlicz spaces. A function $\Phi\colon [0,\infty) \to [0,\infty)$ is a Young function if Φ is increasing, convex, $\Phi(0) = 0$ and $\lim\limits_{t\to\infty} \Phi(t)/t = \infty$.

Denote by $L^\Phi(A, \mathrm{d}\mu)$ the Orlicz space corresponding to a Young function Φ on a set A with a measure μ. If $\mu = \mathcal{L}_n$ we simply write $L^\Phi(A)$. Similarly to [17] we use the norm on $L^\Phi(A,\, \mathrm{d}\mu)$ given by

$$\|f\|_{L^\Phi(A,\mathrm{d}\mu)} = \inf\left\{\lambda > 0\colon \int_A \Phi\Big(\frac{|f(x)|}{\lambda}\Big)\,\mathrm{d}\mu(x) \leqslant \Phi(1)\right\}. \tag{2.1}$$

By Ψ we denote the associated Young function to Φ. The dual space to $L^\Phi(A, \mathrm{d}\mu)$ is the Orlicz space $L^\Psi(A, \mathrm{d}\mu)$.

If we have $\Phi(1) + \Psi(1) = 1$ (and if the norm is defined by (2.1)), then the following generalization of Hölder's inequality is valid (see [21, page 58] for the proof):

$$\int_A |f(y)g(y)|\,\mathrm{d}\mu(y) \leqslant \|f\|_{L^\Phi(A,\mathrm{d}\mu)} \|g\|_{L^\Psi(A,\mathrm{d}\mu)}. \tag{2.2}$$

The reason why we use the norm (2.1) instead of the Luxemburg norm (version of (2.1) with the bound $\Phi(1)$ replaced by 1) is that a version of inequality (2.2) for the Luxemburg norm differs by a multiplicative constant 2 on the right-hand side and this constant is not sharp.

For an introduction to Orlicz spaces see e.g. [21].

Δ_2-condition. We say that the Young function Φ satisfies the Δ_2-condition, if there are $t_\Delta \geqslant 0$ and $C_\Delta > 1$ such that

$$\Phi(2t) \leqslant C_\Delta \Phi(t) \quad \text{whenever } t \geqslant t_\Delta.$$

It is easy to see that if Φ satisfies the Δ_2-condition for one fixed $t_\Delta > 0$ then it satisfies this condition with arbitrary $\tilde{t}_\Delta > 0$ with a different constant $\tilde{C}_\Delta > 1$. It is not difficult to check the Δ_2-condition for our Young functions satisfying (1.3). Therefore one easily proves that for any $\eta > 0$ there is $\varepsilon > 0$ such that

$$\Phi((1+\varepsilon)t) \leqslant (1+\eta)\Phi(t) \quad \text{for } t \geqslant t_\Delta, \tag{2.3}$$

$$\|f\|_{L^\Phi(A,\mathrm{d}\mu)} = 1 \Longleftrightarrow \int_A \Phi(|f|)\,\mathrm{d}\mu(x) = \Phi(1), \tag{2.4}$$

$$\|f_k\|_{L^\Phi(A,\mathrm{d}\mu)} \overset{k\to\infty}{\longrightarrow} 0 \Longleftrightarrow \int_A \Phi(|f_k|)\,\mathrm{d}\mu(x) \overset{k\to\infty}{\longrightarrow} 0. \tag{2.5}$$

Orlicz-Sobolev spaces. Let A be a nonempty open bounded set in $\mathbb{R}^n$ and let Φ be a Young function. In this subsection we consider Orlicz spaces only with the Lebesgue measure. We define the Orlicz-Sobolev space $WL^\Phi(A)$ as the set

$$WL^\Phi(A) := \{u\colon\ u, |\nabla u| \in L^\Phi(A)\}$$

equipped with the norm

$$\|u\|_{WL^\Phi(A)}\colon = \|u\|_{L^\Phi(A)} + \|\nabla u\|_{L^\Phi(A)}$$

where ∇u is the gradient of u and we use its Euclidean norm in $\mathbb{R}^n$. The space $WL^\Phi(A)$ is a reflexive Banach space. We write that $f_k \rightharpoonup f$ in $WL^\Phi(A)$, if

$$\int_A f_k g \to \int_A fg \quad \text{and} \quad \int_A \frac{\partial f_k}{\partial x_i}\, g \to \int_A \frac{\partial f}{\partial x_i}\, g$$
$$\text{for every } g \in L^\Psi(A) \text{ and } i \in \{1,\dots,n\}.$$

We put $W_0L^\Phi(A)$ for the closure of $C_0^\infty(A)$ in $WL^\Phi(A)$.

Tools from Measure Theory. We need a version of [1, Lemma 2.3].

Lemma 2.1. *Let $l \in \mathbb{N}$, $K > 0$, $\gamma > 0$ and let $\{u_k\}_{k=1}^\infty$ be a sequence of measurable functions such that $u_k \to u$ a.e. in Ω. Suppose that there are $\delta > 0$ and $C_1 > 0$ such that*

$$\int_\Omega \exp_{[l]}(K(1+\delta)|u_k|^\gamma) \leqslant C_1 \quad \textit{for all } k \in \mathbb{N}. \tag{2.6}$$

Let F be a continuous function such that

$$\sup_{|t|\in(t_0,\infty)} \frac{|F(t)|}{\exp_{[l]}(K|t|^\gamma)} < \infty \quad \textit{for some } t_0 > 0.$$

Then

$$\int_\Omega F(u_k) \overset{k\to\infty}{\longrightarrow} \int_\Omega F(u).$$

In particular,

$$\int_\Omega \exp_{[l]}(K|u_k|^\gamma) \overset{k\to\infty}{\longrightarrow} \int_\Omega \exp_{[l]}(K|u|^\gamma).$$

In the original version of the previous lemma given in [1, Lemma 2.3], the function F is supposed to be an even function. However, the original proof is valid without such an assumption.

Isoperimetric function and generalized Pólya-Szegö principle. In this subsection we recall some partial results and estimates used in [8].

Let $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$, be a bounded connected domain. We define the isoperimetric function $h_\Omega\colon (0, \mathcal{L}_n(\Omega)) \to [0, +\infty)$ by

$$h_\Omega(y) = \inf\{P(E;\Omega)\colon E \subset \Omega, \mathcal{L}_n(E) = y\}, \quad y \in (0, \mathcal{L}_n(\Omega)),$$

where $P(E;\Omega)$ is the perimeter of $E \subset \mathbb{R}^n$ in Ω defined by

$$P(E;\Omega) = \sup\biggl\{\int_E \operatorname{div}\varphi \,\mathrm{d}x\colon \varphi \in [C_0^1(\Omega)]^n, \|\varphi\|_{L^\infty(\Omega)} \leqslant 1\biggr\}.$$

The function h_Ω satisfies

$$h_\Omega(y) = h_\Omega(\mathcal{L}_n(\Omega) - y), \quad y \in (0, \mathcal{L}_n(\Omega)). \tag{2.7}$$

By [8, Theorem 1.3 and Corollary 2.4] we have the following estimate.

Proposition 2.2. *Let $n \geqslant 2$ and let Ω be a bounded connected domain in $\mathbb{R}^n$ of class $C^{1,\theta}$ for some $\theta \in (0,1]$. Then there are $C_0 > 0$, $\beta > 0$ and $y_1 \in (0, \frac{1}{2}\mathcal{L}_n(\Omega)]$ such that for the function*

$$h(y) = \begin{cases} n^{(n-1)/n}\Bigl(\dfrac{\omega_{n-1}}{2}\Bigr)^{1/n} y^{(n-1)/n}(1 - C_0 y^\beta) & \textit{for } y \in (0, y_1], \\ h(y_1) & \textit{for } y \in \Bigl[y_1, \dfrac{\mathcal{L}_n(\Omega)}{2}\Bigr], \\ h(\mathcal{L}_n(\Omega) - y) & \textit{for } y \in \Bigl[\dfrac{\mathcal{L}_n(\Omega)}{2}, \mathcal{L}_n(\Omega)\Bigr), \end{cases} \tag{2.8}$$

we have that $h(y)$ and $y/h(y)$ are non-negative and non-decreasing on $(0, \frac{1}{2}\mathcal{L}_n(\Omega)]$ *and*

$$h_\Omega(y) \geqslant h(y), \quad y \in (0, \mathcal{L}_n(\Omega)). \tag{2.9}$$

Finally, let us define by $u^\circ\colon (0, \mathcal{L}_n(\Omega)) \to \mathbb{R}$ the signed non-increasing rearrangement of u, given by

$$u^\circ(y) = \sup\{t \in \mathbb{R}\colon \mathcal{L}_n(\{x \in \Omega\colon u(x) > t\}) > y\} \quad \text{for } y \in (0, \mathcal{L}_n(\Omega)),$$

and recall the generalized Pólya-Szegö principle for the space $WL^\Phi(\Omega)$ which follows from [9, Lemma 4.1 (ii)] by replacing the function h_Ω by h (recall $h \leqslant h_\Omega$).

Lemma 2.3. *Let $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$, be a bounded connected open set and let $u \in WL^\Phi(\Omega)$. Then u° is locally absolutely continuous and*

$$\int_0^{\mathcal{L}_n(\Omega)} \Phi\Big(h(y)\Big(-\frac{\mathrm{d}u^\circ}{\mathrm{d}y}(y)\Big)\Big)\,\mathrm{d}y \leqslant \int_\Omega \Phi(|\nabla u|)\,\mathrm{d}x,$$
$$\Big\|h(y)\Big(-\frac{\mathrm{d}u^\circ}{\mathrm{d}y}(y)\Big)\Big\|_{L^\Phi((0,\mathcal{L}_n(\Omega)))} \leqslant \|\nabla u\|_{L^\Phi(\Omega)}.$$

Let us note that in the original statement of Lemma 2.3 in [9], only the norm-estimate is given. However, since such an estimate holds for any norm on the Orlicz space $L^\Phi(\Omega)$, one easily proves the modular-estimate by passing to the Luxemburg norm with respect to a suitable multiple (1 over the right-hand side of the desired modular-estimate) of Φ and then using the norm-estimate for the corresponding norm. The same method works when showing that the modulars are weakly lower semicontinuous using the weak lower semicontinuity of a norm.

3. Estimates concerning the associated Young function

In the sequel, we follow the ideas from [17] and [5] used in the proof of Theorem 1.1 (ii) and (iv). If our Young function Φ satisfies (1.3) and (1.4), in a standard way we can prove that there is a Young function $\Phi_1\colon [0,\infty) \to [0,\infty)$ such that

(3.1) Φ_1' is continuous and increasing on $(0,\infty)$,

$$\Phi_1(t) = \frac{1}{n}\,t^n \quad \text{for } t \in [0,1],$$

and there is $G > t_0$ such that for every $t \geqslant G$ we have

$$\Phi_1(t) = \frac{1}{n}\,t^n \Big(\prod_{j=1}^{l-1} \log_{[j]}^{n-1}(t)\Big) \log_{[l]}^{\alpha}(t)(1 + \log_{[l]}^{-a}(t)) \leqslant \frac{1}{n}\,\Phi(t).$$

If Φ satisfies only (1.3), we find $\Phi_1\colon [0,\infty)\to[0,\infty)$ such that

$$\begin{gathered}\Phi_1' \text{ is continuous and increasing on } (0,\infty),\\ \lim_{t\to\infty}\frac{\Phi_1(t)}{n^{-1}\,t^n\Bigl(\prod_{j=1}^{l-1}\log_{[j]}^{n-1}(t)\Bigr)\log_{[l]}^{\alpha}(t)}=1,\\ \lim_{t\to\infty}\frac{\Phi_1'(t)}{t^{n-1}\Bigl(\prod_{j=1}^{l-1}\log_{[j]}^{n-1}(t)\Bigr)\log_{[l]}^{\alpha}(t)}=1,\\ \Phi_1(t)=\frac{1}{n}\,t^n \quad \text{for } t\in[0,1],\\ \text{and there is } G>\exp_{[l]}(1) \text{ such that for every } t\geqslant G \text{ we have}\\ \Phi_1(t)\leqslant\frac{1}{n}\,\Phi(t).\end{gathered}\tag{3.2}$$

Denote by Ψ_1 the Young function associated with the function Φ_1. In both the above cases, clearly $\Psi_1(t)=n^{-1}(n-1)t^{n/(n-1)}$ for $t\in[0,1]$. Hence $\Phi_1(1)+\Psi_1(1)=1$. Therefore (Φ_1,Ψ_1) is a normalized complementary Young pair and we can use the generalized Hölder's inequality (2.2).

We need to be able to estimate the norm with respect to Φ_1 by a modular with respect to Φ.

Lemma 3.1. *Let $\delta>0$, $0<C_1<C_2$ and let $A\subset(0,\infty)$ be a measurable set. Suppose that the Young function Φ satisfies* (1.3) *and let Φ_1 be given by* (3.2)*. Then there is $\tilde G=\tilde G(C_1,C_2,\delta)>G$ with the following property:*

If $v\in L^{\Phi}(A)$ is such that

$$C_1\leqslant\|v\|_{L^{\Phi_1}(A)}\leqslant C_2$$

and $|v|\geqslant\tilde G$ on A, then

$$\|v\|^n_{L^{\Phi_1}(A)}\leqslant(1+\delta)^3\int_A\Phi(|v|)\,\mathrm{d}x.$$

The proof is an easy exercise using (1.3), (3.2) and (2.4) (moreover, it is very similar to the proof of [2, Lemma 3.2]). Therefore we omit it.

By [17, Lemma 3.1], [5, Lemma 3.1], [17, Lemma 4.3] and [5, Lemma 4.3] we have the following estimates concerning Ψ_1.

Lemma 3.2.

(i) *Let* Φ *satisfy* (1.3) *and let* Φ_1 *be constructed so that* (3.2) *is satisfied. Then there is* $E > 0$ *such that if* $t \in (0, \infty)$, *then*

$$\Psi_1(t) < \hat{\Psi}_1(t) := Et^{n/(n-1)}(1 + |\log(t)|^E). \tag{3.3}$$

Moreover, for every $\delta > 0$ *there is* $G_2 > G$ *such that if* $t \in [G_2, \infty)$, *then*

$$\Psi_1(t) \leqslant \tilde{\Psi}_1(t) := \begin{cases} (1+\delta)\dfrac{(n-1)^{1+(\alpha/(n-1))}}{n}\, t^{n/(n-1)} \log^{-\alpha/(n-1)}(t) & \text{if } l = 1, \\ (1+\delta)\dfrac{(n-1)^2}{n}\, t^{n/(n-1)} \Big(\prod_{j=1}^{l-1} \log_{[j]}^{-1}(t)\Big) \log_{[l]}^{-\alpha/(n-1)}(t) & \text{if } l \geqslant 2. \end{cases} \tag{3.4}$$

(ii) *Let* Φ *satisfy* (1.3) *and* (1.4), *and let* Φ_1 *be constructed so that* (3.1) *is satisfied. Then there is* $E > 0$ *such that if* $t \in (0, \infty)$, *then*

$$\Psi_1(t) < \hat{\Psi}_1(t) := Et^{n/(n-1)}(1 + |\log(t)|^E). \tag{3.5}$$

Moreover, there are $G_2 > G$ *and* $b \in (a, \min\{1, 1/\gamma\})$ *such that for every* $t \in [G_2, \infty)$ *we have*

$$\Psi_1(t) \leqslant \tilde{\Psi}_1(t) := \begin{cases} \dfrac{(n-1)^{1+(\alpha/(n-1))}}{n}\, t^{n/(n-1)} \log^{-\alpha/(n-1)}(t)(1 - \log^{-b}(t)) & \text{if } l = 1, \\ \dfrac{(n-1)^2}{n} t^{n/(n-1)} \Big(\prod_{j=1}^{l-1} \log_{[j]}^{-1}(t)\Big) \log_{[l]}^{-\alpha/(n-1)}(t)(1 - \log_{[l]}^{-b}(t)) & \text{if } l \geqslant 2. \end{cases} \tag{3.6}$$

The main result of this section is an estimate of $\|1/h(y)\|_{L^{\Psi_1}((t, \frac{1}{2}\mathcal{L}_n(\Omega)))}$ for $t > 0$ sufficiently small.

Lemma 3.3. *Let us define*

$$D = \begin{cases} \Big(\dfrac{\omega_{n-1}}{2}\Big)^{-1/n} B^{-(n-1)/n}\, n^{-1/\gamma} & \text{for } l = 1, \\ \Big(\dfrac{\omega_{n-1}}{2}\Big)^{-1/n} B^{-(n-1)/n} & \text{for } l \geqslant 2. \end{cases} \tag{3.7}$$

(i) *Let* Φ *satisfy* (1.3) *and let* Φ_1 *be constructed so that* (3.2) *is satisfied. Let* $\varepsilon > 0$. *Then there is* $t_{\Psi_1} \in (0, \frac{1}{2}\mathcal{L}_n(\Omega))$ *such that for every* $t \in (0, t_{\Psi_1})$ *we have*

$$\Big\| \frac{1}{h(y)} \Big\|_{L^{\Psi_1}((t,\frac{1}{2}\mathcal{L}_n(\Omega)))} \leqslant (1+\varepsilon) D \log_{[l]}^{1/\gamma}\Big(\frac{1}{t}\Big).$$

(ii) *Let* Φ *satisfy* (1.3) *and* (1.4), *and let* Φ_1 *be constructed so that* (3.1) *is satisfied. Then there are* $t_{\Psi_1} \in (0, \frac{1}{2}\mathcal{L}_n(\Omega))$ *and* $c \in (b, \min\{1, 1/\gamma\})$ *such that for every* $t \in (0, t_{\Psi_1})$ *we have*

$$\Big\| \frac{1}{h(y)} \Big\|_{L^{\Psi_1}((t,\frac{1}{2}\mathcal{L}_n(\Omega)))} \leqslant D \log_{[l]}^{1/\gamma}\Big(\frac{1}{t}\Big)\Big(1 - \log_{[l]}^{-c}\Big(\frac{1}{t}\Big)\Big).$$

Before we prove Lemma 3.3, let us note that the function $\log_{[j]}$ has asymptotic behavior similar to the function log. We recall [5, Lemma 2.2].

Lemma 3.4. *Let* $t_1, p, q, \delta, E, L > 0$ *and* $l \in \mathbb{N}$ *and let functions* $f, h\colon \mathbb{R} \to (0,\infty)$ *and* $g\colon \mathbb{R} \to \mathbb{R}$ *satisfy*

$$g(t) + Ef(t) > \exp_{[l]}(0) \quad \text{and} \quad Eh^q(t) f^p(t) > \exp_{[l]}(0) \quad \text{on } (t_1,\infty),$$
$$\lim_{t\to\infty} f(t) = \infty, \quad \frac{g(t)}{f(t)} \in [-E+\delta, L] \quad \text{and} \quad \frac{\log(h(t))}{\log(f(t))} \in \Big[-\frac{p}{q} + \delta, L\Big] \quad \text{on } (t_1,\infty).$$

Then there is $t_0 > t_1$ *such that if* $t > t_0$ *then*

$$(3.8)\quad 1 - \frac{C}{\log_{[l]}(f(t))} < \frac{\log_{[j]}(g(t) + Ef(t))}{\log_{[j]}(f(t))} < 1 + \frac{C}{\log_{[l]}(f(t))} \quad \text{for } j \in \{1,\ldots,l\}$$

and

$$(3.9)\quad 1 - \frac{C}{\log_{[l]}(f(t))} < \frac{\log_{[j]}(Eh^q(t) f^p(t))}{\log_{[j]}(f(t))} < 1 + \frac{C}{\log_{[l]}(f(t))} \quad \text{for } j \in \{2,\ldots,l\}.$$

P r o o f of Lemma 3.3. (i) For fixed $t \in (0, \exp_{[l]}^{-1}(1))$ we set

$$(3.10)\qquad \lambda = (1+\varepsilon) D \log_{[l]}^{1/\gamma}\Big(\frac{1}{t}\Big)$$

and our aim is to prove that

$$(3.11)\qquad \int_t^{\frac{1}{2}\mathcal{L}_n(\Omega)} \Psi_1\Big(\frac{1}{\lambda h(y)}\Big)\,\mathrm{d}y \leqslant \frac{n-1}{n} = \Psi_1(1)$$

provided $t > 0$ is sufficiently small. Let $\delta > 0$ be so small that

$$\frac{(1+\delta)^{l+2}}{(1+\varepsilon)^{n/(n-1)}} < 1 - \frac{1}{2}\varepsilon. \tag{3.12}$$

For $t \in (0, \exp_{[l]}^{-1}(1))$ put

$$M = M(t) = \exp\Bigl(-\log_{[l]}^{(\min\{1,1/\gamma\})/((E+2)(n-1))}\Bigl(\frac{1}{t}\Bigr)\Bigr). \tag{3.13}$$

Since $1/t \gg 1/(M^{n/(n-1)}) \gg \lambda$ and $1/h(M) \approx 1/(M^{(n-1)/n})$ (see (2.8)) for $t > 0$ small, there is $t_1 \in (0, \exp_{[l]}^{-1}(1))$ such that for $0 < t < t_1$ we have

$$t < M < \frac{\mathcal{L}_n(\Omega)}{2}, \quad \frac{1}{\lambda h(M)} > G_2 \quad \text{and} \quad \log_{[l]}(1/M) > 0, \tag{3.14}$$

where G_2 (depending on δ) comes from Lemma 3.2 (i). Therefore from Lemma 3.2 (i) we have

$$\begin{aligned} &\int_t^{\frac{1}{2}\mathcal{L}_n(\Omega)} \Psi_1\Bigl(\frac{1}{\lambda h(y)}\Bigr)\,\mathrm{d}y \\ &\quad \leqslant \int_t^M \tilde{\Psi}_1\Bigl(\frac{1}{\lambda h(y)}\Bigr)\,\mathrm{d}y + \int_M^{\frac{1}{2}\mathcal{L}_n(\Omega)} \hat{\Psi}_1\Bigl(\frac{1}{\lambda h(y)}\Bigr)\,\mathrm{d}y = I_1 + I_2. \end{aligned} \tag{3.15}$$

By (2.8) we have $1/h(y) \leqslant Cy^{-(n-1)/n}$ and thus (3.3) gives

$$\begin{aligned} I_2 &\leqslant CE \int_M^{\frac{1}{2}\mathcal{L}_n(\Omega)} \frac{1}{\lambda^{n/(n-1)}}\Bigl(1 + \Bigl|\log\Bigl(C\frac{1}{\lambda y^{(n-1)/n}}\Bigr)\Bigr|^E\Bigr)\frac{\mathrm{d}y}{y} \\ &\leqslant \frac{C}{\lambda^{n/(n-1)}} \int_M^{\frac{1}{2}\mathcal{L}_n(\Omega)} (1 + |\log(\lambda)|^E + |\log(y)|^E)\frac{\mathrm{d}y}{y} = J_1 + J_2, \end{aligned} \tag{3.16}$$

where (see (3.10) and (3.13))

$$\begin{aligned} J_1 &= \frac{C}{\lambda^{n/(n-1)}} \int_M^{\frac{1}{2}\mathcal{L}_n(\Omega)} (1 + |\log(\lambda)|^E)\frac{\mathrm{d}y}{y} \\ &\leqslant \frac{C}{\log_{[l]}^{n/((n-1)\gamma)}(1/t)}\Bigl(1 + \log^E\Bigl(\log_{[l]}\Bigl(\frac{1}{t}\Bigr)\Bigr)\Bigr)(C + \log(1/M)) \\ &\leqslant C\frac{\log_{[l+1]}^E(1/t)\log_{[l]}^{\min\{1,1/\gamma\}/((E+2)(n-1))}(1/t)}{\log_{[l]}^{n/((n-1)\gamma)}(1/t)} \overset{t\to 0_+}{\longrightarrow} 0 \end{aligned} \tag{3.17}$$

and

$$J_2 = \frac{C}{\lambda^{n/(n-1)}} \int_M^{\frac{1}{2}\mathcal{L}_n(\Omega)} |\log(y)|^E \frac{\mathrm{d}y}{y} \leqslant \frac{C}{\log_{[l]}^{n/((n-1)\gamma)}(1/t)} \Big(C + \log^{E+1}\Big(\frac{1}{M}\Big)\Big) \leqslant C \frac{\log_{[l]}^{\min\{1,1/\gamma\}/(n-1)}(1/t)}{\log_{[l]}^{n/((n-1)\gamma)}(1/t)} \xrightarrow{t\to 0_+} 0. \tag{3.18}$$

Hence there is $t_2 \in (0, t_1)$ such that if $0 < t < t_2$ then we have

$$I_2 < \frac{n-1}{n}\frac{\varepsilon}{2}. \tag{3.19}$$

Next, we need to estimate I_1. We distinguish two cases.

Case $l \geqslant 2$. Since, by (2.8), (3.10) and (3.13), we have $\log(1/h(M)) \gg \log(\lambda) > 1$ and $1/h(y) \approx 1/y^{(n-1)/n}$ for small t and $y \in [t, M]$, we can find $t_3 \in (0, t_2)$ such that for all $0 < t < t_3$ and $y \in [t, M]$ we have

$$\log^{-1}\Big(\frac{1}{\lambda h(y)}\Big) \leqslant (1+\delta)\log^{-1}\Big(\frac{1}{y^{(n-1)/n}}\Big) = (1+\delta)\frac{n}{n-1}\log^{-1}\Big(\frac{1}{y}\Big). \tag{3.20}$$

Similarly, we can find $t_4 \in (0, t_3)$ such that for every $0 < t < t_4$ and every $y \in [t, M]$ we obtain

$$\log_{[j]}^{-1}\Big(\frac{1}{\lambda h(y)}\Big) \leqslant (1+\delta)\log_{[j]}^{-1}\Big(\frac{1}{y}\Big) \quad \text{for } j \in \{2, \ldots, l-1\} \tag{3.21}$$

(indeed, $\log(y^{-(n-1)/n}) = ((n-1)/n)\log(1/y)$ while $\log_{[j]}(y^{-(n-1)/n})$ is very close to $\log_{[j]}(1/y)$ for $j \geqslant 2$),

$$\log_{[l]}^{-\alpha/(n-1)}\Big(\frac{1}{\lambda h(y)}\Big) \leqslant (1+\delta)\log_{[l]}^{-\alpha/(n-1)}\Big(\frac{1}{y}\Big) \tag{3.22}$$

and

$$\Big(\frac{1}{1 - C_0 y^\beta}\Big)^{n/(n-1)} \leqslant 1 + \delta \quad \text{for } y \in [t, M]. \tag{3.23}$$

Therefore (2.8), (3.4), (3.10), (3.20), (3.21), (3.22) and (3.23) imply that for $0 < t < t_4$ we have

$$I_1 \leqslant (1+\delta)\int_t^M \frac{(n-1)^2}{n}\Big(\frac{1}{\lambda h(y)}\Big)^{n/(n-1)} \times \Big(\prod_{j=1}^{l-1}\log_{[j]}^{-1}\Big(\frac{1}{\lambda h(y)}\Big)\Big)\log_{[l]}^{-\alpha/(n-1)}\Big(\frac{1}{\lambda h(y)}\Big)\,\mathrm{d}y$$

$$\leqslant (1+\delta)^{l+2}\frac{n-1}{n}\Big(\frac{\omega_{n-1}}{2}\Big)^{-1/(n-1)}\frac{1}{\lambda^{n/(n-1)}}$$
$$\times\int_t^M \log^{-1}\Big(\frac{1}{y}\Big)\Big(\prod_{j=2}^{l-1}\log_{[j]}^{-1}\Big(\frac{1}{y}\Big)\Big)\log_{[l]}^{-\alpha/(n-1)}\Big(\frac{1}{y}\Big)\frac{\mathrm{d}y}{y}$$
$$=\frac{(1+\delta)^{l+2}\frac{n-1}{n}\big(\frac{1}{2}\omega_{n-1}\big)^{-1/(n-1)}}{(1+\varepsilon)^{n/(n-1)}D^{n/(n-1)}\log_{[l]}^{n/((n-1)\gamma)}(1/t)}\Big[-\frac{\log_{[l]}^{1-\alpha/(n-1)}(1/y)}{1-\alpha/(n-1)}\Big]_t^M.$$

Thus, as $\log_{[l]}^{1-\alpha/(n-1)}(1/M)>0$ (by (3.14)), using (3.12), $1-\alpha/(n-1)=B=n/\gamma(n-1)$ (see (1.2)) and (3.7) we obtain

$$I_1\leqslant\frac{(1-\frac{1}{2}\omega)\frac{n-1}{n}(\frac{1}{2}\omega_{n-1})^{-1/(n-1)}}{D^{n/(n-1)}B}=\frac{n-1}{n}\Big(1-\frac{1}{2}\varepsilon\Big). \tag{3.24}$$

From (3.15), (3.19) and (3.24) we obtain that for $0<t<t_4$ we have

$$\int_t^{\frac{1}{2}\mathcal{L}_n(\Omega)}\Psi\Big(\frac{1}{\lambda h(y)}\Big)\,\mathrm{d}y\leqslant I_1+I_2\leqslant\frac{n-1}{n}\Big(1-\frac{1}{2}\varepsilon\Big)+\frac{n-1}{n}\frac{\varepsilon}{2}=\frac{n-1}{n}=\Psi_1(1).$$

This is (3.11) and the proof of (i) is complete in the case $l\geqslant 2$.

Case $l=1$. Since, by (2.8), (3.10) and (3.13), we have $\log(1/h(M))\gg\log(\lambda)>1$ and $1/h(y)\approx 1/y^{(n-1)/n}$ for small t and $y\in[t,M]$, we can find $t_3\in(0,t_2)$ such that for all $0<t<t_3$ and $y\in[t,M]$ we have

$$\log^{-\alpha/(n-1)}\Big(\frac{1}{\lambda h(y)}\Big)<(1+\delta)\log^{-\alpha/(n-1)}\Big(\frac{1}{y^{(n-1)/n}}\Big) \tag{3.25}$$
$$=(1+\delta)\Big(\frac{n-1}{n}\Big)^{-\alpha/(n-1)}\log^{-\alpha/(n-1)}\Big(\frac{1}{y}\Big).$$

Moreover, we can suppose that t_3 is so small that (3.23) is satisfied. Therefore (2.8), (3.4), (3.10), (3.23) and (3.25) imply that for $0<t<t_3$ we have

$$I_1\leqslant(1+\delta)\int_t^M\frac{(n-1)^{1+\alpha/(n-1)}}{n}\Big(\frac{1}{\lambda h(y)}\Big)^{n/(n-1)}\log^{-\alpha/(n-1)}\Big(\frac{1}{\lambda h(y)}\Big)\,\mathrm{d}y$$
$$\leqslant(1+\delta)^3\frac{n-1}{n^{2-(\alpha/(n-1))}}\Big(\frac{\omega_{n-1}}{2}\Big)^{-1/(n-1)}\frac{1}{\lambda^{n/(n-1)}}\int_t^M\log^{-\alpha/(n-1)}\Big(\frac{1}{y}\Big)\frac{\mathrm{d}y}{y}$$
$$=\frac{(1+\delta)^3(n-1)(\frac{1}{2}\omega_{n-1})^{-1/(n-1)}}{(1+\varepsilon)^{n/(n-1)}n^{2-\alpha/(n-1)}D^{n/(n-1)}\log^{n/((n-1)\gamma)}(1/t)}\Big[-\frac{\log^{1-\alpha/(n-1)}(1/y)}{1-\alpha/(n-1)}\Big]_t^M.$$

Thus, as $\log_{[l]}^{1-\alpha/(n-1)}(1/M)>0$ (by (3.14)), using (3.12), $1-\alpha/(n-1)=B=n/(\gamma(n-1))$ (see (1.2)) and (3.7) we obtain

$$I_1\leqslant\frac{(1-\frac{1}{2}\varepsilon)(n-1)(\frac{1}{2}\omega_{n-1})^{-1/(n-1)}}{n^{2-\alpha/(n-1)}D^{n/(n-1)}B}=\frac{n-1}{n}\Big(1-\frac{1}{2}\varepsilon\Big).$$

This, (3.15) and (3.19) imply (3.11) for $0 < t < t_3$ and the proof of (i) is complete also in the case $l = 1$.

(ii) The proof is similar to the proof of part (i), but we need more careful estimates. This time we set

$$\lambda = D \log_{[l]}^{1/\gamma}\Big(\frac{1}{t}\Big)\Big(1 - \log_{[l]}^{-c}\Big(\frac{1}{t}\Big)\Big) \tag{3.26}$$

and we want to obtain (3.11) for this λ. The point $M = M(t)$ is defined by (3.13) again, (3.14) still holds, and we split the integral as in (3.15). According to the fact that estimates (3.3) and (3.5) coincide we obtain from (3.16), (3.17) and (3.18) for $0 < t < t_2$

$$I_2 \leqslant \frac{C}{\log_{[l]}^{n/((n-1)\gamma)-\min\{1,1/\gamma\}/(n-1)}(1/t)}. \tag{3.27}$$

It remains to estimate I_1. We distinguish two cases.

Case $l \geqslant 2$. Since $b \in (0,1)$ and thus $\log^{1-b}(1/M) \gg \log(\lambda) > 1$ for small $t > 0$ (by (3.13) and (3.26)), we can choose $t_3 \in (0, t_2)$ such that if $0 < t < t_3$ and $y \in [t, M]$, then we have by (2.8)

$$\begin{aligned}\log^{-1}\Big(\frac{1}{\lambda h(y)}\Big) &= \log^{-1}\Big(\frac{1}{y^{(n-1)/n}}\Big)\frac{1}{1 + \dfrac{\log(\lambda^{-1} y^{(n-1)/n}/h(y))}{\log(1/y^{(n-1)/n})}} \\ &\leqslant \frac{n}{n-1}\log^{-1}\Big(\frac{1}{y}\Big)\Big(1 + \frac{C\log(\lambda)}{\log(1/y)}\Big) \\ &\leqslant \frac{n}{n-1}\log^{-1}\Big(\frac{1}{y}\Big)\Big(1 + \frac{C\log(\lambda)}{\log^{1-b}(1/M)}\frac{1}{\log^{b}(1/y)}\Big) \\ &\leqslant \Big(1 + \frac{1/4}{\log_{[l]}^{b}(1/y)}\Big)\frac{n}{n-1}\log^{-1}\Big(\frac{1}{y}\Big).\end{aligned} \tag{3.28}$$

Further, estimate (3.9) from Lemma 3.4 together with (2.8) gives us $t_4 \in (0, t_3)$ such that if $0 < t < t_4$ and $y \in [t, M]$, hence we have

$$\log_{[j]}^{-1}\Big(\frac{1}{\lambda h(y)}\Big) \leqslant \Big(1 + \frac{C}{\log_{[l]}(1/y)}\Big)\log_{[j]}^{-1}\Big(\frac{1}{y}\Big) \quad \text{for } j \in \{2, \ldots, l-1\}, \tag{3.29}$$

$$\log_{[l]}^{-\alpha/(n-1)}\Big(\frac{1}{\lambda h(y)}\Big) \leqslant \Big(1 + \frac{C}{\log_{[l]}(1/y)}\Big)\log_{[l]}^{-\alpha/(n-1)}\Big(\frac{1}{y}\Big), \tag{3.30}$$

$$1 - \log_{[l]}^{-b}\Big(\frac{1}{\lambda h(y)}\Big) \leqslant 1 - \frac{1}{2}\log_{[l]}^{-b}\Big(\frac{1}{y}\Big) \tag{3.31}$$

and

$$\Big(1 - \log_{[l]}^{-c}\Big(\frac{1}{t}\Big)\Big)^{-n/(n-1)} \leqslant 1 + C\log_{[l]}^{-c}\Big(\frac{1}{t}\Big). \tag{3.32}$$

Finally, we observe that for $y \in [t, M]$

$$\Big(\frac{1}{1 - C_0 y^\beta}\Big)^{n/(n-1)} \leqslant 1 + C y^\beta \leqslant 1 + C \log_{[l]}^{-1}\Big(\frac{1}{y}\Big). \tag{3.33}$$

Hence (2.8), (3.6), (3.26), (3.28), (3.29), (3.30), (3.31), (3.32) and (3.33) give us that

$$\begin{aligned}
I_1 &\leqslant \frac{(n-1)^2}{n} \int_t^M \Big(\frac{1}{\lambda h(y)}\Big)^{n/(n-1)} \\
&\quad \times \Big(\prod_{j=1}^{l-1} \log_{[j]}^{-1}\Big(\frac{1}{\lambda h(y)}\Big)\Big) \log_{[l]}^{-\alpha/(n-1)}\Big(\frac{1}{\lambda h(y)}\Big)\Big(1 - \log_{[l]}^{-b}\Big(\frac{1}{\lambda h(y)}\Big)\Big)\,\mathrm{d}y \\
&\leqslant \frac{((n-1)/n)(\frac{1}{2}\omega_{n-1})^{-1/(n-1)}\big(1 + C\log_{[l]}^{-c}(1/t)\big)}{D^{n/(n-1)} \log_{[l]}^{n/((n-1)\gamma)}(1/t)} \\
&\quad \times \int_t^M \Big(\prod_{j=1}^{l-1} \log_{[j]}^{-1}\Big(\frac{1}{y}\Big)\Big) \log_{[l]}^{-\alpha/(n-1)}\Big(\frac{1}{y}\Big) \\
&\quad \times \Big(1 + \frac{1}{4}\log_{[l]}^{-b}\Big(\frac{1}{y}\Big)\Big)\Big(1 + \frac{C}{\log_{[l]}(1/y)}\Big)^l \Big(1 - \frac{1}{2}\log_{[l]}^{-b}\Big(\frac{1}{y}\Big)\Big)\frac{\mathrm{d}y}{y}.
\end{aligned}$$

Further, as $0 < b < c < 1$, there is $t_5 \in (0, t_4)$ such that for $0 < t < t_5$ and $y \in [t, M]$ we obtain

$$\begin{aligned}
&\Big(1 + C\log_{[l]}^{-c}\Big(\frac{1}{t}\Big)\Big)\Big(1 + \frac{C}{\log_{[l]}(1/y)}\Big)^l\Big(1 + \frac{1}{4}\log_{[l]}^{-b}\Big(\frac{1}{y}\Big)\Big)\Big(1 - \frac{1}{2}\log_{[l]}^{-b}\Big(\frac{1}{y}\Big)\Big) \\
&\qquad \leqslant \Big(1 + C\log_{[l]}^{-c}\Big(\frac{1}{t}\Big)\Big)\Big(1 - \frac{1}{8}\log_{[l]}^{-b}\Big(\frac{1}{y}\Big)\Big) \\
&\qquad \leqslant \Big(1 + C\log_{[l]}^{-c}\Big(\frac{1}{t}\Big)\Big)\Big(1 - \frac{1}{8}\log_{[l]}^{-b}\Big(\frac{1}{t}\Big)\Big) \leqslant 1 - \frac{1}{16}\log_{[l]}^{-b}\Big(\frac{1}{t}\Big).
\end{aligned}$$

Therefore (3.7) and $-\alpha/(n-1) = B - 1 \neq -1$ (see (1.2)) imply

$$\begin{aligned}
I_1 &\leqslant \frac{((n-1)/n)(\frac{1}{2}\omega_{n-1})^{-1/(n-1)}}{D^{n/(n-1)}} \frac{1 - \frac{1}{16}\log_{[l]}^{-b}(1/t)}{\log_{[l]}^{n/((n-1)\gamma)}(1/t)} \\
&\quad \times \int_t^M \Big(\prod_{j=1}^{l-1}\log_{[j]}^{-1}\Big(\frac{1}{y}\Big)\Big)\log_{[l]}^{-\alpha/(n-1)}\Big(\frac{1}{y}\Big)\frac{1}{y}\,\mathrm{d}y \\
&= \frac{n-1}{n} B \frac{1 - \frac{1}{16}\log_{[l]}^{-b}(1/t)}{\log_{[l]}^{n/((n-1)\gamma)}(1/t)}\Big[-\frac{\log_{[l]}^{1-\alpha/(n-1)}(1/y)}{1 - \alpha/(n-1)}\Big]_t^M.
\end{aligned}$$

Since $1 - \alpha/(n-1) = B = n/((n-1)\gamma)$ (see (1.2)) and $\log_{[l]}(1/M) > 0$ (see (3.14)), we have

$$I_1 \leqslant \frac{n-1}{n}\Big(1 - \frac{1}{16}\log_{[l]}^{-b}\Big(\frac{1}{t}\Big)\Big). \tag{3.34}$$

From (3.15), (3.27), (3.34) and

$$0 < b < \min\{1, 1/\gamma\} \leqslant n/((n-1)\gamma) - (\min\{1, 1/\gamma\}/(n-1))$$

we obtain that there is $t_6 \in (0, t_5)$ such that for $0 < t < t_6$ we have

$$\begin{aligned}\int_t^{\mathcal{L}_n(\Omega)/2} \Psi_1\Big(\frac{1}{\lambda h(y)}\Big)\,\mathrm{d}y &\leqslant I_1 + I_2 \\ &\leqslant C\frac{1}{\log_{[l]}^{n/((n-1)\gamma)-\min\{1,\gamma\}/(n-1)}(1/t)} + \frac{n-1}{n}\Big(1 - \frac{1}{16}\log_{[l]}^{-b}\Big(\frac{1}{t}\Big)\Big) \\ &\leqslant \frac{n-1}{n} = \Psi_1(1).\end{aligned}$$

Case $l = 1$. First, by (2.8), (3.13) and (3.26), there are $t_3 \in (0, t_2)$ and $\tilde{C} > 0$ such that for $0 < t < t_3$ and $y \in [t, M]$ we have

$$1 - \log^{-b}\Big(\frac{1}{\lambda h(y)}\Big) \leqslant 1 - \tilde{C}\log^{-b}\Big(\frac{1}{y}\Big) \tag{3.35}$$

and

$$\frac{1}{(1 - \log^{-c}(1/y))^{n/(n-1)}} \leqslant 1 + C\log^{-c}\Big(\frac{1}{y}\Big). \tag{3.36}$$

Further, let us prove that there is $t_4 \in (0, t_3)$ such that for $0 < t < t_4$ and $y \in [t, M]$ we have the estimate

$$\log^{-\alpha/(n-1)}\Big(\frac{1}{\lambda h(y)}\Big) \leqslant \Big(\frac{n}{n-1}\Big)^{\alpha/(n-1)}\log^{-\alpha/(n-1)}\Big(\frac{1}{y}\Big)\Big(1 + \frac{\tilde{C}}{2}\log^{-b}\Big(\frac{1}{y}\Big)\Big). \tag{3.37}$$

For $\alpha \geqslant 0$ estimate (3.37) is obtained in the same way as (3.28). Now, let $\alpha < 0$. This time for $y > 0$ and $t > 0$ small enough we have $\lambda h(y) \geqslant y^{(n-1)/n}$ (see (2.8) and (3.26)). Thus

$$\log\Big(\frac{1}{\lambda h(y)}\Big) \leqslant \log\Big(\frac{1}{y^{(n-1)/n}}\Big) = \frac{n-1}{n}\log\Big(\frac{1}{y}\Big)$$

and (3.37) follows.

Hence (2.8), (3.6), (3.26), (3.35), (3.36), (3.37) and (3.33) give us that

$$\begin{aligned}I_1 &= \frac{(n-1)^{1+\alpha/(n-1)}}{n}\int_t^M \Big(\frac{1}{\lambda h(y)}\Big)^{n/(n-1)}\log^{-\alpha/(n-1)}\frac{1}{\lambda h(y)}\Big(1 - \log^{-b}\frac{1}{\lambda h(y)}\Big)\,\mathrm{d}y \\ &\leqslant \frac{(n-1)(\frac{1}{2}\omega_{n-1})^{-1/(n-1)}(1 + C\log^{-c}(1/t))}{n^{2-\alpha/(n-1)}D^{n/(n-1)}\log^{n/((n-1)\gamma)}(1/t)} \\ &\quad \int_t^M \log^{-\alpha/(n-1)}\Big(\frac{1}{y}\Big)\Big(1 + C\log^{-1}\frac{1}{y}\Big)\Big(1 + \frac{\tilde{C}}{2}\log^{-b}\frac{1}{y}\Big)\Big(1 - \tilde{C}\log^{-b}\frac{1}{y}\Big)\frac{\mathrm{d}y}{y}.\end{aligned}$$

Further, as $0 < b < c < 1$, there is $t_5 \in (0, t_4)$ such that for $0 < t < t_5$ and $y \in [t, M]$ we have

$$\Big(1 + C \log^{-c}\Big(\frac{1}{t}\Big)\Big)\Big(1 + C \log^{-1}\Big(\frac{1}{y}\Big)\Big)\Big(1 + \frac{\tilde{C}}{2}\log^{-b}\Big(\frac{1}{y}\Big)\Big)\Big(1 - \tilde{C}\log^{-b}\Big(\frac{1}{y}\Big)\Big) \leqslant 1 - \frac{\tilde{C}}{4}\log^{-b}\Big(\frac{1}{t}\Big).$$

Therefore (3.7) and $-\alpha/(n-1) = B - 1 \neq -1$ (see (1.2)) imply

$$\begin{aligned} I_1 &\leqslant \frac{n-1}{n} B \frac{1 - \frac{1}{4}\tilde{C}\log^{-b}(1/t)}{\log^{n/((n-1)\gamma)}(1/t)} \int_t^M \log^{-\alpha/(n-1)}\Big(\frac{1}{y}\Big)\frac{1}{y}\,\mathrm{d}y \\ &= \frac{n-1}{n} B \frac{1 - \frac{1}{4}\tilde{C}\log_{[l]}^{-b}(1/t)}{\log^{n/((n-1)\gamma)}(1/t)} \Big[-\frac{\log^{1-(\alpha/(n-1))}(1/y)}{1 - \alpha/(n-1)}\Big]_t^M. \end{aligned}$$

Since $1 - \alpha/(n-1) = B = n/((n-1)\gamma)$ (by (1.2)) and $\log(1/M) > 0$ (by (3.14)), we have

$$I_1 \leqslant \frac{n-1}{n}\Big(1 - \frac{\tilde{C}}{4}\log^{-b}\Big(\frac{1}{t}\Big)\Big)$$

and we complete the proof in the same way as in the previous case. □

4. Concentrating sequences

Let $R > 0$. We make use of the following sequences of $W_0L^\Phi(B(R))$-functions from [1], [5] and [17] that played an important role in the proof of Theorem 1.1 (iii) and (v), respectively. For $l = 1$ we set

$$w_k(x) = g_k(|x|), \tag{4.1}$$

where

$$g_k(y) = \begin{cases} 0 & \text{for } y \in [R, \infty), \\ \Big(-\frac{2}{R}y + 2\Big)K_{1,n,\alpha}^{-1/\gamma} n^B \log^B(2) k^{1/\gamma - B}\Big(1 + \frac{\log(k)}{k}\Big)^{1/\gamma} & \text{for } y \in \Big[\frac{R}{2}, R\Big], \\ K_{1,n,\alpha}^{-1/\gamma} n^B \log^B\Big(\frac{R}{y}\Big) k^{1/\gamma - B}\Big(1 + \frac{\log(k)}{k}\Big)^{1/\gamma} & \text{for } y \in \Big[R\mathrm{e}^{-k/n}, \frac{R}{2}\Big], \\ K_{1,n,\alpha}^{-1/\gamma} k^{1/\gamma}\Big(1 + \frac{\log(k)}{k}\Big)^{1/\gamma} & \text{for } y \in [0, R\mathrm{e}^{-k/n}]. \end{cases}$$

In case $l \geqslant 2$ we fix $T > \exp_{[l]}(1)$ and define

$$w_k(x) = g_k(|x|), \tag{4.2}$$

where

$$g_k(y) = \begin{cases} 0 & \text{for } y \in [R, \infty), \\ \Big(-\frac{2}{R}y + 2\Big) K_{l,n,\alpha}^{-1/\gamma} \log_{[l]}^{B}(T+2) k^{1/\gamma - B}\Big(1 + \frac{\log k}{k}\Big)^{1/\gamma} & \\ & \text{for } y \in \Big[\frac{R}{2}, R\Big], \\ K_{l,n,\alpha}^{-1/\gamma} \log_{[l]}^{B}(T + R/y) k^{1/\gamma - B}\Big(1 + \frac{\log k}{k}\Big)^{1/\gamma} & \text{for } y \in \Big[R \exp_{[l]}^{-1/n}(k), \frac{R}{2}\Big], \\ K_{l,n,\alpha}^{-1/\gamma} \log_{[l]}^{B}(T + \exp_{[l]}^{1/n}(k)) k^{1/\gamma - B}\Big(1 + \frac{\log k}{k}\Big)^{1/\gamma} & \\ & \text{for } y \in [0, R \exp_{[l]}^{-1/n}(k)]. \end{cases}$$

For a given $K > K_{l,n,\alpha}$ we fix $A \in (K_{l,n,\alpha}, K)$ and define for $l = 1$

$$\tilde{w}_k(x) = g_k(|x|) \tag{4.3}$$

where

$$g_k(y) = \begin{cases} 0 & \text{for } y \in [R, \infty), \\ \Big(-\frac{2}{R}y + 2\Big) A^{-1/\gamma} n^B \log^B(2) k^{1/\gamma - B} & \text{for } y \in \Big[\frac{R}{2}, R\Big], \\ A^{-1/\gamma} n^B \log^B\Big(\frac{R}{y}\Big) k^{1/\gamma - B} & \text{for } y \in \Big[R\mathrm{e}^{-k/n}, \frac{R}{2}\Big], \\ A^{-1/\gamma} k^{1/\gamma} & \text{for } y \in [0, R\mathrm{e}^{-k/n}]. \end{cases}$$

In case $l \geqslant 2$ we fix $T > \exp_{[l]}(1)$ and we define

$$\tilde{w}_k(x) = g_k(|x|) \tag{4.4}$$

where

$$g_k(y) = \begin{cases} 0 & \text{for } y \in [R, \infty), \\ \Big(-\frac{2}{R}y + 2\Big) A^{-1/\gamma} \log_{[l]}^{B}(T+2) k^{(1/\gamma) - B} & \text{for } y \in \Big[\frac{R}{2}, R\Big], \\ A^{-1/\gamma} \log_{[l]}^{B}\Big(T + \frac{R}{y}\Big) k^{(1/\gamma) - B} & \text{for } y \in \Big[R \exp_{[l]}^{-1/n}(k), \frac{R}{2}\Big], \\ A^{-1/\gamma} \log_{[l]}^{B}(T + \exp_{[l]}^{1/n}(k)) k^{(1/\gamma) - B} & \text{for } y \in [0, R \exp_{[l]}^{-1/n}(k)]. \end{cases}$$

The proofs of [1, Example 5.1], [17, Theorem 1.2] and [5, Theorem 1.2 and Theorem 4.1] give us the following results.

We have

$$\int_{B(R)} \exp_{[l]}(K_{l,n,\alpha}|w_k|^\gamma) \overset{k\to\infty}{\longrightarrow} \infty \tag{4.5}$$

and if $K_{l,n,\alpha} < A < K$ then

$$\int_{B(R)} \exp_{[l]}(K|\tilde{w}_k|^\gamma) \overset{k\to\infty}{\longrightarrow} \infty. \tag{4.6}$$

If Φ satisfies (1.3) then there is $k_0 \in \mathbb{N}$ such that

$$\|\Phi(\nabla\tilde{w}_k)\|_{L^1(B(R))} \leqslant 1 \quad \text{for } k \geqslant k_0, \tag{4.7}$$

and if Φ satisfies (1.3) and (1.5) then there is $k_0 \in \mathbb{N}$ such that

$$\|\Phi(\nabla w_k)\|_{L^1(B(R))} \leqslant 1 \quad \text{for } k \geqslant k_0. \tag{4.8}$$

Further, one easily modifies the proofs so that for $\theta > 0$ fixed there is $k_0 \in \mathbb{N}$ such that

$$\int_{B(R)} \Phi(\theta|\nabla\tilde{w}_k|) \leqslant \theta^n \quad \text{for every } acek \geqslant k_0 \tag{4.9}$$

provided Φ satisfies (1.3), and if Φ satisfies (1.3) and (1.5), then

$$\int_{B(R)} \Phi(\theta|\nabla w_k|) \leqslant \theta^n \quad \text{for every } k \geqslant k_0. \tag{4.10}$$

Let us note that [17, Theorem 1.2] (which concerns the sequence $\{\tilde{w}_k\}_{k\in\mathbb{N}}$ in the case $l = 1$) contains an assumption concerning the behavior of Φ near the origin, but this assumption can be removed using the convexity of Φ (cf. [5, Proof of Theorem 1.2]).

5. Proof of Theorem 1.2

P r o o f of Theorem 1.2 (iii) and (v). Let $\tilde{\Omega} \subset \mathbb{R}^n$ be a smooth domain such that there is $R > 0$ satisfying

$$\tilde{\Omega} \cap B(R) = \{x \in B(R)\colon\ x_n > 0\}. \tag{5.1}$$

As $K > K_{l,n,\alpha}(\frac{1}{2})^{\gamma/n}$, we can find $A \in (K_{l,n,\alpha}, 2^{\gamma/n}K)$. We define

$$u_k = 2^{1/n}\tilde{w}_k, \quad k \geqslant k_0,$$

where $\tilde{w}_k$ are given by (4.3) and (4.4) (with the parameter A chosen above), respectively, and $k_0 \in \mathbb{N}$ comes from (4.9) for $\theta = 2^{1/n}$. It is not difficult to see from (4.3) and (4.4), respectively, that

$$0 \leqslant \operatorname{med}(u_k) \overset{k\to\infty}{\longrightarrow} 0.$$

Next, by $A > K_{l,n,\alpha}$, (5.1) and (4.9) we have

$$\int_{\tilde{\Omega}} \Phi(|\nabla u_k|) = \frac{1}{2}\int_{B(R)} \Phi(|\nabla u_k|) = \frac{1}{2}\int_{B(R)} \Phi(2^{1/n}|\nabla \tilde{w}_k|) \leqslant 1.$$

Finally, we use (5.1), (4.6) and $2^{\gamma/n}K > A$ to obtain

$$\int_{\tilde{\Omega}} \exp_{[l]}(K|u_k|^\gamma) = \frac{1}{2}\int_{B(R)} \exp_{[l]}\big(K2^{\gamma/n}|\tilde{w}_k|^\gamma\big) \overset{k\to\infty}{\longrightarrow} \infty.$$

This completes the proof of Theorem 1.2 (iii).

Theorem 1.2 (v) is proved in the same way. We define

$$u_k = 2^{1/n}w_k, \quad k \geqslant k_0,$$

where w_k are given by (4.1) and (4.2), respectively. The properties of the sequence $\{u_k\}$ are verified using (4.10) and (4.5). □

Remark 5.1. One can easily see from the above proof that in Theorem 1.2 (iii), it is enough to suppose that $\tilde{\Omega}$ is from the class $C^{1,\theta}$, $\theta \in (0,1]$. Indeed, for any $x_0 \in \partial\tilde{\Omega}$ we can find the radius $R > 0$ so small that $\mathcal{L}_n(B(x_0,R) \cap \tilde{\Omega})$ is as close to $\frac{1}{2}\mathcal{L}_n(B(x_0,R))$ as we need.

Now we proceed to the proof of Theorem 1.2 (i), (ii) and (iv). We start with some estimates common for all three proofs. First, we define

$$v = u - \operatorname{med}(u).$$

This implies that $\operatorname{med}(v) = 0$, $\nabla v = \nabla u$ on Ω and

$$\int_\Omega \exp_{[l]}(K(|u|)^\gamma) \leqslant \int_\Omega \exp_{[l]}(K(|v| + |\operatorname{med}(u)|)^\gamma).$$

In the rest of the proof we estimate the right-hand side of the above inequality.

Changing the sign of u if necessary, we can suppose that

$$\int_{\Omega\cap\{v\geqslant 0\}} \exp_{[l]}(K(|v| + |\operatorname{med}(u)|)^\gamma) \geqslant \int_{\Omega\cap\{v\leqslant 0\}} \exp_{[l]}(K(|v| + |\operatorname{med}(u)|)^\gamma).$$

We make use of the following estimate of $v^\circ(s_1) - v^\circ(s_2)$ for $0 < s_1 < s_2 \leqslant \frac{1}{2}\mathcal{L}_n(\Omega)$. From (2.8) and the generalized Hölder's inequality (2.2) we have

$$\begin{aligned}
(5.2)\quad & v^\circ(s_1) - v^\circ(s_2) \\
&= \int_{s_1}^{s_2} -\frac{\mathrm{d}v^\circ}{\mathrm{d}y}(y)\,\mathrm{d}y = \int_{s_1}^{s_2} -\frac{\mathrm{d}v^\circ}{\mathrm{d}y}(y)h(y)\frac{1}{h(y)}\,\mathrm{d}y \\
&\leqslant \int_{(s_1,s_2)\cap\{(-\mathrm{d}v^\circ/\mathrm{d}y)(y)h(y)\geqslant G\}} -\frac{\mathrm{d}v^\circ}{\mathrm{d}y}(y)h(y)\frac{1}{h(y)}\,\mathrm{d}y + \int_0^{\frac{1}{2}\mathcal{L}_n(\Omega)} \frac{G}{h(y)}\,\mathrm{d}y \\
&\leqslant \Big\| -\frac{\mathrm{d}v^\circ}{\mathrm{d}y}(y)h(y)\Big\|_{L^{\Phi_1}((s_1,s_2)\cap\{-(\mathrm{d}v^\circ/\mathrm{d}y)(y)h(y)\geqslant G\})} \\
&\quad\times \Big\|\frac{1}{h(y)}\Big\|_{L^{\Psi_1}((s_1,\frac{1}{2}\mathcal{L}_n(\Omega)))} + C,
\end{aligned}$$

where G comes from (3.1) and (3.2), respectively. Moreover, if we assume that $\int_\Omega \Phi(|\nabla u|) \leqslant 1$, then Lemma 2.3 together with (3.1) and (3.2), respectively, implies

$$\begin{aligned}
\int_{(s_1,s_2)\cap\{(-\mathrm{d}v^\circ/\mathrm{d}y)(y)h(y)\geqslant G\}} \Phi_1\Big(-\frac{\mathrm{d}v^\circ}{\mathrm{d}y}(y)h(y)\Big)\,\mathrm{d}y &\leqslant \frac{1}{n}\int_0^{\mathcal{L}_n(\Omega)} \Phi\Big(-\frac{\mathrm{d}v^\circ}{\mathrm{d}y}(y)h(y)\Big)\,\mathrm{d}y \\
&\leqslant \frac{1}{n}\int_\Omega \Phi(|\nabla v|)\,\mathrm{d}x \leqslant \frac{1}{n} = \Phi_1(1)
\end{aligned}$$

and thus

$$(5.3)\qquad \Big\| -\frac{\mathrm{d}v^\circ}{\mathrm{d}y}(y)h(y)\Big\|_{L^{\Phi_1}((s_1,s_2)\cap\{(-\mathrm{d}v^\circ/\mathrm{d}y)(y)h(y)\geqslant G\})} \leqslant 1.$$

P r o o f of Theorem 1.2 (i). From $u \in WL^{\Phi}(\Omega)$, (3.2), and Lemma 2.3 we can see that $\int_0^{\mathcal{L}_n(\Omega)} \Phi_1(-(\mathrm{d}v^\circ/\mathrm{d}y)(y)h(y))$ is finite and thus we can find $s_2 \in (0, \frac{1}{2}\mathcal{L}_n(\Omega))$ so small that

$$\Big\| -\frac{\mathrm{d}v^\circ}{\mathrm{d}y}(y)h(y) \Big\|_{L^{\Phi_1}((s_1,s_2))} < \delta \quad \text{for every } s_1 \in (0, s_2),$$

where $\delta > 0$ is a small fixed number specified below. Thus, decreasing s_2 if necessary, we obtain from (5.2) and Lemma 3.3 (i) (where we set $\varepsilon = 1$)

$$v^\circ(s_1) \leqslant v^\circ(s_2) + C\delta \log_{[l]}^{1/\gamma}\Big(\frac{1}{s_1}\Big) + C.$$

Hence, if δ is small enough, we can find $y_0 \in (0, s_2)$ so small that

$$K(v^\circ(y) + |\mathrm{med}(u)|)^\gamma \leqslant \frac{1}{2}\log_{[l]}\Big(\frac{1}{y}\Big) \leqslant \log_{[l]}\Big(\frac{1}{\sqrt{y}}\Big) \quad \text{for every } y \in (0, y_0).$$

Therefore, as $0 = \mathrm{med}(v) = v^\circ(\frac{1}{2}\mathcal{L}_n(\Omega))$,

$$\begin{aligned}\int_{\Omega\cap\{v\geqslant 0\}} &\exp_{[l]}(K(|v(x)| + |\mathrm{med}(u)|)^\gamma)\,\mathrm{d}x \\ &= \int_0^{\frac{1}{2}\mathcal{L}_n(\Omega)} \exp_{[l]}(K(|v^\circ(y)| + |\mathrm{med}(u)|)^\gamma)\,\mathrm{d}y \\ &\leqslant \int_0^{y_0} y^{-1/2}\,\mathrm{d}y + \int_{y_0}^{\frac{1}{2}\mathcal{L}_n(\Omega)} \exp_{[l]}(K(|v^\circ(y_0)| + |\mathrm{med}(u)|)^\gamma)\,\mathrm{d}y < \infty\end{aligned}$$

and we are done. □

P r o o f of Theorem 1.2 (ii). Since $K < K_{l,n,\alpha}(\frac{1}{2})^{\gamma/n}$, there is $\varepsilon > 0$ so small that

$$(1+\varepsilon)^{2\gamma} < \frac{K_{l,n,\alpha}(\frac{1}{2})^{\gamma/n}}{K}. \tag{5.4}$$

By (5.2) with $s_2 = \frac{1}{2}\mathcal{L}_n(\Omega)$ (recall $0 = \mathrm{med}(v) = v^\circ\big(\frac{1}{2}\mathcal{L}_n(\Omega)\big)$), (5.3), Lemma 3.3 (i) with our ε and $|\mathrm{med}(u)| \leqslant M$ we have

$$v^\circ(s_1) + |\mathrm{med}(u)| \leqslant (1+\varepsilon)D\log_{[l]}^{1/\gamma}\Big(\frac{1}{s_1}\Big) + C + M$$

provided $s_1 > 0$ is sufficiently small. Hence there is $y_0 \in (0, \frac{1}{2}\mathcal{L}_n(\Omega))$ such that

$$v^\circ(y) + |\mathrm{med}(u)| \leqslant (1+\varepsilon)^2 D\log_{[l]}^{1/\gamma}\Big(\frac{1}{y}\Big) \quad \text{for every } y \in (0, y_0).$$

Finally, using (5.4) and $D^\gamma = K_{l,n,\alpha}^{-1} 2^{\gamma/n}$ (see (1.2) and (3.7)) we conclude

$$\begin{aligned}
&\int_{\Omega\cap\{v\geqslant 0\}} \exp_{[l]}(K(|v(x)|+|\mathrm{med}(u)|)^\gamma)\,\mathrm{d}x \\
&\qquad = \int_0^{\frac12\mathcal{L}_n(\Omega)} \exp_{[l]}(K(|v^\circ(y)|+|\mathrm{med}(u)|)^\gamma)\,\mathrm{d}y \\
&\qquad \leqslant \int_0^{y_0} \exp_{[l]}\Big(K(1+\varepsilon)^{2\gamma} D^\gamma \log_{[l]}\Big(\frac{1}{y}\Big)\Big)\,\mathrm{d}y \\
&\qquad\quad + \int_{y_0}^{\frac12\mathcal{L}_n(\Omega)} \exp_{[l]}(K(|v^\circ(y_0)|+|\mathrm{med}(u)|)^\gamma)\,\mathrm{d}y \\
&\qquad \leqslant C + \int_0^{y_0} \exp_{[l]}\Big(\frac{K(1+\varepsilon)^{2\gamma}}{K_{l,n,\alpha}(\frac12)^{\gamma/n}} \log_{[l]}\Big(\frac{1}{y}\Big)\Big)\,\mathrm{d}y \leqslant C.
\end{aligned}$$

□

P r o o f of Theorem 1.2 (iv). By (5.2) with $s_2 = \frac12\mathcal{L}_n(\Omega)$, (5.3), Lemma 3.3 (ii), and $|\mathrm{med}(u)| \leqslant M$ we have

$$v^\circ(s_1) + |\mathrm{med}(u)| \leqslant D\log_{[l]}^{1/\gamma}\Big(\frac{1}{s_1}\Big)\Big(1-\log_{[l]}^{-c}\Big(\frac{1}{s_1}\Big)\Big) + C + M$$

provided $s_1 > 0$ is sufficiently small. Hence, as $c < 1/\gamma$, there is $y_0 \in (0, \frac12\mathcal{L}_n(\Omega))$ such that

$$\text{(5.5)}\qquad v^\circ(y) + |\mathrm{med}(u)| \leqslant D\log_{[l]}^{1/\gamma}\Big(\frac{1}{y}\Big)\Big(1-\frac12\log_{[l]}^{-c}\Big(\frac{1}{y}\Big)\Big) \quad \text{for every } y\in(0,y_0).$$

Next, there is $C_1 > 0$ such that

$$\text{(5.6)}\qquad \Big(1-\frac12\log_{[l]}^{-c}\Big(\frac{1}{y}\Big)\Big)^\gamma \leqslant 1 - C_1\log_{[l]}^{-c}\Big(\frac{1}{y}\Big) \quad \text{for every } y\in(0,y_0).$$

Finally, using (5.5), (5.6) and $D^\gamma = K_{l,n,\alpha}^{-1}2^{\gamma/n} = 1/K$ (see (1.2) and (3.7)) we conclude

$$\begin{aligned}
&\int_{\Omega\cap\{v\geqslant 0\}} \exp_{[l]}(K(|v(x)|+|\mathrm{med}(u)|)^\gamma)\,\mathrm{d}x \\
&\qquad \leqslant \int_0^{y_0} \exp_{[l]}\Big(\log_{[l]}\Big(\frac{1}{y}\Big)\Big(1-C_1\log_{[l]}^{-c}\Big(\frac{1}{y}\Big)\Big)\Big)\,\mathrm{d}y \\
&\qquad\quad + \int_{y_0}^{\frac12\mathcal{L}_n(\Omega)} \exp_{[l]}(K(|v^\circ(y_0)|+|\mathrm{med}(u)|)^\gamma)\,\mathrm{d}y \leqslant C.
\end{aligned}$$

Indeed, the latter integral is plainly finite and for the former in the case $l = 1$ we have from $0 < c < 1$ for $y > 0$ very small

$$\exp\Big(\log\Big(\frac{1}{y}\Big) - C_1 \log^{1-c}\Big(\frac{1}{y}\Big)\Big) \leqslant \exp\Big(\log\Big(\frac{1}{y}\Big) - \log_{[2]}\Big(\frac{1}{y}\Big)\Big)$$
$$= \frac{1}{y}\,\frac{1}{\log^2(1/y)} \in L^1\Big(\Big(0, \frac{1}{2}\Big)\Big).$$

For $l \geqslant 2$ and y small enough we can use estimate (3.9) from Lemma 3.4 and $0 < c < 1$ to obtain

$$\log_{[l]}\Big(\frac{1}{y}\Big) - C_1 \log_{[l]}^{1-c}\Big(\frac{1}{y}\Big) \leqslant \log_{[l]}\Big(\frac{1}{y}\Big) - C \leqslant \log_{[l]}\Big(\frac{1}{y}\,\frac{1}{\log^2(1/y)}\Big)$$

and thus

$$\exp_{[l]}\Big(\log_{[l]}\Big(\frac{1}{y}\Big)\Big(1 - C_1 \log_{[l]}^{-c}\Big(\frac{1}{y}\Big)\Big)\Big) \leqslant \frac{1}{y}\,\frac{1}{\log^2(1/y)} \in L^1\Big(\Big(0, \frac{1}{2}\Big)\Big).$$

□

Remark 5.2. By the previous proof we can see that for any fixed $\tilde{C} \geqslant 0$, we have versions of Theorem 1.2 (ii) and (iv) with

$$\int_\Omega \exp_{[l]}(K(\tilde{C} + |u(x)|)^\gamma) \leqslant C(\tilde{C}, l, n, \alpha, \Phi, \mathcal{L}_n(\Omega), K, M).$$

6. Concentration-Compactness Principle

The main ingredient in the proof of Theorem 1.3 is the following result.

Proposition 6.1. *Let $l \in \mathbb{N}$, $n \geqslant 2$ and $\alpha < n - 1$. Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded connected domain of the class $C^{1,\theta}$ for some $\theta \in (0, 1]$. Let Φ be a Young function satisfying* (1.3). *Let $\{u_k\}_{k=1}^\infty \subset WL^\Phi(\Omega)$ satisfy $\|\Phi(|\nabla u_k|)\|_{L^1(\Omega)} \leqslant 1$ and*

$$\operatorname{med}(u_k) = 0, \quad u_k \rightharpoonup u \text{ in } WL^\Phi(\Omega), \quad \text{and} \quad u_k \to u \text{ a.e. in } \Omega$$

for some $u \in WL^\Phi(\Omega)$. Then for every

$$p < P := \begin{cases} (1 - \int_\Omega \Phi(|\nabla u|))^{-\gamma/n} & \text{if } \int_\Omega \Phi(|\nabla u|) < 1, \\ \infty & \text{if } \int_\Omega \Phi(|\nabla u|) = 1, \end{cases}$$

we have

$$\int_\Omega \exp_{[l]}\Big(K_{l,n,\alpha}\Big(\frac{1}{2}\Big)^{\gamma/n} p|u_k|^\gamma\Big) \leqslant C \quad \textit{where } C \textit{ is independent of } k \in \mathbb{N}.$$

P r o o f. We distinguish three cases.

Case $\int_\Omega \Phi(|\nabla u|) = 0$. We have $P = 1$ and our statement is just a weaker version of Theorem 1.2 (ii).

Case $0 < \int_\Omega \Phi(|\nabla u|) < 1$. We proceed by contradiction. Suppose that there exists a sequence $\{u_k\}$ satisfying the assumptions of the proposition and

$$\int_\Omega \exp_{[l]}\Big(K_{l,n,\alpha}\Big(\frac{1}{2}\Big)^{\gamma/n} p_1|u_k|^\gamma\Big) \stackrel{k\to\infty}{\longrightarrow} \infty \quad \text{for some } p_1 < P.$$

Passing to a subsequence and changing the sign of the entire sequence if necessary we can suppose that

$$\int_{\Omega\cap\{u_k\geqslant 0\}} \exp_{[l]}\Big(K_{l,n,\alpha}\Big(\frac{1}{2}\Big)^{\gamma/n} p_1|u_k|^\gamma\Big) \stackrel{k\to\infty}{\longrightarrow} \infty.$$

That is, for a signed non-increasing rearrangement which is equimeasurable we have

$$\int_0^{\frac{1}{2}\mathcal{L}_n(\Omega)} \exp_{[l]}\Big(K_{l,n,\alpha}\Big(\frac{1}{2}\Big)^{\gamma/n} p_1|u_k^\circ|^\gamma\Big) \stackrel{k\to\infty}{\longrightarrow} \infty. \tag{6.1}$$

Next, we fix $p_2, p_3 \in (p_1, P)$ such that $p_2 < p_3$. Let us fix $\tilde{G} > G$ (G comes from (3.2)) so that Lemma 3.1 holds with this $\tilde{G}$ for $C_1 = (1/p_3)^\gamma$, $C_2 = 1$ and some $\delta > 0$ so small that

$$\frac{p_3}{P}(1+\delta)^{3\gamma/n} < 1. \tag{6.2}$$

Finally, fix $\varepsilon > 0$ such that

$$1+\varepsilon < \Big(\frac{p_3}{p_2}\Big)^{1/\gamma}. \tag{6.3}$$

The rest of the proof is divided into several steps.

Step 1 (*Upper estimates of* u_k°). As the assumptions of our proposition are more restrictive than the assumptions of Theorem 1.2 (ii), we can use all partial results from its proof. In particular, for all $k \in \mathbb{N}$ we have the estimate (see (5.2))

$$u_k^\circ(s_1) - u_k^\circ(s_2) \leqslant \Big\|-\frac{\mathrm{d}u_k^\circ}{\mathrm{d}y}(y)h(y)\Big\|_{L^{\Phi_1}((s_1,s_2)\cap A_k)}\Big\|\frac{1}{h(y)}\Big\|_{L^{\Psi_1}((s_1,s_2))} + C, \tag{6.4}$$

where $0 < s_1 < s_2 \leqslant \frac{1}{2}\mathcal{L}_n(\Omega)$, $C = C(G, \mathcal{L}_n(\Omega))$ and

$$A_k = \Big\{ y \in \Big(0, \frac{\mathcal{L}_n(\Omega)}{2}\Big) \colon -\frac{\mathrm{d}u_k^\circ}{\mathrm{d}y}(y)h(y) \geqslant G \Big\}.$$

We also have (see (5.3))

$$\Big\| -\frac{\mathrm{d}u_k^\circ}{\mathrm{d}y}(y)h(y) \Big\|_{L^{\Phi_1}((s_1,s_2)\cap A_k)} \leqslant 1 \tag{6.5}$$

for $0 < s_1 < s_2 \leqslant \frac{1}{2}\mathcal{L}_n(\Omega)$ and $k \in \mathbb{N}$. In particular, if $s_2 = \frac{1}{2}\mathcal{L}_n(\Omega)$, estimates (6.4), (6.5), $u_k^\circ(\frac{1}{2}\mathcal{L}_n(\Omega)) = \operatorname{med}(u_k) = 0$ and Lemma 3.3 (i) with $\varepsilon = 1$ give for every $y \in (0, \frac{1}{2}\mathcal{L}_n(\Omega))$

$$u_k^\circ(y) \leqslant C + C \log_{[l]}^{1/\gamma}\Big(\frac{1}{y}\Big). \tag{6.6}$$

Step 2 (*Lower estimates of* u_k°). We claim that for every $k_0 \in \mathbb{N}$ and every $t_0 \in (0, \frac{1}{2}\mathcal{L}_n(\Omega))$ there exist $k \in \mathbb{N}$, $k > k_0$, and $t \in (0, t_0)$ such that

$$u_k^\circ(t) \geqslant \Big(\frac{1}{p_2 K_{l,n,\alpha}(\frac{1}{2})^{\gamma/n}}\Big)^{1/\gamma} \log_{[l]}^{1/\gamma}\Big(\frac{1}{t}\Big).$$

We prove this claim by contradiction. Suppose that there exist $k_0 \in \mathbb{N}$ and $t_0 \in (0, \frac{1}{2}\mathcal{L}_n(\Omega))$ such that

$$u_k^\circ(t) < \Big(\frac{1}{p_2 K_{l,n,\alpha}(\frac{1}{2})^{\gamma/n}}\Big)^{1/\gamma} \log_{[l]}^{1/\gamma}\Big(\frac{1}{t}\Big) \quad \text{for every } t \in (0, t_0) \text{ and } k \geqslant k_0.$$

Then by the this estimate, $p_1 < p_2$ and inequality (6.6), one has that, if $k \geqslant k_0$, then

$$\begin{aligned}\int_0^{\frac{1}{2}\mathcal{L}_n(\Omega)} & \exp_{[l]}\Big(K_{l,n,\alpha}\Big(\frac{1}{2}\Big)^{\gamma/n} p_1 |u_k^\circ|^\gamma\Big)\,\mathrm{d}x \\ & \leqslant \int_0^{t_0} \exp_{[l]}\Big(\frac{p_1}{p_2}\log_{[l]}\Big(\frac{1}{y}\Big)\Big)\,\mathrm{d}y + \int_{t_0}^{\frac{1}{2}\mathcal{L}_n(\Omega)} \exp_{[l]}\Big(C + C\log_{[l]}\Big(\frac{1}{t_0}\Big)\Big)\,\mathrm{d}y \leqslant C,\end{aligned}$$

an estimate which contradicts (6.1). Therefore, our claim is proved. Thus, possibly after passing to a subsequence, there exist $t_k \in (0, \frac{1}{2}\mathcal{L}_n(\Omega))$, $k \in \mathbb{N}$, such that

$$u_k^\circ(t_k) \geqslant \Big(\frac{1}{p_2 K_{l,n,\alpha}(\frac{1}{2})^{\gamma/n}}\Big)^{1/\gamma} \log_{[l]}^{1/\gamma}\Big(\frac{1}{t_k}\Big) \quad \text{and} \quad t_k \leqslant \frac{1}{k} \text{ for every } k \in \mathbb{N}. \tag{6.7}$$

Step 3 (*Decomposition of functions* u *and* u_k). Given $L > 0$, we define functions u^L and $\tilde{u}^L$ in Ω as

$$u^L(x) = \min\{|u(x)|, L\} \operatorname{sign}(u) \quad \text{and} \quad \tilde{u}^L(x) = u(x) - u^L(x) \text{ for } x \in \Omega.$$

The functions u_k^L and $\tilde{u}_k^L$ are defined analogously for $k \in \mathbb{N}$. It is not difficult to verify that

$$\begin{gathered} \int_\Omega \Phi(|\nabla u_k|) = \int_\Omega \Phi(|\nabla u_k^L|) + \int_\Omega \Phi(|\nabla \tilde{u}_k^L|), \\ u_k^L \to u^L \text{ a.e. in } \Omega \text{ and } \tilde{u}_k^L \to \tilde{u}^L \text{ a.e. in } \Omega. \end{gathered} \tag{6.8}$$

Moreover, $\{u_k^L\}$ is a bounded sequence in $WL^\Phi(\Omega)$, and hence there exists a weakly convergent subsequence. Since it converges almost everywhere to u^L, one also has that

$$u_k^L \rightharpoonup u^L \text{ in } WL^\Phi(\Omega) \quad \text{and} \quad \tilde{u}_k^L \rightharpoonup \tilde{u}^L \text{ in } WL^\Phi(\Omega).$$

Finally, we choose L so large that

$$\Big(\frac{1 - \int_\Omega \Phi(|\nabla u|)}{1 - \int_\Omega \Phi(|\nabla u^L|)}\Big)^{\gamma/n} > \frac{p_3}{P}(1+\delta)^{3\gamma/n}. \tag{6.9}$$

By (6.7), passing to a subsequence if necessary, we may suppose that $u_k^\circ(t_k) > L$ for every $k \in \mathbb{N}$. Consequently, there exist $s_k \in (t_k, \frac{1}{2}\mathcal{L}_n(\Omega))$ such that $u_k^\circ(s_k) = L$ for every $k \in \mathbb{N}$.

Step 4 (*Final computation leading to a contradiction*). By (6.7), (6.4), and Lemma 3.3 (i) we have

$$\begin{aligned} &\Big(\frac{1}{p_2 K_{l,n,\alpha}(\frac{1}{2})^{\gamma/n}}\Big)^{1/\gamma} \log_{[l]}^{1/\gamma}\Big(\frac{1}{t_k}\Big) - L \\ &\qquad \leqslant u_k^\circ(t_k) - u_k^\circ(s_k) \\ &\qquad \leqslant C + \Big\| -\frac{\mathrm{d}u_k^\circ}{\mathrm{d}y}(y)h(y)\Big\|_{L^{\Phi_1}((t_k,s_k)\cap A_k)} \Big((1+\varepsilon)D \log_{[l]}^{1/\gamma}\Big(\frac{1}{t_k}\Big)\Big). \end{aligned}$$

Hence, from $p_3 > p_2$, (6.3) and $D = K_{l,n,\alpha}^{-1/\gamma} 2^{1/n}$ (see (1.2) and (3.7)), for all k large enough we obtain

$$\Big(\frac{1}{p_3}\Big)^{1/\gamma} \leqslant \Big\| -\frac{\mathrm{d}u_k^\circ}{\mathrm{d}y}(y)h(y)\Big\|_{L^{\Phi_1}((t_k,s_k)\cap A_k)}. \tag{6.10}$$

From (6.10), Lemma 3.1 (the assumptions are satisfied by the choices preceding Step 1, (6.5) and (6.10)), Lemma 2.3 (applied to each function $\tilde{u}_k$, $k \in \mathbb{N}$) and (6.8) we see that for $k \in \mathbb{N}$ large enough

$$\begin{aligned} p_3 &\geqslant \frac{1}{\|-(\mathrm{d}u_k^\circ/\mathrm{d}y)(y)h(y)\|^{\gamma}_{L^{\Phi_1}((t_k,s_k)\cap A_k)}} \\ &\geqslant \frac{1}{(1+\delta)^{3\gamma/n}\big(\int_{(t_k,s_k)\cap A_k}\Phi(|-(\mathrm{d}u_k^\circ/\mathrm{d}y)(y)h(y)|)\big)^{\gamma/n}} \\ &\geqslant \frac{(1+\delta)^{-3\gamma/n}}{\big(\int_\Omega \Phi(|\nabla \tilde{u}_k^L|)\big)^{\gamma/n}} \geqslant \frac{(1+\delta)^{-3\gamma/n}}{\big(1-\int_\Omega \Phi(|\nabla u_k^L|)\big)^{\gamma/n}}. \end{aligned}$$

This last inequality, the weak lower semicontinuity of the modular and (6.9) yield

$$\begin{aligned} p_3 &\geqslant \frac{(1+\delta)^{-3\gamma/n}}{\Big(1-\liminf_{k\to\infty}\int_\Omega \Phi(|\nabla u_k^L|)\Big)^{\gamma/n}} \\ &\geqslant \frac{(1+\delta)^{-3\gamma/n}}{\big(1-\int_\Omega \Phi(|\nabla u^L|)\big)^{\gamma/n}} > \frac{p_3}{P}\frac{1}{\big(1-\int_\Omega \Phi(|\nabla u|)\big)^{\gamma/n}} = p_3, \end{aligned}$$

which gives us the desired contradiction.

Case $\int_\Omega \Phi(|\nabla u|) = 1$. The proof is similar to the previous one, therefore we only sketch it pointing out the differences.

First, we do not have any upper bound of p_1 and we just fix any p_2, p_3 such that $p_1 < p_2 < p_3$. The level $L > 0$ is chosen such that

$$\Big(\frac{1}{1-\int_\Omega \Phi(|\nabla u^L|)}\Big)^{\gamma/n} > p_3(1+\delta)^{3\gamma/n}$$

and thus the final computation takes the form

$$p_3 \geqslant \frac{(1+\delta)^{-3\gamma/n}}{\Big(1-\liminf_{k\to\infty}\int_\Omega \Phi(|\nabla u_k^L|)\Big)^{\gamma/n}} \geqslant \frac{(1+\delta)^{-3\gamma/n}}{\big(1-\int_\Omega \Phi(|\nabla u^L|)\big)^{\gamma/n}} > p_3.$$

□

We also need a version of [4, Lemma 3.1] for the space $WL^\Phi(\Omega)$.

Lemma 6.2. *Let* $l \in \mathbb{N}$, $n \geqslant 2$, $\alpha < n-1$ *and* $u_0 \in \mathbb{R}$. *Let* Ω *be a bounded connected domain in* $\mathbb{R}^n$ *of class* $C^{1,\theta}$ *for some* $\theta \in (0,1]$ *and let* Φ *be a Young function satisfying* (1.3). *Let* $\{u_k\}_{k=1}^\infty \subset WL^\Phi(\Omega)$ *satisfy* $\|\Phi(\nabla u_k)\|_{L^1(\Omega)} \leqslant 1$. *Suppose that*

$$u_k \rightharpoonup u_0 \quad \text{in } WL^\Phi(\Omega) \quad \text{and} \quad \Phi(|\nabla u_k|) \overset{*}{\rightharpoonup} \mu \quad \text{in } \mathcal{M}(\bar{\Omega}).$$

Let $F, N \subset \bar{\Omega}$ be compact sets such that $F \cap N = \emptyset$ and $\mu(N) > 0$. Then there is $\delta > 0$ such that

$$\left\| \exp_{[l]} \Big(K_{l,n,\alpha} \Big(\frac{1}{2}\Big)^{\gamma/n} (1+\delta)|u_k|^\gamma \Big) \right\|_{L^1(F)} \quad \text{is bounded.} \tag{6.11}$$

P r o o f. Let us briefly outline the idea of the proof. Since $\mu(N) > 0$ we obtain that $\int_N \Phi(|\nabla u_k|)$ cannot be small for k big enough and thus we can find $\delta > 0$ such that $\|\Phi((1+2\delta)|\nabla u_k|)\|_{L^1(F)} \leqslant 1$. Then, using Theorem 1.2 (ii) for some modification of the function $(1+2\delta)u_k$ we obtain (6.11).

First, let us give the proof in the case $u_0 = 0$. We use $\Phi(|\nabla u_k|) \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}(\bar{\Omega})$ for the test function $\psi \equiv 1$ to obtain

$$1 \geqslant \int_{\bar{\Omega}} \Phi(|\nabla u_k|) = \int_{\bar{\Omega}} \psi \Phi(|\nabla u_k|) \overset{k\to\infty}{\longrightarrow} \int_{\bar{\Omega}} \psi \, \mathrm{d}\mu = \mu(\bar{\Omega}). \tag{6.12}$$

Set $\sigma = \frac{1}{5}\mu(N)$ and recall that C_Δ, t_Δ are the constants from the Δ_2-condition (i.e. $\Phi(2t) \leqslant C_\Delta \Phi(t)$ for $t \geqslant t_\Delta$). By Preliminaries, we can suppose that t_Δ is so small that

$$\Phi(2t_\Delta) \leqslant \frac{\sigma}{2\mathcal{L}_n(\Omega)}. \tag{6.13}$$

For $\tau > 0$ denote $G_\tau = \{x \in \mathbb{R}^n \colon \operatorname{dist}(x, F) > \tau\}$. Clearly, we can find $0 < a < b < \operatorname{dist}(F, N)$ such that

$$\mu(G_a \setminus G_b) \leqslant \frac{\sigma}{2C_\Delta^2} \quad \text{and} \quad \mathcal{L}_n(G_a \setminus G_b) < \frac{\sigma}{C_\Delta^2 \Phi(t_\Delta)}. \tag{6.14}$$

Set $M_1 = \bar{\Omega} \setminus G_a$ and $M_2 = \bar{\Omega} \setminus G_b$. We observe that $F \subset M_1 \subset M_2$ and $M_2 \cap N = \emptyset$.

If $\psi \in C(\bar{\Omega})$ is such that $0 \leqslant \psi \leqslant 1$, $\psi \equiv 0$ on N and $\psi \equiv 1$ on M_2 then

$$\int_{M_2} \Phi(|\nabla u_k|) \leqslant \int_{\bar{\Omega}} \psi \Phi(|\nabla u_k|) \overset{k\to\infty}{\longrightarrow} \int_{\bar{\Omega}} \psi \, \mathrm{d}\mu \leqslant 1 - \mu(N) = 1 - 5\sigma.$$

Hence there is $k_1 \in \mathbb{N}$ such that

$$\int_{M_2} \Phi(|\nabla u_k|) \leqslant 1 - 4\sigma \quad \text{for } k > k_1. \tag{6.15}$$

Using (6.14) in the same way as above we can find $k_2 > k_1$ such that

$$\int_{M_2 \setminus M_1} \Phi(|\nabla u_k|) \leqslant \frac{\sigma}{C_\Delta^2} \quad \text{for } k > k_2. \tag{6.16}$$

We claim that there is $\delta \in (0, \frac{1}{2})$ such that

$$\int_{M_2} \Phi((1+2\delta)|\nabla u_k|) \leqslant 1 - 3\sigma \quad \text{for } k > k_1. \tag{6.17}$$

Indeed, by (6.13) we have $\Phi(2t_\Delta) \leqslant \sigma/(2\mathcal{L}_n(M_2))$ and we set

$$\eta = \Big(1 - \Big(3 + \frac{1}{2}\Big)\sigma\Big)/(1 - 4\sigma).$$

Then there is $\varepsilon \in (0,1)$ such that (2.3) holds on $[t_\Delta, \infty)$ (see Preliminaries). Thus setting $\delta = \frac{1}{2}\varepsilon$ we can use (6.15) to obtain

$$\begin{aligned}
&\int_{M_2} \Phi((1+2\delta)|\nabla u_k|) \\
&\qquad = \int_{M_2 \cap \{|\nabla u_k| \geqslant t_\Delta\}} \Phi((1+\varepsilon)|\nabla u_k|) + \int_{M_2 \cap \{|\nabla u_k| < t_\Delta\}} \Phi((1+\varepsilon)t_\Delta) \\
&\qquad \leqslant 1 - \Big(3 + \frac{1}{2}\Big)\sigma + \frac{1}{2}\sigma = 1 - 3\sigma
\end{aligned}$$

and (6.17) is proved.

Now we can define v_k. Take $\psi \in C^1(\bar{\Omega})$ such that $0 \leqslant \psi \leqslant 1$, $\psi \equiv 1$ on M_1 and $\psi \equiv 0$ on $\bar{\Omega} \setminus \operatorname{Int}(M_2)$. Set $v_k = (1+2\delta)\psi u_k$. Our aim is to apply Theorem 1.2 (ii) to v_k, thus we need to prove that there is $k_3 > k_2$ such that

$$I := \int_\Omega \Phi(|\nabla v_k|) \leqslant 1 \quad \text{for } k > k_3. \tag{6.18}$$

We have $I = I_1 + I_2 + I_3$, where

$$\begin{aligned}
I_1 &= \int_{M_1} \Phi(|\nabla v_k|) = \int_{M_1} \Phi((1+2\delta)|\nabla u_k|) \leqslant 1 - 3\sigma \quad \text{for } k > k_1 \text{ by (6.17)},\\
I_2 &= \int_{\Omega \setminus M_2} \Phi(|\nabla v_k|) = \int_{\Omega \setminus M_2} \Phi(0) = 0,
\end{aligned}$$

and

$$I_3 = \int_{M_2 \setminus M_1} \Phi(|\nabla v_k|).$$

Set $P = \max_{x \in \bar{\Omega}} |\nabla \psi(x)|$. From $\delta \in (0, \frac{1}{2})$ we have on $M_2 \setminus M_1$

$$\Phi(|\nabla v_k|) \leqslant \Phi((1+2\delta)\psi|\nabla u_k| + (1+2\delta)|u_k||\nabla\psi|) \leqslant \Phi(2|\nabla u_k| + 2P|u_k|). \tag{6.19}$$

It is convenient for us to decompose $M_2 \setminus M_1$ into three sets

$$\begin{aligned}
A_k^1 &= \{x \in M_2 \setminus M_1 \colon\ t_\Delta \geqslant |\nabla u_k(x)|, t_\Delta \geqslant P|u_k(x)|\},\\
A_k^2 &= \{x \in M_2 \setminus M_1 \colon\ |\nabla u_k(x)| \geqslant t_\Delta, |\nabla u_k(x)| \geqslant P|u_k(x)|\},\\
A_k^3 &= \{x \in M_2 \setminus M_1 \colon\ P|u_k(x)| \geqslant t_\Delta, P|u_k(x)| \geqslant |\nabla u_k(x)|\}.
\end{aligned}$$

As $M_2 \setminus M_1 = A_k^1 \cup A_k^2 \cup A_k^3$, we have

$$I_3 = \int_{M_2\setminus M_1} \Phi(|\nabla v_k|) \leqslant \int_{A_k^1} + \int_{A_k^2} + \int_{A_k^3}.$$

First, by (6.14) and (6.19) we have

$$\begin{aligned}
\int_{A_k^1} \Phi(|\nabla v_k|) &\leqslant \int_{A_k^1} \Phi(4t_\Delta) \leqslant C_\Delta^2 \Phi(t_\Delta) \mathcal{L}_n(G_a \setminus G_b)\\
&\leqslant C_\Delta^2 \Phi(t_\Delta) \frac{\sigma}{C_\Delta^2 \Phi(t_\Delta)} = \sigma.
\end{aligned} \tag{6.20}$$

Second, (6.16) and (6.19) imply

$$\int_{A_k^2} \Phi(|\nabla v_k|) \leqslant \int_{A_k^2} \Phi(4|\nabla u_k|) \leqslant C_\Delta^2 \int_{M_2\setminus M_1} \Phi(|\nabla u_k|) \leqslant C_\Delta^2 \frac{\sigma}{C_\Delta^2} = \sigma. \tag{6.21}$$

Third, by the compact embedding of $WL^\Phi(\Omega)$ into $L^\Phi(\Omega)$ we see that the weak convergence $u_k \rightharpoonup 0$ in $WL^\Phi(\Omega)$ implies $u_k \to 0$ in $L^\Phi(\Omega)$. Then, using (2.5), we find $k_3 > k_2$ such that for $k > k_3$ we have

$$\int_{A_k^3} \Phi(|\nabla v_k|) \leqslant \int_{A_k^3} \Phi(4P|u_k|) < \sigma. \tag{6.22}$$

Estimates (6.20), (6.21) and (6.22) imply $I_3 < 3\sigma$ and (6.18) follows.

Therefore $v_k \in WL^\Phi(\Omega)$ and $\|\Phi(|\nabla v_k|)\|_{L^1(\Omega)} \leqslant 1$ for $k > k_3$. Moreover, as u_k weakly converge in $WL^\Phi(\Omega)$, they are bounded. Plainly v_k are also bounded and so are their medians. Indeed, for every $k \in \mathbb{N}$ we have

$$\int_\Omega \Phi(|v_k|) \geqslant \Phi(|\{\mathrm{med}(v_k)\}|) \frac{1}{2} \mathcal{L}_n(\Omega).$$

Thus using Theorem 1.2 (ii) with $K = ((1+\delta)/(1+2\delta))^\gamma K_{l,n,\alpha}(\frac{1}{2})^{\gamma/n}$ and the fact that $v_k = (1+2\delta)u_k$ on F we obtain for $k > k_3$

$$\begin{aligned}
\Big\|\exp\Big(K_{l,n,\alpha}\Big(\frac{1}{2}\Big)^{\gamma/n}(1+\delta)^\gamma |u_k|^\gamma\Big)\Big\|_{L^1(F)} &= \|\exp(K(1+2\delta)^\gamma |u_k|^\gamma)\|_{L^1(F)}\\
&\leqslant \|\exp(K|v_k|^\gamma)\|_{L^1(\Omega)} \leqslant C_K.
\end{aligned}$$

Moreover, for every fixed $k < k_3$ there is C_k such that $\|\exp(K_{l,n,\alpha}(\frac{1}{2})^{\gamma/n}(1+\delta)^{\gamma}|u_k|^{\gamma})\|_{L^1(F)} \leqslant C_k$ by Theorem 1.2 (i). Hence we obtain (6.11) for $\tilde{\delta} = (1+\delta)^{\gamma} - 1$ with the bound $\max(C_1, \dots, C_{k_3}, C_K)$. Thus, we are done in the case $u_0 = 0$.

In the general case, we write $u_k = (u_k - u_0) + u_0$. In view of Remark 5.2, the constant u_0 does not influence the boundedness of the integrals while for the functions $u_k - u_0$ we can use the procedure from the previous part of the proof. □

Remark 6.3. It can be easily seen that if we have $\mu(\bar{\Omega}) < 1$, then there is a simplified version of the above proof giving us $\delta > 0$ such that

$$\left\|\exp_{[l]}\Bigl(K_{l,n,\alpha}\Bigl(\frac{1}{2}\Bigr)^{\gamma/n}(1+\delta)|u_k|^{\gamma}\Bigr)\right\|_{L^1(\Omega)} \text{ is bounded.}$$

Proof of Theorem 1.3.

(i) *Case* $u \equiv u_0$ *and* $\mu = \delta_{x_0}$. First, we claim that

$$\text{(6.23)}\quad \eta > 0 \Longrightarrow \int_{\Omega\setminus B(x_0,\eta)} \Bigl(\exp_{[l]}\Bigl(K_{l,n,\alpha}\Bigl(\frac{1}{2}\Bigr)^{\gamma/n}|u_k|^{\gamma}\Bigr) - \exp_{[l]}\Bigl(K_{l,n,\alpha}\Bigl(\frac{1}{2}\Bigr)^{\gamma/n}|u_0|^{\gamma}\Bigr)\Bigr) \stackrel{k\to\infty}{\longrightarrow} 0.$$

Indeed, from Lemma 6.2 for $N = \overline{B(x_0, \eta/2)}$ we obtain that

$$\int_{\Omega\setminus B(x_0,\eta)} \exp_{[l]}\Bigl(K_{l,n,\alpha}\Bigl(\frac{1}{2}\Bigr)^{\gamma/n}(1+\delta)|u_k|^{\gamma}\Bigr)$$

is bounded for some $\delta > 0$ and thus we can use Lemma 2.1 to obtain (6.23).

Further we observe that (6.23) and assumption (1.7) imply

$$\text{(6.24)}\quad \eta > 0 \Longrightarrow \int_{B(x_0,\eta)} \Bigl(\exp_{[l]}\Bigl(K_{l,n,\alpha}\Bigl(\frac{1}{2}\Bigr)^{\gamma/n}|u_k|^{\gamma}\Bigr) - \exp_{[l]}\Bigl(K_{l,n,\alpha}\Bigl(\frac{1}{2}\Bigr)^{\gamma/n}|u_0|^{\gamma}\Bigr)\Bigr) \stackrel{k\to\infty}{\longrightarrow} c.$$

Fix an arbitrary test function $\psi \in C(\bar{\Omega})$ and let $\varepsilon > 0$. Then there is $\eta > 0$ such that

$$\text{(6.25)}\qquad |\psi(x) - \psi(x_0)| < \frac{\varepsilon}{2\max(c,1)} \quad \text{whenever } |x - x_0| < \eta.$$

We have

$$
\begin{aligned}
I &:= \left|\int_{\bar\Omega} \psi \,\mathrm{d}(c\delta_{x_0}) - \int_\Omega \psi\Big(\exp_{[l]}\Big(K_{l,n,\alpha}\Big(\frac{1}{2}\Big)^{\gamma/n}|u_k|^\gamma\Big) - \exp_{[l]}\Big(K_{l,n,\alpha}\Big(\frac{1}{2}\Big)^{\gamma/n}|u_0|^\gamma\Big)\Big)\right| \\
&= \left|c\psi(x_0) - \int_\Omega \psi\Big(\exp_{[l]}\Big(K_{l,n,\alpha}\Big(\frac{1}{2}\Big)^{\gamma/n}|u_k|^\gamma\Big) - \exp_{[l]}\Big(K_{l,n,\alpha}\Big(\frac{1}{2}\Big)^{\gamma/n}|u_0|^\gamma\Big)\Big)\right| \\
&\leqslant \int_{\Omega\setminus B(x_0,\eta)} |\psi|\Big(\exp_{[l]}\Big(K_{l,n,\alpha}\Big(\frac{1}{2}\Big)^{\gamma/n}|u_k|^\gamma\Big) - \exp_{[l]}\Big(K_{l,n,\alpha}\Big(\frac{1}{2}\Big)^{\gamma/n}|u_0|^\gamma\Big)\Big) \\
&\quad + \int_{B(x_0,\eta)} |\psi-\psi(x_0)|\Big(\exp_{[l]}\Big(K_{l,n,\alpha}\Big(\frac{1}{2}\Big)^{\gamma/n}|u_k|^\gamma\Big) - \exp_{[l]}\Big(K_{l,n,\alpha}\Big(\frac{1}{2}\Big)^{\gamma/n}|u_0|^\gamma\Big)\Big) \\
&\quad + |\psi(x_0)|\cdot\bigg|c - \int_{B(x_0,\eta)}\Big(\exp_{[l]}\Big(K_{l,n,\alpha}\Big(\frac{1}{2}\Big)^{\gamma/n}|u_k|^\gamma\Big) \\
&\qquad\qquad - \exp_{[l]}\Big(K_{l,n,\alpha}\Big(\frac{1}{2}\Big)^{\gamma/n}|u_0|^\gamma\Big)\Big)\bigg| = I_1 + I_2 + I_3.
\end{aligned}
$$

By (6.23) and $\sup\limits_\Omega |\psi| < \infty$ we see that there is $k_1 \in \mathbb{N}$ such that $I_1 < \varepsilon$ for $k > k_1$. Further, using (6.24) and (6.25) we obtain

$$
\begin{aligned}
I_2 &= \int_{B(x_0,\eta)} |\psi-\psi(x_0)|\Big(\exp_{[l]}\Big(K_{l,n,\alpha}\Big(\frac{1}{2}\Big)^{\gamma/n}|u_k|^\gamma\Big) - \exp_{[l]}\Big(K_{l,n,\alpha}\Big(\frac{1}{2}\Big)^{\gamma/n}|u_0|^\gamma\Big)\Big) \\
&\leqslant \frac{\varepsilon}{2\max(c,1)} \int_{B(x_0,\eta)}\Big(\exp_{[l]}\Big(K_{l,n,\alpha}\Big(\frac{1}{2}\Big)^{\gamma/n}|u_k|^\gamma\Big) \\
&\qquad\qquad - \exp_{[l]}\Big(K_{l,n,\alpha}\Big(\frac{1}{2}\Big)^{\gamma/n}|u_0|^\gamma\Big)\Big) \overset{k\to\infty}{\longrightarrow} \frac{\varepsilon}{2}\,\frac{c}{\max(c,1)}.
\end{aligned}
$$

Therefore we can find $k_2 > k_1$ such that $I_2 < \varepsilon$ for $k > k_2$. Finally, from (6.24) and $|\psi(x_0)| < \infty$ we obtain $k_3 > k_2$ such that $I_3 < \varepsilon$ for $k > k_3$. Hence we have $I < 3\varepsilon$ for k large and the assertion is proved.

(ii) *Case $u \equiv u_0$ and μ is not a Dirac mass at one point.* As $\mu(\bar\Omega) \leqslant 1$ (see (6.12)), we distinguish two cases. If $\mu(\bar\Omega) < 1$, then the assertion follows from Remark 6.3. Now, let $\mu(\Omega) = 1$. As μ is not a Dirac mass at one point, there is $N_1 \subset \bar\Omega$ compact such that $\mu(N_1) \in (0,1)$. We denote $G = \mathbb{R}^n \setminus N_1$ and $G_\tau = \{x \in \mathbb{R}^n\colon \operatorname{dist}(x, N_1) > \tau\}$ for $\tau > 0$. Considering μ as a Radon measure on $\mathbb{R}^n$ supported in $\bar\Omega$ we obtain

$$\lim_{\tau\to 0_+} \mu(G_\tau) = \mu(G) = 1 - \mu(N_1) \in (0,1).$$

Therefore there is $\tau > 0$ such that

$$0 < \mu(G_{2\tau}) \leqslant \mu(G_\tau) < 1.$$

Set $F_1 = \bar{\Omega}\setminus G_\tau$, $F_2 = \bar{\Omega}\cap\overline{G}_\tau$ and $N_2 = \bar{\Omega}\cap\overline{G}_{2\tau}$. Clearly F_1, F_2, N_1, N_2 are compact sets, $F_1 \cup F_2 = \bar{\Omega}$. Moreover $\mu(N_2) \geqslant \mu(G_{2\tau}) > 0$, $N_2 \cap F_1 = \emptyset$ and $N_1 \cap F_2 = \emptyset$.

Applying Lemma 6.2 to $F = F_1$ and $N = N_2$ we obtain that there is $\delta_1 > 0$ such that

$$\Big\|\exp_{[l]}\Big(K_{l,n,\alpha}\Big(\frac{1}{2}\Big)^{\gamma/n}(1+\delta_1)|u_k|^\gamma\Big)\Big\|_{L^1(F_1)} \text{ is bounded.}$$

If $F = F_2$ and $N = N_1$ then Lemma 6.2 gives us $\delta_2 > 0$ such that

$$\Big\|\exp_{[l]}\Big(K_{l,n,\alpha}\Big(\frac{1}{2}\Big)^{\gamma/n}(1+\delta_2)|u_k|^\gamma\Big)\Big\|_{L^1(F_2)} \text{ is bounded.}$$

From $F_1 \cup F_2 = \bar{\Omega}$ we conclude that $\|\exp_{[l]}(K_{l,n,\alpha}(\frac{1}{2})^{\gamma/n}(1+\delta)|u_k|^\gamma)\|_{L^1(\Omega)}$ is bounded for $\delta = \min(\delta_1, \delta_2)$.

(iii) *Case with u not being a constant function.* Since the sequence $\{u_k\}$ is weakly convergent in $WL^\Phi(\Omega)$, it is bounded in $WL^\Phi(\Omega)$, and thus the sequence $\{\mathrm{med}(u_k)\}$ is also bounded. Thus there are $C_k \in \mathbb{R}$, $k \in \mathbb{N}$, such that $|C_k| < \tilde{C}$ and $\mathrm{med}(u_k - C_k) = 0$. For the sake of contradiction assume that the statement is not true. That is, there is $p_1 < P$ such that passing to a subsequence if necessary

$$\int_\Omega \exp_{[l]}\Big(K_{l,n,\alpha}\Big(\frac{1}{2}\Big)^{\gamma/n} p_1|u_k|^\gamma\Big) \overset{k\to\infty}{\longrightarrow} \infty. \tag{6.26}$$

Next fix $p_2 \in (p_1, P)$. The sequence $\{u_k - C_k\}$ is bounded in $WL^\Phi(\Omega)$ and $\{C_k\}$ is bounded, thus we can pass to a subsequence such that $u_k - C_k$ weakly converge in $WL^\Phi(\Omega)$, $C_k \to C_0$ for some $C_0 \in [-\tilde{C}, \tilde{C}]$ and $u_k - C_k$ converge to $u - C_0$ a.e. in Ω. Hence our sequence $\{u_k - C_k\}$ now satisfies the assumptions of Proposition 6.1 and, as $\nabla(u - C_0) = \nabla u$ in Ω, we obtain

$$\int_\Omega \exp_{[l]}\Big(K_{l,n,\alpha}\Big(\frac{1}{2}\Big)^{\gamma/n} p_2|u_k - C_k|^\gamma\Big) \leqslant C. \tag{6.27}$$

On the set where $((p_2/p_1)^{1/\gamma} - 1)|u_k - C_k| \geqslant \tilde{C}$ we have

$$\begin{aligned} p_1|u_k|^\gamma &\leqslant p_1(|u_k - C_k| + \tilde{C})^\gamma \\ &\leqslant p_1\Big(|u_k - C_k| + \Big(\Big(\frac{p_2}{p_1}\Big)^{1/\gamma} - 1\Big)|u_k - C_k|\Big)^\gamma \leqslant p_2|u_k - C_k|^\gamma, \end{aligned}$$

while on the set where $((p_2/p_1)^{1/\gamma} - 1)|u_k - C_k| \leqslant \tilde{C}$ we estimate $|u_k|$ by the constant $p_2^{1/\gamma}/(p_2^{1/\gamma} - p_1^{1/\gamma})\tilde{C}$. Therefore estimate (6.27) contradicts (6.26).

Finally, we apply Lemma 2.1 to prove (1.8). □

Proof of Theorem 1.4. Put

$$S := \sup\{\Lambda_F(u)\colon\ u \in WL^\Phi(\Omega), \|\Phi(|\nabla u|)\|_{L^1(\Omega)} + \lambda\|\Phi(|u|)\|_{L^1(\Omega)} \leqslant 1\}.$$

If $S = \mathcal{L}_n(\Omega)F(T)$, where $T \in \mathbb{R}$ is such that $\lambda\|\Phi(|T|)\|_{L^1(\Omega)} \leqslant 1$, then the proof is trivial, because for $u \equiv T$ we have $\Lambda_F(u) = \mathcal{L}_n(\Omega)F(T)$. Otherwise there is a maximizing sequence $\{u_k\} \subset WL^\Phi(\Omega)$ such that

$$\|\Phi(|\nabla u_k|)\|_{L^1(\Omega)} + \lambda\|\Phi(|u_k|)\|_{L^1(\Omega)} \leqslant 1 \quad \text{and} \quad \Lambda_F(u_k) \overset{k\to\infty}{\longrightarrow} S.$$

We can further suppose that

$$u_k \rightharpoonup u \quad \text{in } WL^\Phi(\Omega), \quad u_k \to u \text{ a.e. in } \Omega \quad \text{and} \quad \Phi(|\nabla u_k|) \overset{*}{\rightharpoonup} \mu \text{ in } \mathcal{M}(\bar{\Omega}),$$

otherwise we pass to a subsequence (notice that $WL^\Phi(\Omega)$ is reflexive). Next, we claim that we have the estimate

$$\|\Phi(|\nabla u|)\|_{L^1(\Omega)} + \lambda\|\Phi(|u|)\|_{L^1(\Omega)} \leqslant 1. \tag{6.28}$$

Let us prove this claim. If $\nabla u = 0$ a.e. in Ω, then the proof of (6.28) plainly follows from the compact embedding of $WL^\Phi(\Omega)$ into $L^\Phi(\Omega)$. Otherwise we set

$$\Phi_1(t) = \frac{\Phi(t)}{\int_\Omega \Phi(|\nabla u|)}.$$

Next, denoting by $\|\cdot\|_{\tilde{L}^{\Phi_1}(\Omega)}$ the Luxemburg norm with respect to Φ_1 and using the weak lower semicontinuity of the norm $\|\nabla\cdot\|_{\tilde{L}^{\Phi_1}(\Omega)} + \lambda\|\cdot\|_{\tilde{L}^{\Phi_1}(\Omega)}$ together with the compact embedding of $WL^\Phi(\Omega)$ into $L^\Phi(\Omega)$ we see that

$$\begin{aligned}\|\nabla u\|_{\tilde{L}^{\Phi_1}(\Omega)} + \lambda\|u\|_{\tilde{L}^{\Phi_1}(\Omega)} &\leqslant \liminf_{k\to\infty}\big(\|\nabla u_k\|_{\tilde{L}^{\Phi_1}(\Omega)} + \lambda\|u_k\|_{\tilde{L}^{\Phi_1}(\Omega)}\big)\\ &= \liminf_{k\to\infty}\|\nabla u_k\|_{\tilde{L}^{\Phi_1}(\Omega)} + \lambda\|u\|_{\tilde{L}^{\Phi_1}(\Omega)}.\end{aligned}$$

Therefore, by $\int_\Omega \Phi_1(|\nabla u|) = 1$ (see the definition of Φ_1) and a version of (2.4) for the Luxemburg norm (the constant $\Phi_1(1)$ is replaced by 1 on the right-hand side of (2.4)), we obtain

$$1 = \|\nabla u\|_{\tilde{L}^{\Phi_1}(\Omega)} \leqslant \liminf_{k\to\infty}\|\nabla u_k\|_{\tilde{L}^{\Phi_1}(\Omega)}.$$

The above mentioned version of (2.4) now yields

$$\liminf_{k\to\infty}\int_\Omega \Phi_1(|\nabla u_k|) \geqslant 1$$

and thus by the definition of Φ_1

$$\liminf_{k\to\infty} \int_\Omega \Phi(|\nabla u_k|) \geqslant \int_\Omega \Phi(|\nabla u|).$$

Hence (6.28) follows and thus all we need to show is $\Lambda_F(u) = S$.

If (1.9) is satisfied, then we find $\delta > 0$ such that $(1+\delta)K < K_{l,n,\alpha}(\frac{1}{2})^{\gamma/n}$. Now, we can use Lemma 2.1 (the medians are bounded and thus we can use Theorem 1.2 (ii) to verify the assumptions of Lemma 2.1) to complete the proof.

The rest of the proof is devoted to the case when (1.10) is satisfied. By Theorem 1.3 we have either

$$\exp_{[l]}\Big(K_{l,n,\alpha}\Big(\frac{1}{2}\Big)^{\gamma/n}(1+\delta)|u_k|^\gamma\Big) \quad \text{is bounded in } L^1(\Omega)$$

or

$$u \equiv u_0 \quad \text{and} \quad \Phi(|\nabla u_k|) \overset{*}{\rightharpoonup} \delta_{x_0} \quad \text{in } \mathcal{M}(\bar{\Omega}).$$

In the former case we easily complete the proof using Lemma 2.1 because we obtain $\Lambda_F(u) = S$.

Now, it is enough to prove that in the latter case we have

$$\lim_{k\to\infty} \Lambda_F(u_k) = \mathcal{L}_n(\Omega)F(u_0). \tag{6.29}$$

Fix $\varepsilon > 0$. As $\|\Phi(|\nabla u_k|)\|_{L^1(\Omega)} \leqslant 1$, the medians are bounded and (1.4) is satisfied, we can use Theorem 1.2 (iv) to obtain $C_2 > 0$ such that

$$\int_\Omega \exp_{[l]}\Big(K_{l,n,\alpha}\Big(\frac{1}{2}\Big)^{\gamma/n}|u_k|^\gamma\Big) \leqslant C_2. \tag{6.30}$$

Next, by (1.10), there is $t_0 > |u_0|$ such that

$$|F(t)| \leqslant \frac{\varepsilon}{C_2}\exp_{[l]}\Big(K_{l,n,\alpha}\Big(\frac{1}{2}\Big)^{\gamma/n}|t|^\gamma\Big) \quad \text{for } |t| \geqslant t_0. \tag{6.31}$$

Now, we have

$$\begin{aligned}
&|\Lambda_F(u_k) - \mathcal{L}_n(\Omega)F(u_0)| \\
&\quad\leqslant \int_\Omega |F(u_k) - F(u_0)| \\
&\quad\leqslant \int_\Omega |F(u_k)\chi_{\{|u_k|\leqslant t_0\}} - F(u_0)| + \int_\Omega |F(u_k)|\chi_{\{|u_k|>t_0\}} = I_1 + I_2.
\end{aligned}$$

Since F is continuous and $u_k \to u_0$ a.e. in Ω, by the Lebesgue Dominated Convergence Theorem we obtain $I_1 \to 0$. By (6.30) and (6.31) we see that $I_2 \leqslant \varepsilon$. We have proved (6.29) and we are done. □

In the rest of the paper we show that we cannot improve the estimate concerning p neither in Proposition 6.1 nor in Theorem 1.3 (iii).

Example 6.4. Let $l \in \mathbb{N}$, $n \geqslant 2$, $\alpha < n-1$, $R > 0$ and suppose that the Young function Φ satisfies (1.3). Let $\Omega \subset \mathbb{R}^n$ be a smooth domain such that

$$\Omega \cap B(3R) = \{x \in B(3R)\colon\ x_n > 0\}.$$

For every $\varrho \in [0,1)$ and $p > P := (1/(1-\varrho))^{\gamma/n}$ there is a sequence $\{u_k\} \subset WL^{\Phi}(\Omega)$ and a function $u \in WL^{\Phi}(\Omega)$ such that

$$\|\Phi(\nabla u_k)\|_{L^1(\Omega)} \leqslant 1, \quad u_k \rightharpoonup u \text{ in } WL^{\Phi}(\Omega), \quad u_k \to u \text{ a.e. in } \Omega,$$
$$\int_\Omega \Phi(|\nabla u|) = \varrho \quad \text{and} \quad \int_\Omega \exp_{[l]}\Big(K_{l,n,\alpha}\Big(\frac{1}{2}\Big)^{\gamma/n} p|u_k|^\gamma\Big) \overset{k\to\infty}{\longrightarrow} \infty.$$

P r o o f. We define the function $u \in W_0L^{\Phi}(B(3R))$ by $u(x) = h(|x|)$, where

$$h(y) = \begin{cases} 0 & \text{for } y \in [3R,\infty), \\ 3\Theta - \dfrac{\Theta}{R}y & \text{for } y \in [2R,3R], \\ \Theta & \text{for } y \in [0,2R], \end{cases}$$

where $\Theta \geqslant 0$ is chosen such that

$$\int_\Omega \Phi(|\nabla u|) = \frac{1}{2}\int_{B(3R)} \Phi(|\nabla u|) = \varrho.$$

Next, let $\tilde{w}_k$, $k \in \mathbb{N}$, be the functions given by (4.3) and (4.4), respectively, with the parameter $A \in (K_{l,n,\alpha}, (p/P)K_{l,n,\alpha})$. We set

$$u_k = u + 2^{1/n}(1-\varrho)^{1/n}\tilde{w}_k, \quad k \in \mathbb{N}.$$

By the definition of u_k and by (4.9) we have for $k \in \mathbb{N}$ large enough

$$\begin{aligned}\int_\Omega \Phi(|\nabla u_k|) &= \frac{1}{2}\int_{B(3R)} \Phi(|\nabla u_k|) \\ &= \frac{1}{2}\int_{B(3R)\setminus B(2R)} \Phi(|\nabla u|) + \frac{1}{2}\int_{B(R)} \Phi(2^{1/n}(1-\varrho)^{1/n}|\nabla \tilde{w}_k|) \\ &\leqslant \varrho + (1-\varrho) = 1.\end{aligned}$$

Next, from (4.6), $P=(1-\varrho)^{-\gamma/n}$ and $A<(p/P)K_{l,n,\alpha}$ we have

$$\begin{aligned}\int_\Omega \exp_{[l]}\Bigl(K_{l,n,\alpha}\Bigl(\frac{1}{2}\Bigr)^{\gamma/n} p|u_k|^\gamma\Bigr) &= \frac{1}{2}\int_{B(3R)} \exp_{[l]}\Bigl(K_{l,n,\alpha}\Bigl(\frac{1}{2}\Bigr)^{\gamma/n} p|u_k|^\gamma\Bigr)\\ &\geqslant \frac{1}{2}\int_{B(R)} \exp_{[l]}\Bigl(K_{l,n,\alpha} p(1-\varrho)^{\gamma/n}|\tilde{w}_k|^\gamma\Bigr)\\ &= \frac{1}{2}\int_{B(R)} \exp_{[l]}\Bigl(K_{l,n,\alpha}\frac{p}{P}|\tilde{w}_k|^\gamma\Bigr) \overset{k\to\infty}{\longrightarrow} \infty.\end{aligned}$$

The remaining properties of the sequence $\{u_k\}$ are easily verified. □

Notice that if in addition the function Φ satisfies condition (1.5), then, by (4.5) and (4.10), we can use sequences from (4.1) and (4.2) in the definition of $\{u_k\}$ in the proof of Example 6.4. Such a version of the example gives that we cannot have $p=P$ in Proposition 6.1.

By the same argument as in Remark 5.1, it was not necessary to suppose that $\partial\Omega$ is flat in Example 6.4 (when showing that we cannot have $p>P$).

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