Czechoslovak Mathematical Journal

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*Czechoslovak Mathematical Journal*, Vol. 62 (2012), No. 3, 849–861

Persistent URL: `http://dml.cz/dmlcz/143029`

Czechoslovak Mathematical Journal, 62 (137) (2012), 849–861

# REAL HYPERSURFACES IN COMPLEX TWO-PLANE GRASSMANNIANS WITH CERTAIN COMMUTING CONDITION

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(Received August 8, 2011)

*Abstract.* In this paper, first we introduce a new notion of commuting condition that $\varphi\varphi_1 A = A\varphi_1\varphi$ between the shape operator $A$ and the structure tensors $\varphi$ and $\varphi_1$ for real hypersurfaces in $G_2(\mathbb{C}^{m+2})$. Suprisingly, real hypersurfaces of type $(A)$, that is, a tube over a totally geodesic $G_2(\mathbb{C}^{m+1})$ in complex two plane Grassmannians $G_2(\mathbb{C}^{m+2})$ satisfy this commuting condition. Next we consider a complete classification of Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ satisfying the commuting condition. Finally we get a characterization of Type $(A)$ in terms of such commuting condition $\varphi\varphi_1 A = A\varphi_1\varphi$.

*Keywords*: real hypersurface, complex two-plane Grassmannians, Hopf hypersurface, commuting shape operator

*MSC 2010*: 53C50, 53C55

## Introduction

We denote by $G_2(\mathbb{C}^{m+2})$ the set of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. This Riemannian symmetric space $G_2(\mathbb{C}^{m+2})$ has a remarkable geometric structure. It is the unique compact irreducible Riemannian manifold with both a Kähler structure $J$ and a quaternionic Kähler structure $\mathfrak{J}$ not containing $J$. Namely, $G_2(\mathbb{C}^{m+2})$ is a unique compact, irreducible, Kähler, quaternionic Kähler manifold which is not a hyper-Kähler manifold. Accordingly, in $G_2(\mathbb{C}^{m+2})$ we have the two natural geometric conditions for real hypersurfaces $M$ that the 1-dimensional distribution $[\xi] = \mathrm{Span}\{\xi\}$ and the 3-dimensional distribution $\mathfrak{D}^\perp = \mathrm{Span}\{\xi_1, \xi_2, \xi_3\}$ are invariant under the shape operator $A$ of $M$ (see [2], [3] and [4]).

This work was supported by grant Proj. No. NRF-2011-220-C00002 from National Research Foundation of Korea. The first author by grant Proj. No. BSRP-2012-002031 and the third by BSRP-2012-0007402 and KNU 2012.

The almost contact structure vector field $\xi$ defined by $\xi = -JN$ is said to be a Reeb vector field, where $N$ denotes a local unit normal vector field of $M$ in $G_2(\mathbb{C}^{m+2})$. The almost contact 3-structure vector fields $\{\xi_1, \xi_2, \xi_3\}$ for the 3-dimensional distribution $\mathfrak{D}^\perp$ of $M$ in $G_2(\mathbb{C}^{m+2})$ are defined by $\xi_\nu = -J_\nu N$ ($\nu = 1, 2, 3$), where $J_\nu$ denotes a canonical local basis of a quaternionic Kähler structure $\mathfrak{J}$ and $T_x M = \mathfrak{D} \oplus \mathfrak{D}^\perp$, $x \in M$.

By using the two invariant conditions mentioned above and the result in Alekseevskii [1], Berndt and Suh [3] proved the following:

**Theorem A.** *Let $M$ be a connected orientable real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geqslant 3$. Then both $[\xi]$ and $\mathfrak{D}^\perp$ are invariant under the shape operator of $M$ if and only if*

(A) *$M$ is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$, or*

(B) *$m$ is even, say $m = 2n$, and $M$ is an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$.*

The Reeb vector field $\xi$ is said to be Hopf if it is invariant under the shape operator $A$. The 1-dimensional foliation of $M$ by the integral manifolds of the Reeb vector field $\xi$ is said to be a Hopf foliation of $M$. We say that $M$ is a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ if and only if the Hopf foliation of $M$ is totally geodesic. By the formulas in Section 2 it can be easily checked that $M$ is Hopf if and only if the Reeb vector field $\xi$ is Hopf.

On the other hand, we say that the Reeb flow on $M$ in $G_2(\mathbb{C}^{m+2})$ is isometric, when the Reeb vector field $\xi$ on $M$ is Killing. In [4], Berndt and Suh gave some equivalent conditions for isometric Reeb flow. Among them, we want to introduce a commuting condition between the shape operator $A$ and the structure tensor $\varphi$, that is, $A\varphi = \varphi A$. By such a commuting condition, a characterization of real hypersurfaces of Type $(A)$ in Theorem A was given in terms of the Reeb flow on $M$ as follows:

**Theorem B.** *Let $M$ be a connected orientable real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geqslant 3$. Then the Reeb flow on $M$ is isometric if and only if $M$ is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.*

In [7], Suh considered a condition that the almost contact 3-structure tensors $\{\varphi_1, \varphi_2, \varphi_3\}$ commute with the shape operator $A$ of real hypersurface $M$ in $G_2(\mathbb{C}^{m+2})$, and he proved that there does not exist any real hypersurface $M$ in $G_2(\mathbb{C}^{m+2})$ with $A\varphi_\nu X = \varphi_\nu AX$, $\nu = 1, 2, 3$, for any tangent vector field $X$ on $M$. In addition, he gave a characterization of real hypersurface of Type $(B)$ under assumption that $M$ is a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ with $A\varphi_\nu X = \varphi_\nu AX$,

$\nu = 1, 2, 3$, for any tangent vector field $X$ on $T_0$. Here, the distribution $T_0$ is defined by $T_0 = \{X \in T_pM \mid \xi \perp X\}$ (see [7]).

Summing up these statements, naturally we ask what can we say about the commuting condition between the shape operator $A$ and the two structure tensors $\varphi$ and $\varphi_1$. According to such a problem, in this paper we consider a new condition that the shape operator $A$ commutes with two kinds of structure tensors $\varphi$ and $\varphi_1$ for a real hypersurface $M$ in $G_2(\mathbb{C}^{m+2})$ as follows:

$$(*) \qquad \varphi\varphi_1 AX = A\varphi_1\varphi X$$

for any tangent vector field $X$ on $M$.

Suprisingly, by Proposition A in Section 3, we know that real hypersurfaces of Type $(A)$ in $G_2(\mathbb{C}^{m+2})$ in Theorem A satisfy the formula $(*)$. From such a point of view, we give another characterization of real hypersurface of Type $(A)$ in $G_2(\mathbb{C}^{m+2})$ as follows:

**Main Theorem.** *Let $M$ be a connected orientable Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geqslant 3$. Then the shape operator satisfies the commuting condition $(*)$ if and only if $M$ is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.*

## 1. Riemannian geometry of $G_2(\mathbb{C}^{m+2})$

In this section we summarize basic material about $G_2(\mathbb{C}^{m+2})$, for details we refer to [2], [3] and [4]. By $G_2(\mathbb{C}^{m+2})$ we denote the set of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. The special unitary group $G = SU(m+2)$ acts transitively on $G_2(\mathbb{C}^{m+2})$ with stabilizer isomorphic to $K = S(U(2) \times U(m)) \subset G$. Then $G_2(\mathbb{C}^{m+2})$ can be identified with the homogeneous space $G/K$, which we equip with the unique analytic structure for which the natural action of $G$ on $G_2(\mathbb{C}^{m+2})$ becomes analytic. Denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebra of $G$ and $K$, respectively, and by $\mathfrak{m}$ the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Cartan-Killing form $B$ of $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is an $Ad(K)$-invariant reductive decomposition of $\mathfrak{g}$. We put $o = eK$ and identify $T_oG_2(\mathbb{C}^{m+2})$ with $\mathfrak{m}$ in the usual manner. Since $B$ is negative definite on $\mathfrak{g}$, its negative restricted to $\mathfrak{m} \times \mathfrak{m}$ yields a positive definite inner product on $\mathfrak{m}$. By $Ad(K)$-invariance of $B$ this inner product can be extended to a $G$-invariant Riemannian metric $g$ on $G_2(\mathbb{C}^{m+2})$. In this way $G_2(\mathbb{C}^{m+2})$ becomes a Riemannian homogeneous space, even a Riemannian symmetric space. For computational reasons we normalize $g$ such that the maximal sectional curvature of $(G_2(\mathbb{C}^{m+2}), g)$ is eight.

When $m = 1$, $G_2(\mathbb{C}^3)$ is isometric to the two-dimensional complex projective space $\mathbb{C}P^2$ with constant holomorphic sectional curvature eight.

When $m = 2$, we note that the isomorphism $\mathrm{Spin}(6) \simeq SU(4)$ yields an isometry between $G_2(\mathbb{C}^4)$ and the real Grassmann manifold $G_2^+(\mathbb{R}^6)$ of oriented two-dimensional linear subspaces in $\mathbb{R}^6$. In this paper, we will assume $m \geqslant 3$.

The Lie algebra $\mathfrak{k}$ has the direct sum decomposition $\mathfrak{k} = \mathfrak{su}(m) \oplus \mathfrak{su}(2) \oplus \mathfrak{R}$, where $\mathfrak{R}$ denotes the center of $\mathfrak{k}$. Viewing $\mathfrak{k}$ as the holonomy algebra of $G_2(\mathbb{C}^{m+2})$, the center $\mathfrak{R}$ induces a Kähler structure $J$ and the $\mathfrak{su}(2)$-part a quaternionic Kähler structure $\mathfrak{J}$ on $G_2(\mathbb{C}^{m+2})$. If $J_\nu$ is any almost Hermitian structure in $\mathfrak{J}$, then $JJ_\nu = J_\nu J$, and $JJ_\nu$ is a symmetric endomorphism with $(JJ_\nu)^2 = I$ and $\mathrm{tr}(JJ_\nu) = 0$ for $\nu = 1, 2, 3$.

A canonical local basis $\{J_1, J_2, J_3\}$ of $\mathfrak{J}$ consists of three local almost Hermitian structures $J_\nu$ in $\mathfrak{J}$ such that $J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1}J_\nu$, where the index $\nu$ is taken modulo three. Since $\mathfrak{J}$ is parallel with respect to the Riemannian connection $\widetilde{\nabla}$ of $(G_2(\mathbb{C}^{m+2}), g)$, there exist for any canonical local basis $\{J_1, J_2, J_3\}$ of $\mathfrak{J}$ three local one-forms $q_1, q_2, q_3$ such that

$$\widetilde{\nabla}_X J_\nu = q_{\nu+2}(X)J_{\nu+1} - q_{\nu+1}(X)J_{\nu+2} \tag{1.1}$$

for all vector fields $X$ on $G_2(\mathbb{C}^{m+2})$.

The Riemannian curvature tensor $\widetilde{R}$ of $G_2(\mathbb{C}^{m+2})$ is locally given by

$$\begin{aligned} \widetilde{R}(X,Y)Z = {} & g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX \\ & - g(JX,Z)JY - 2g(JX,Y)JZ \\ & + \sum_{\nu=1}^{3}\Big\{g(J_\nu Y,Z)J_\nu X - g(J_\nu X,Z)J_\nu Y - 2g(J_\nu X,Y)J_\nu Z\Big\} \\ & + \sum_{\nu=1}^{3}\Big\{g(J_\nu JY,Z)J_\nu JX - g(J_\nu JX,Z)J_\nu JY\Big\}, \end{aligned} \tag{1.2}$$

where $\{J_1, J_2, J_3\}$ denotes a canonical local basis of $\mathfrak{J}$.

## 2. Some fundamental formulas

In this section we derive some basic formulas and the Codazzi equation for a real hypersurface in $G_2(\mathbb{C}^{m+2})$ (see [5], [6] and [7]).

Let $M$ be a real hypersurface of $G_2(\mathbb{C}^{m+2})$, that is, a hypersurface of $G_2(\mathbb{C}^{m+2})$ with real codimension one. The induced Riemannian metric on $M$ will also be denoted by $g$, and $\nabla$ denotes the Riemannian connection of $(M, g)$. Let $N$ be a local unit normal vector field of $M$ and $A$ the shape operator of $M$ with respect to $N$.

Now let us put

$$JX = \varphi X + \eta(X)N, \quad J_\nu X = \varphi_\nu X + \eta_\nu(X)N \tag{2.1}$$

for any tangent vector field $X$ of a real hypersurface $M$ in $G_2(\mathbb{C}^{m+2})$, where $N$ denotes a unit normal vector field of $M$ in $G_2(\mathbb{C}^{m+2})$. From the Kähler structure $J$ of $G_2(\mathbb{C}^{m+2})$ there exists an almost contact metric structure $(\varphi, \xi, \eta, g)$ induced on $M$ in such a way that

$$\varphi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \varphi\xi = 0, \quad \eta(X) = g(X, \xi) \tag{2.2}$$

for any vector field $X$ on $M$. Furthermore, let $\{J_1, J_2, J_3\}$ be a canonical local basis of $\mathfrak{J}$. Then the quaternionic Kähler structure $J_\nu$ of $G_2(\mathbb{C}^{m+2})$, together with the condition $J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1}J_\nu$ in Section 1, induces an almost contact metric 3-structure $(\varphi_\nu, \xi_\nu, \eta_\nu, g)$ on $M$ as follows:

$$\begin{cases} \varphi_\nu^2 X = -X + \eta_\nu(X)\xi_\nu, \quad \eta_\nu(\xi_\nu) = 1, \quad \varphi_\nu\xi_\nu = 0, \\ \varphi_{\nu+1}\xi_\nu = -\xi_{\nu+2}, \quad \varphi_\nu\xi_{\nu+1} = \xi_{\nu+2}, \\ \varphi_\nu\varphi_{\nu+1}X = \varphi_{\nu+2}X + \eta_{\nu+1}(X)\xi_\nu, \\ \varphi_{\nu+1}\varphi_\nu X = -\varphi_{\nu+2}X + \eta_\nu(X)\xi_{\nu+1} \end{cases} \tag{2.3}$$

for any vector field $X$ tangent to $M$. Moreover, from the commuting property of $J_\nu J = JJ_\nu$, $\nu = 1, 2, 3$ in Section 1 and (2.1), the relation between these two contact metric structures $(\varphi, \xi, \eta, g)$ and $(\varphi_\nu, \xi_\nu, \eta_\nu, g)$, $\nu = 1, 2, 3$, can be given by

$$\begin{gathered} \varphi\varphi_\nu X = \varphi_\nu\varphi X + \eta_\nu(X)\xi - \eta(X)\xi_\nu, \\ \eta_\nu(\varphi X) = \eta(\varphi_\nu X), \quad \varphi\xi_\nu = \varphi_\nu\xi. \end{gathered} \tag{2.4}$$

On the other hand, from the Kähler structure $J$, that is, $\widetilde{\nabla} J = 0$ and the quaternionic Kähler structure $J_\nu$ (see (1.1)), together with Gauss and Weingarten formulas it follows that

$$(\nabla_X\varphi)Y = \eta(Y)AX - g(AX, Y)\xi, \quad \nabla_X\xi = \varphi AX, \tag{2.5}$$

$$\nabla_X\xi_\nu = q_{\nu+2}(X)\xi_{\nu+1} - q_{\nu+1}(X)\xi_{\nu+2} + \varphi_\nu AX, \tag{2.6}$$

$$(\nabla_X\varphi_\nu)Y = -q_{\nu+1}(X)\varphi_{\nu+2}Y + q_{\nu+2}(X)\varphi_{\nu+1}Y + \eta_\nu(Y)AX - g(AX, Y)\xi_\nu \tag{2.7}$$

Summing up these formulas, we find the following

$$\begin{aligned} \nabla_X(\varphi_\nu\xi) &= \nabla_X(\varphi\xi_\nu) \\ &= (\nabla_X\varphi)\xi_\nu + \varphi(\nabla_X\xi_\nu) \\ &= q_{\nu+2}(X)\varphi_{\nu+1}\xi - q_{\nu+1}(X)\varphi_{\nu+2}\xi + \varphi_\nu\varphi AX \\ &\quad - g(AX, \xi)\xi_\nu + \eta(\xi_\nu)AX. \end{aligned} \tag{2.8}$$

Using the above expression (1.2) for the curvature tensor $\widetilde{R}$ of $G_2(\mathbb{C}^{m+2})$, the equation of Codazzi becomes

$$(2.9) \qquad \begin{aligned}(\nabla_X A)Y - (\nabla_Y A)X &= \eta(X)\varphi Y - \eta(Y)\varphi X - 2g(\varphi X, Y)\xi \\ &+ \sum_{\nu=1}^{3}\{\eta_\nu(X)\varphi_\nu Y - \eta_\nu(Y)\varphi_\nu X - 2g(\varphi_\nu X, Y)\xi_\nu\} \\ &+ \sum_{\nu=1}^{3}\{\eta_\nu(\varphi X)\varphi_\nu\varphi Y - \eta_\nu(\varphi Y)\varphi_\nu\varphi X\} \\ &+ \sum_{\nu=1}^{3}\{\eta(X)\eta_\nu(\varphi Y) - \eta(Y)\eta_\nu(\varphi X)\}\xi_\nu.\end{aligned}$$

## 3. Key lemmas

Now let us assume that $M$ is a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ with commuting shape operator, that is, the shape operator $A$ of $M$ commutes with the structures tensors $\varphi$ and $\varphi_1$ as follows:

$$(*) \qquad \varphi\varphi_1 AX = A\varphi_1\varphi X$$

for any tangent vector field $X$ on $M$.

First of all, we establish one of the key lemmas as follows:

**Lemma 3.1.** *Let $M$ be a Hopf hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m \geqslant 3$. If $M$ has commuting shape operator, then the Reeb vector field $\xi$ belongs to either the distribution $\mathfrak{D}$ or the distribution $\mathfrak{D}^\perp$.*

Proof. In order to prove our lemma, let us put $\xi = \eta(X_0)X_0 + \eta(\xi_1)\xi_1$ for some unit $X_0 \in \mathfrak{D}$ and $\xi_1 \in \mathfrak{D}^\perp$ and $\eta(X_0)\eta(\xi_1) \neq 0$.

From the assumption $(*)$ for $X = \xi$ and (2.2), we have

$$(3.1) \qquad \varphi_1 A\xi = \eta(\varphi_1 A\xi)\xi.$$

On the other hand, from the assumption that $M$ is Hopf, we see that

$$(3.2) \qquad A\xi = \alpha\xi = \alpha\eta(X_0)X_0 + \alpha\eta(\xi_1)\xi_1.$$

Combining with (3.1) and (3.2), we have

$$\alpha\eta(X_0)\varphi_1 X_0 = 0,$$

because $\varphi_1\xi_1 = 0$ and the structure tensor $\varphi_1$ is skew-symmetric.

But we see that $\varphi_1 X_0$ is non-vanishing at all points of $M$. In fact, we obtain

$$\|\varphi_1 X_0\|^2 = g(\varphi_1 X_0, \varphi_1 X_0) = -g(\varphi_1^2 X_0, X_0) = g(X_0, X_0) = 1,$$

where we have used the equation (2.3) and the fact that $X_0$ is unit.

Then it follows that

$$\alpha\eta(X_0) = 0. \tag{3.3}$$

Thus we can consider the following two cases:

*Case 1.* $\alpha = 0$, that is, $A\xi = 0$. This case is trivial by Lemma 3.1 due to Pérez and Suh [6].

*Case 2.* $\alpha \neq 0$. From (3.3), we have $\eta(X_0) = 0$. This gives a contradiction.

So we complete the proof of our Lemma. □

Now, we consider another commuting condition for the shape operator $A$ on $M$ when the Reeb vector $\xi$ belongs to the distribution $\mathfrak{D}^\perp$. We prove the following lemma which will be useful in the proof of Lemma 4.2 in Section 4.

**Lemma 3.2.** *Let $M$ be a connected orientable Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geqslant 3$ with $\xi \in \mathfrak{D}^\perp$. If $M$ satisfies the following condition*

$$\varphi\varphi_1 AX = A\varphi\varphi_1 X, \quad X \in \mathfrak{D}^\perp, \tag{$**$}$$

*then the distribution $\mathfrak{D}^\perp$ is invariant under the shape operator $A$ of $M$, that is, $g(A\mathfrak{D}^\perp, \mathfrak{D}) = 0$.*

Proof. From now on, since $\xi \in \mathfrak{D}^\perp$, let us put $\xi = \xi_1$. Taking the covariant derivative along any direction $Y \in TM$, we have

$$\varphi AY = \nabla_Y \xi = \nabla_Y \xi_1 = q_3(Y)\xi_2 - q_2(Y)\xi_3 + \varphi_1 AY. \tag{3.4}$$

From this, taking the inner product with $\xi_2$ and $\xi_3$, we have

$$q_3(Y) = 2g(AY, \xi_3), \quad q_2(Y) = 2g(AY, \xi_2), \tag{3.5}$$

respectively.

Moreover, applying the structure tensor $\varphi$ in (3.4), this equation can be written as

$$AY = \alpha\eta(Y)\xi + 2g(AY, \xi_2)\xi_2 + 2g(AY, \xi_3)\xi_3 - \varphi\varphi_1 AY, \quad Y \in TM, \tag{3.6}$$

where we have used that $M$ is Hopf and the formulas (2.2), (2.3) and (3.5).

Putting $Y = \xi_2$ in (3.6), we get

$$\begin{aligned} A\xi_2 &= \alpha\eta(\xi_2)\xi + 2g(A\xi_2, \xi_2)\xi_2 + 2g(A\xi_2, \xi_3)\xi_3 - \varphi\varphi_1 A\xi_2 \\ &= 2g(A\xi_2, \xi_2)\xi_2 + 2g(A\xi_2, \xi_3)\xi_3 - \varphi\varphi_1 A\xi_2 \\ &= 2g(A\xi_2, \xi_2)\xi_2 + 2g(A\xi_2, \xi_3)\xi_3 - A\xi_2. \end{aligned}$$

Here from the condition $(**)$ we see that $\varphi\varphi_1 A\xi_2 = A\varphi\varphi_1\xi_2 = A\xi_2$, because $\xi_2 \in \mathfrak{D}^\perp$. Therefore the third equality in the above equation holds. Consequently, it implies

$$A\xi_2 = g(A\xi_2, \xi_2)\xi_2 + g(A\xi_2, \xi_3)\xi_3. \tag{3.7}$$

Similarly, if we consider $Y = \xi_3$ in (3.6), we get

$$A\xi_3 = g(A\xi_3, \xi_2)\xi_2 + g(A\xi_3, \xi_3)\xi_3, \tag{3.8}$$

because $\varphi\varphi_1 A\xi_3 = A\varphi\varphi_1\xi_3 = A\xi_3$.

From the two equations (3.7), (3.8) and the assumption $A\xi_1 = A\xi = \alpha\xi = \alpha\xi_1$, we have $A\xi_\nu \in \mathfrak{D}^\perp$ for any $\nu = 1, 2, 3$. So we conclude that the distribution $\mathfrak{D}^\perp$ is invariant under the shape operator $A$ of $M$, that is, $A\mathfrak{D}^\perp \subset \mathfrak{D}^\perp$. This gives a complete proof of our lemma. □

Before giving the proof of our Main Theorem from the Introduction, let us check whether the shape operator $A$ of real hypersurfaces of Type $(A)$ or of Type $(B)$ in Theorem A satisfies the condition $(*)$ or not.

First let us check for the case that $M$ is locally congruent to a real hypersurface of Type $(A)$, an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$. We recall a proposition due to Berndt and Suh [3] as follows:

**Proposition A.** *Let $M$ be a connected real hypersurface of $G_2(\mathbb{C}^{m+2})$. Suppose that $A\mathfrak{D} \subset \mathfrak{D}$, $A\xi = \alpha\xi$, and $\xi$ is tangent to $\mathfrak{D}^\perp$. Let $J_1 \in \mathfrak{J}$ be the almost Hermitian structure such that $JN = J_1N$. Then $M$ has three (if $r = \pi/2\sqrt{8}$) or four (otherwise) distinct constant principal curvatures*

$$\alpha = \sqrt{8}\cot(\sqrt{8}r), \quad \beta = \sqrt{2}\cot(\sqrt{2}r), \quad \lambda = -\sqrt{2}\tan(\sqrt{2}r), \quad \mu = 0$$

*with some $r \in (0, \pi/\sqrt{8})$. The corresponding multiplicities are*

$$m(\alpha) = 1, \quad m(\beta) = 2, \quad m(\lambda) = 2m - 2 = m(\mu),$$

*and the corresponding eigenspaces are*

$$\begin{aligned}
T_\alpha &= \mathbb{R}\xi = \mathbb{R}JN = \mathbb{R}\xi_1 = \mathrm{Span}\{\xi\} = \mathrm{Span}\{\xi_1\},\\
T_\beta &= \mathbb{C}^\perp\xi = \mathbb{C}^\perp N = \mathbb{R}\xi_2 \oplus \mathbb{R}\xi_3 = \mathrm{Span}\{\xi_2, \xi_3\},\\
T_\lambda &= \{X \mid X \perp \mathbb{H}\xi,\ JX = J_1X\},\\
T_\mu &= \{X \mid X \perp \mathbb{H}\xi,\ JX = -J_1X\}
\end{aligned}$$

*where $\mathbb{R}\xi$, $\mathbb{C}\xi$ and $\mathbb{H}\xi$ respectively denotes real, complex and quaternionic span of the structure vector field $\xi$ and $\mathbb{C}^\perp\xi$ denotes the orthogonal complement of $\mathbb{C}\xi$ in $\mathbb{H}\xi$.*

Now let us check case by case whether the two sides in $(*)$ are equal to each other:

*Case A-1.* $X \in T_\alpha$ (i.e. $X = \xi = \xi_1$). It can easily be checked that the two sides are equal to each other.

*Case A-2.* $X \in T_\beta$, (i.e. $X = \xi_2$ or $X = \xi_3$). Then we put $A\xi_2 = \beta\xi_2$, $A\xi_3 = \beta\xi_3$, where $\beta = \sqrt{2}\cot(\sqrt{2}r)$. Then by putting $X = \xi_2$ in $(*)$ we have

$$\text{Left-Hand Side} = \varphi\varphi_1 A\xi_2 = \beta\varphi\varphi_1\xi_2 = \beta\varphi\xi_3 = \beta\varphi_3\xi_1 = \beta\xi_2,$$

and

$$\text{Right-Hand Side} = A\varphi_1\varphi\xi_2 = A\varphi_1\varphi_2\xi = A\varphi_1\varphi_2\xi_1 = -A\varphi_1\xi_3 = A\xi_2 = \beta\xi_2.$$

From this we see that both sides are equal to $\beta\xi_2$. Similarly, by putting $X = \xi_3$ in $(*)$ we know that they are equal to $\beta\xi_3$.

*Case A-3.* $X \in T_\lambda = \{X \mid X \perp \mathbb{H}\xi,\ \varphi X = \varphi_1 X\}$. For any $X \in T_\lambda$, $\lambda = -\sqrt{2}\tan(\sqrt{2}r)$ we get

$$\varphi\varphi_1 X = \varphi^2 X = -X, \quad \varphi_1\varphi X = {\varphi_1}^2 X = -X.$$

From this we know that the formula $(*)$ is equal to $-\lambda X$.

*Case A-4.* $X \in T_\mu = \{X \mid X \perp \mathbb{H}\xi,\ \varphi X = -\varphi_1 X\}$. We have $\varphi\varphi_1 X = -\varphi^2 X = X$, $\varphi_1\varphi X = -{\varphi_1}^2 X = X$ for any $X \in T_\mu$. So we know that they are equal to $\mu X = 0$, because $\mu = 0$.

Hence we conclude with a remark as follows:

**Remark 3.3.** The shape operator $A$ of real hypersurfaces of Type $(A)$ in $G_2(\mathbb{C}^{m+2})$ satisfies the condition $(*)$.

Second, let us check whether the shape operator $A$ of real hypersurfaces of Type $(B)$ satisfies the condition $(*)$. As is well known to us, a real hypersurface of Type $(B)$ has five distinct constant principal curvatures as follows [3]:

**Proposition B.** *Let $M$ be a connected real hypersurface of $G_2(\mathbb{C}^{m+2})$. Suppose that $A\mathfrak{D} \subset \mathfrak{D}$, $A\xi = \alpha\xi$, and $\xi$ is tangent to $\mathfrak{D}$. Then the quaternionic dimension $m$ of $G_2(\mathbb{C}^{m+2})$ is even, say $m = 2n$, and $M$ has five distinct constant principal curvatures*

$$\alpha = -2\tan(2r), \quad \beta = 2\cot(2r), \quad \gamma = 0, \quad \lambda = \cot(r), \quad \mu = -\tan(r)$$

*with some $r \in (0, \pi/4)$. The corresponding multiplicities are*

$$m(\alpha) = 1, \quad m(\beta) = 3 = m(\gamma), \quad m(\lambda) = 4n - 4 = m(\mu)$$

*and the corresponding eigenspaces are*

$$\begin{aligned}
&T_\alpha = \mathbb{R}\xi = \mathrm{Span}\{\xi\},\\
&T_\beta = \mathfrak{J}J\xi = \mathrm{Span}\{\xi_\nu \mid \nu = 1,2,3\},\\
&T_\gamma = \mathfrak{J}\xi = \mathrm{Span}\{\varphi_\nu\xi \mid \nu = 1,2,3\},\\
&T_\lambda, \quad T_\mu,
\end{aligned}$$

*where*

$$T_\lambda \oplus T_\mu = (\mathbb{H}\mathbb{C}\xi)^\perp, \quad \mathfrak{J}T_\lambda = T_\lambda, \quad \mathfrak{J}T_\mu = T_\mu, \quad JT_\lambda = T_\mu.$$

Here we suppose that a real hypersurface of Type $(B)$ has the commuting shape operator $A$, that is, the shape operator $A$ of $M$ satisfies the commuting condition $\varphi\varphi_1 AX = A\varphi_1\varphi X$ for any tangent vector field $X$ on $M$. Then we see that

$$\begin{aligned}
\varphi\varphi_1 A\xi = A\varphi_1\varphi\xi &\Leftrightarrow \varphi\varphi_1 A\xi - A\varphi_1\varphi\xi = 0\\
&\Leftrightarrow \varphi\varphi_1 A\xi = 0\\
&\Leftrightarrow \alpha\varphi\varphi_1\xi = 0 \quad (\text{because } \xi \in \mathrm{T}_\alpha)\\
&\Leftrightarrow \alpha\varphi^2\xi_1 = 0 \quad (\text{by eq: (2.4)})\\
&\Leftrightarrow -\alpha\xi_1 = 0 \quad (\text{by eq: (2.2)})\\
&\Leftrightarrow \alpha = 0. \quad (\text{because } \xi_1\text{: unit})
\end{aligned}$$

But this case can not occur for any $r \in (0, \pi/4)$. In fact, $\alpha = -2\tan(2r)$ is non-vanishing in $(0, \pi/4)$. So we also state the following remark:

**Remark 3.4.** The shape operators $A$ of real hypersurfaces of Type $(B)$ in $G_2(\mathbb{C}^{m+2})$ do not satisfy the commuting condition $(*)$.

## 4. The proof of the Main Theorem

In this section, we assume that $M$ is a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ with commuting shape operator, that is, the shape operator satisfies the condition $(*)$. Then by Lemma 3.1 we consider the following two cases:

*Case I:* the Reeb vector field $\xi$ belongs to the distribution $\mathfrak{D}$,

*Case II:* the Reeb vector field $\xi$ belongs to the distribution $\mathfrak{D}^{\perp}$.

First, let us consider Case I, that is, $\xi \in \mathfrak{D}$.

To consider this case, we recall a one theorem by Lee and Suh [5] as follows:

**Theorem C.** *Let $M$ be a connected orientable Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geqslant 3$. Then the Reeb vector field $\xi$ belongs to the distribution $\mathfrak{D}$ if and only if $M$ is locally congruent to an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$, where $m = 2n$.*

Then from Theorem C, we see that $M$ is locally congruent to a real hypersurface of Type $(B)$ under our assumption. But in Section 3 we have checked that the shape operator $A$ of real hypersurface of Type $(B)$ does not satisfy the condition $(*)$ (see Remark 3.4). From these facts, first we assert the following:

**Theorem 4.1.** *There does not exist any Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geqslant 3$, with the commuting shape operator $\varphi\varphi_1 A = A\varphi_1\varphi$ if the Reeb vector field $\xi$ belongs to the distribution $\mathfrak{D}$.*

Next we consider the case $\xi \in \mathfrak{D}^{\perp}$. Accordingly, we may put $\xi = \xi_1$. Then we have the following:

**Lemma 4.2.** *Let $M$ be a hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m \geqslant 3$ with $\xi \in \mathfrak{D}^{\perp}$. If $M$ has commuting shape operator, that is, the shape operator $A$ on $M$ satisfies the condition $(*)$, then the distribution $\mathfrak{D}^{\perp}$ is invariant under the shape operator $A$ on $M$.*

Proof. Since $\xi \in \mathfrak{D}^{\perp}$, let us assume $\xi = \xi_1$. Substituting $X = \xi$ in our assumption $(*)$, we have

$$\varphi\varphi_1 A\xi = 0.$$

Applying $\varphi$ in the above equation, it becomes

$$\varphi_1 A\xi = \eta(\varphi_1 A\xi)\xi.$$

Taking an inner product with $\xi_1$, we obtain $\eta(\varphi_1 A\xi)\eta(\xi_1) = 0$. Since $\xi = \xi_1$, it means that $\eta(\varphi_1 A\xi) = 0$. So, we have

$$\varphi_1 A\xi = 0.$$

From this, we have $A\xi = \alpha\xi$ where $\alpha = g(A\xi, \xi_1) = g(A\xi, \xi)$, because $\xi = \xi_1$.

Moreover, from (2.4), we see that

$$\begin{aligned} \varphi_1\varphi X &= \varphi\varphi_1 X - \eta_1(X)\xi + \eta(X)\xi_1 \\ &= \varphi\varphi_1 X \end{aligned} \tag{4.1}$$

for any tangent vector field $X$ on $M$.

Thus we can write the condition $(*)$ as

$$\varphi\varphi_1 AX = A\varphi_1\varphi X = A\varphi\varphi_1 X \tag{4.2}$$

for any tangent vector field $X$ on $M$.

Now putting $X = \xi_\nu$, $\nu = 2, 3$ in (4.2), this equation can be written as

$$\varphi\varphi_1 A\xi_\nu = A\varphi\varphi_1\xi_\nu, \quad \nu = 2, 3. \tag{4.3}$$

From Lemma 3.2, we have $A\xi_\nu \in \mathfrak{D}^\perp$, $\nu = 2, 3$ under our assumption. This completes the proof of our Lemma. □

Therefore from Theorem A in the Introduction, we conclude the following:

**Lemma 4.3.** *Let $M$ be a connected hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m \geqslant 3$ satisfying the commuting condition $(*)$. If the Reeb vector field $\xi$ belongs to the distribution $\mathfrak{D}^\perp$, then $M$ is locally congruent to an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.*

As mentioned in Remark 3.3 in Section 3, the shape operator $A$ for real hypersurfaces of Type $(A)$ satisfies the commuting condition (*) for any tangent vector field on $M$. From this fact and Lemma 4.3, we arrive at the following:

**Theorem 4.4.** *Let $M$ be a connected hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m \geqslant 3$ satisfying the commuting condition $(*)$. Then the Reeb vector field $\xi$ belongs to the distribution $\mathfrak{D}^\perp$ if and only if $M$ is locally congruent to an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.*

Summing up Lemma 3.1, and Theorems 4.1 and 4.4, we give a complete proof of our Main Theorem from the Introduction. □

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