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A HAVEL-HAKIMI TYPE PROCEDURE AND A SUFFICIENT CONDITION FOR A SEQUENCE TO BE POTENTIALLY $S_{r,s}$-GRAPHIC

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Abstract. The split graph $K_r + \overline{K_s}$ on $r+s$ vertices is denoted by $S_{r,s}$. A non-increasing sequence $\pi = (d_1, d_2, \ldots, d_n)$ of nonnegative integers is said to be potentially $S_{r,s}$-graphic if there exists a realization of π containing $S_{r,s}$ as a subgraph. In this paper, we obtain a Havel-Hakimi type procedure and a simple sufficient condition for π to be potentially $S_{r,s}$-graphic. They are extensions of two theorems due to A. R. Rao (The clique number of a graph with given degree sequence, Graph Theory, Proc. Symp., Calcutta 1976, ISI Lect. Notes Series 4 (1979), 251–267 and An Erdős-Gallai type result on the clique number of a realization of a degree sequence, unpublished).

Keywords: graph, split graph, degree sequence

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1. Introduction

A sequence $\pi = (d_1, d_2, \ldots, d_n)$ of nonnegative integers is said to be graphic if it is the degree sequence of a simple graph G on n vertices, and such a graph G is referred to as a realization of π. The following well-known result due to Hakimi [1] and Havel [2] gives a necessary and sufficient condition for π to be graphic.

Let $\pi = (d_1, d_2, \ldots, d_n)$ be a non-increasing sequence of nonnegative integers. Let $d'_1 \geqslant d'_2 \geqslant \ldots \geqslant d'_{n-1}$ be the rearrangement in non-increasing order of $d_2 - 1$, $d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n$. Then $\pi' = (d'_1, d'_2, \ldots, d'_{n-1})$ is called the residual sequence of π.

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Theorem 1.1 (Hakimi [1] and Havel [2]). *π is graphic if and only if π' is graphic.*

A sequence $\pi = (d_1, d_2, \ldots, d_n)$ is said to be *potentially K_{r+1}-graphic* if there is a realization G of π containing K_{r+1} as a subgraph.

Definition. If π has a realization G containing K_{r+1} on those vertices having degree $d_1, \ldots, d_{r+1}$, then π is potentially A_{r+1}-graphic.

In [4], Rao showed that a non-increasing sequence $\pi = (d_1, d_2, \ldots, d_n)$ is potentially A_{r+1}-graphic if and only if it is potentially K_{r+1}-graphic. In [4], Rao considered the problem of characterizing potentially K_{r+1}-graphic sequences and developed a Havel-Hakimi type procedure to determine the maximum clique number of a graph with a given degree sequence π. This procedure can also be used to construct a graph with the degree sequence π and containing K_{r+1} on the first $r+1$ vertices.

Let $n \geqslant r+1$ and let $\pi = (d_1, d_2, \ldots, d_n)$ be a non-increasing sequence of nonnegative integers with $d_{r+1} \geqslant r$. We construct sequences $\pi_1, \ldots, \pi_r$ as follows. We first construct the sequence

$$\pi_1 = (d_2 - 1, \ldots, d_{r+1} - 1, d_{r+2}^{(1)}, \ldots, d_n^{(1)})$$

from π by deleting d_1, reducing the first d_1 remaining terms of π by one, and then reordering the last $n-r-1$ terms to be non-increasing. For $2 \leqslant i \leqslant r$, we construct

$$\pi_i = (d_{i+1} - i, \ldots, d_{r+1} - i, d_{r+2}^{(i)}, \ldots, d_n^{(i)})$$

from

$$\pi_{i-1} = (d_i - i + 1, \ldots, d_{r+1} - i + 1, d_{r+2}^{(i-1)}, \ldots, d_n^{(i-1)})$$

by deleting $d_i - i + 1$, reducing the first $d_i - i + 1$ remaining terms of π_{i-1} by one, and then reordering the last $n-r-1$ terms to be non-increasing.

Theorem 1.2 (Rao [4]). *π is potentially A_{r+1}-graphic if and only if π_r is graphic.*

In [5], Rao gave a simple sufficient condition for a graphic sequence to be potentially A_{r+1}-graphic.

Theorem 1.3 (Rao [5]). *Let $n \geqslant r+1$ and let $\pi = (d_1, d_2, \ldots, d_n)$ be a non-increasing graphic sequence. If $d_{r+1} \geqslant 2r-1$, then π is potentially A_{r+1}-graphic.*

Let $S_{r,s} = K_r + \overline{K_s}$, the split graph on $r+s$ vertices, where $\overline{K_s}$ is the complement of K_s and $+$ denotes the standard join operation. Clearly, $S_{r,1} = K_{r+1}$. Therefore, the graph $S_{r,s}$ is an extension of the graph K_{r+1}. A sequence $\pi = (d_1, d_2, \ldots, d_n)$ is said to be potentially $S_{r,s}$-graphic if there is a realization G of π containing $S_{r,s}$ as a subgraph. If π has a realization G containing $S_{r,s}$ on those vertices having degrees $d_1, d_2, \ldots, d_{r+s}$ such that the vertices of K_r have degrees $d_1, \ldots, d_r$ and the vertices of $\overline{K_s}$ have degrees $d_{r+1}, \ldots, d_{r+s}$, then π is potentially $A_{r,s}$-graphic. Yin [6] showed that a non-increasing sequence $\pi = (d_1, d_2, \ldots, d_n)$ is potentially $A_{r,s}$-graphic if and only if it is potentially $S_{r,s}$-graphic. Related research has been done by Lai et al (see [3]). In the present paper, we develop a Havel-Hakimi type procedure (Theorem 1.4) to determine whether a non-increasing sequence π is potentially $A_{r,s}$-graphic. This is an extension of Theorem 1.2 (which corresponds to $s = 1$). This procedure can also be used to construct a graph with the degree sequence π and containing $S_{r,s}$ on the first $r + s$ vertices.

Let $n \geqslant r + s$ and let $\pi = (d_1, d_2, \ldots, d_n)$ be a non-increasing sequence of nonnegative integers with $d_r \geqslant r + s - 1$ and $d_{r+s} \geqslant r$. We construct sequences $\pi_1, \ldots, \pi_r$ as follows. We first construct the sequence

$$\pi_1 = (d_2 - 1, \ldots, d_r - 1, d_{r+1} - 1, \ldots, d_{r+s} - 1, d^{(1)}_{r+s+1}, \ldots, d^{(1)}_n)$$

from π by deleting d_1, reducing the first d_1 remaining terms of π by one, and then reordering the last $n - r - s$ terms to be non-increasing. For $2 \leqslant i \leqslant r$, we construct

$$\pi_i = (d_{i+1} - i, \ldots, d_r - i, d_{r+1} - i, \ldots, d_{r+s} - i, d^{(i)}_{r+s+1}, \ldots, d^{(i)}_n)$$

from

$$\pi_{i-1} = (d_i - i + 1, \ldots, d_r - i + 1, d_{r+1} - i + 1, \ldots, d_{r+s} - i + 1, d^{(i-1)}_{r+s+1}, \ldots, d^{(i-1)}_n)$$

by deleting $d_i - i + 1$, reducing the first $d_i - i + 1$ remaining terms of π_{i-1} by one, and then reordering the last $n - r - s$ terms to be non-increasing.

Theorem 1.4. *π is potentially $A_{r,s}$-graphic if and only if π_r is graphic.*

Moreover, we also give a simple sufficient condition for a graphic sequence to be potentially $A_{r,s}$-graphic. This is an extension of Theorem 1.3 (which corresponds to $s = 1$).

Theorem 1.5. *Let $n \geqslant r + s$ and let $\pi = (d_1, d_2, \ldots, d_n)$ be a non-increasing graphic sequence. If $d_{r+s} \geqslant 2r + s - 2$, then π is potentially $A_{r,s}$-graphic.*

2. Proofs of Theorems 1.4 and 1.5

Proof of Theorem 1.4. Assume that π is potentially $A_{r,s}$-graphic. Then π has a realization G with a vertex set $V(G)=\{v_1,v_2,\ldots,v_n\}$ such that $d_G(v_i)=d_i$ for $1\leqslant i\leqslant n$ and G contains $S_{r,s}$ on $v_1,v_2,\ldots,v_{r+s}$ so that $V(K_r)=\{v_1,v_2,\ldots,v_r\}$ and $V(\overline{K_s})=\{v_{r+1},\ldots,v_{r+s}\}$. We now show that π has a realization G such that v_1 is adjacent to vertices $v_{r+s+1},\ldots,v_{d_1+1}$. If otherwise, we may choose such a realization H of π such that the number of vertices adjacent to v_1 in $\{v_{r+s+1},\ldots,v_{d_1+1}\}$ is maximum. Let $v_i\in\{v_{r+s+1},\ldots,v_{d_1+1}\}$ and $v_1v_i\notin E(H)$, and let $v_j\in\{v_{d_1+2},\ldots,v_n\}$ and $v_1v_j\in E(H)$. We may assume $d_i>d_j$ since the order of i and j can be interchanged if $d_i=d_j$. Hence there is a vertex v_t, $t\neq i,j$ such that $v_iv_t\in E(H)$ and $v_jv_t\notin E(H)$. Clearly, $G=(H\setminus\{v_1v_j,v_iv_t\})\cup\{v_1v_i,v_jv_t\}$ is a realization of π such that $d_G(v_i)=d_i$ for $1\leqslant i\leqslant n$, G contains $S_{r,s}$ on $v_1,v_2,\ldots,v_{r+s}$ with $V(K_r)=\{v_1,v_2,\ldots,v_r\}$ and $V(\overline{K_s})=\{v_{r+1},\ldots,v_{r+s}\}$, and G has the number of vertices adjacent to v_1 in $\{v_{r+s+1},\ldots,v_{d_1+1}\}$ larger than that of H. This contradicts the choice of H. Clearly, π_1 is the degree sequence of $G-v_1$ and is potentially $A_{r-1,s}$-graphic. Repeating this procedure, we can see that π_i is potentially $A_{r-i,s}$-graphic successively for $i=2,\ldots,r$. In particular, π_r is -graphic.

Suppose that π_r is graphic and is realized by a graph G_r with a vertex set $V(G_r)=\{v_{r+1},\ldots,v_n\}$ such that $d_{G_r}(v_i)=d_i$ for $r+1\leqslant i\leqslant n$. For $i=r,\ldots,1$, form G_{i-1} from G_i by adding a new vertex v_i that is adjacent to each of $v_{i+1},\ldots,v_{r+s}$ and also to the vertices of G_i with degrees $d^{(i-1)}_{r+s+1}-1,\ldots,d^{(i-1)}_{d_i+1}-1$. Then, for each i, G_i has degrees given by π_i, and G_i contains $S_{r-i,s}$ on $r+s-i$ vertices $v_{i+1},\ldots,v_{r+s}$ whose degrees are $d_{i+1}-i,\ldots,d_{r+s}-i$ so that $V(K_{r-i})=\{v_{i+1},\ldots,v_r\}$ and $V(\overline{K_s})=\{v_{r+1},\ldots,v_{r+s}\}$. In particular, G_0 has degrees given by π and contains $S_{r,s}$ on $r+s$ vertices $v_1,\ldots,v_{r+s}$ whose degrees are $d_1,\ldots,d_{r+s}$ so that $V(K_r)=\{v_1,\ldots,v_r\}$ and $V(\overline{K_s})=\{v_{r+1},\ldots,v_{r+s}\}$. □

Proof of Theorem 1.5. Let $n\geqslant r+s$ and let $\pi=(d_1,d_2,\ldots,d_n)$ be a non-increasing graphic sequence with $d_{r+s}\geqslant 2r+s-2$. By Theorem 1.3, π is potentially A_r-graphic. Therefore, we may assume that G is a realization of π with a vertex set $V(G)=\{v_1,v_2,\ldots,v_n\}$ such that $d_G(v_i)=d_i$ for $1\leqslant i\leqslant n$, G contains K_r on $v_1,\ldots,v_r$ and $M=e_G(\{v_1,\ldots,v_r\},\{v_{r+1},\ldots,v_{r+s}\})$ (that is the number of edges between $\{v_1,\ldots,v_r\}$ and $\{v_{r+1},\ldots,v_{r+s}\}$) is maximum. If $M=rs$, then G contains $S_{r,s}$ on $v_1,v_2,\ldots,v_{r+s}$ with $V(K_r)=\{v_1,v_2,\ldots,v_r\}$ and $V(\overline{K_s})=\{v_{r+1},\ldots,v_{r+s}\}$. In other words, π is potentially $A_{r,s}$-graphic. Assume that $M<rs$. Then there exist a $v_k\in\{v_1,v_2,\ldots,v_r\}$ and a $v_m\in\{v_{r+1},\ldots,v_{r+s}\}$ such that $v_kv_m\notin E(G)$. Let

$$A=N_{G\setminus\{v_1,\ldots,v_{r+s}\}}(v_k)\setminus N_{G\setminus\{v_1,\ldots,v_r\}}(v_m),$$
$$B=N_{G\setminus\{v_1,\ldots,v_{r+s}\}}(v_k)\cap N_{G\setminus\{v_1,\ldots,v_r\}}(v_m).$$

Then $xy \in E(G)$ for $x \in N_{G\setminus\{v_1,\dots,v_r\}}(v_m)$ and $y \in N_{G\setminus\{v_1,\dots,v_{r+s}\}}(v_k)$. Otherwise, if $xy \notin E(G)$, then $G' = (G\setminus\{v_k y, v_m x\})\cup\{v_k v_m, xy\}$ is a realization of π and contains $S_{r,s}$ on $v_1, v_2, \dots, v_{r+s}$ with $V(K_r) = \{v_1, v_2, \dots, v_r\}$ and $V(\overline{K_s}) = \{v_{r+1}, \dots, v_{r+s}\}$ such that

$$e_{G'}(\{v_1, \dots, v_r\}, \{v_{r+1}, \dots, v_{r+s}\}) > M,$$

which contradicts the choice of G. Thus, B is complete. We consider the following two cases.

Case 1. $A = \emptyset$. Then $2r + s - 2 \leqslant d_k = d_G(v_k) \leqslant r + s - 2 + |B|$, and so $|B| \geqslant r$. Since each vertex in $N_{G\setminus\{v_1,\dots,v_r\}}(v_m)$ is adjacent to each vertex in B and $|N_{G\setminus\{v_1,\dots,v_r\}}(v_m)| \geqslant 2r+s-2-(r-1) = r+s-1$, it is easy to see that the induced subgraph of $N_{G\setminus\{v_1,\dots,v_r\}}(v_m)\cup\{v_m\}$ in G contains $S_{r,s}$ as a subgraph. Thus, π is potentially $A_{r,s}$-graphic.

Case 2. $A \neq \emptyset$. Let $a \in A$. If there are $x, y \in N_{G\setminus\{v_1,\dots,v_r\}}(v_m)$ such that $xy \notin E(G)$, then

$$G' = (G \setminus \{v_m x, v_m y, v_k a\}) \cup \{v_k v_m, a v_m, xy\}$$

is a realization of π and contains $S_{r,s}$ on $v_1, v_2, \dots, v_{r+s}$ with $V(K_r) = \{v_1, v_2, \dots, v_r\}$ and $V(\overline{K_s}) = \{v_{r+1}, \dots, v_{r+s}\}$ such that

$$e_{G'}(\{v_1, \dots, v_r\}, \{v_{r+1}, \dots, v_{r+s}\}) > M,$$

which contradicts the choice of G. Thus, $N_{G\setminus\{v_1,\dots,v_r\}}(v_m)$ is complete. Since $|N_{G\setminus\{v_1,\dots,v_r\}}(v_m)| \geqslant r + s - 1$ and $v_m z \in E(G)$ for any $z \in N_{G\setminus\{v_1,\dots,v_r\}}(v_m)$, it is easy to see that the induced subgraph of $N_{G\setminus\{v_1,\dots,v_r\}}(v_m)\cup\{v_m\}$ in G is complete, and so contains $S_{r,s}$ as a subgraph. Thus, π is potentially $A_{r,s}$-graphic. □

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