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ON CO-ORDINATED QUASI-CONVEX FUNCTIONS

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Abstract. A function $f\colon I \to \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an interval, is said to be a convex function on I if

$$f(tx + (1-t)y) \leqslant tf(x) + (1-t)f(y)$$

holds for all $x, y \in I$ and $t \in [0,1]$. There are several papers in the literature which discuss properties of convexity and contain integral inequalities. Furthermore, new classes of convex functions have been introduced in order to generalize the results and to obtain new estimations.

We define some new classes of convex functions that we name quasi-convex, Jensen-convex, Wright-convex, Jensen-quasi-convex and Wright-quasi-convex functions on the co-ordinates. We also prove some inequalities of Hadamard-type as Dragomir's results in Theorem 5, but now for Jensen-quasi-convex and Wright-quasi-convex functions. Finally, we give some inclusions which clarify the relationship between these new classes of functions.

Keywords: co-ordinate, quasi-convex, Wright-quasi-convex, Jensen-quasi-convex

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1. Introduction

Let $f\colon I \subset \mathbb{R} \to \mathbb{R}$ be a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$. The double inequality

$$f\Big(\frac{a+b}{2}\Big) \leqslant \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \leqslant \frac{f(a)+f(b)}{2} \tag{1.1}$$

is well-known in the literature as Hadamard's inequality. We recall some definitions:

In [17], Pečarić *et al.* defined quasi-convex functions as follows:

Definition 1. A function $f\colon [a,b] \to \mathbb{R}$ is said to be quasi-convex on $[a,b]$ if

$$f(\lambda x + (1-\lambda)y) \leqslant \max\{f(x), f(y)\}, \qquad (QC)$$

holds for all $x, y \in [a,b]$ and $\lambda \in [0,1]$.

Clearly, any convex function is a quasi-convex function. Furthermore, there exist quasi-convex functions which are not convex.

Definition 2 (See [8], [19]). We say that $f\colon I \to \mathbb{R}$ is a Wright-convex function or that f belongs to the class $W(I)$, if for all x, $y+\delta \in I$ with $x < y$ and $\delta > 0$ we have

$$f(x+\delta) + f(y) \leqslant f(y+\delta) + f(x).$$

Definition 3 (See [8]). For $I \subseteq \mathbb{R}$, a mapping $f\colon I \to \mathbb{R}$ is a Wright-quasi-convex function if, for all $x, y \in I$ and $t \in [0,1]$, one has the inequality

$$\frac{1}{2}[f(tx + (1-t)y) + f((1-t)x + ty)] \leqslant \max\{f(x), f(y)\}, \qquad (WQC)$$

or equivalently

$$\frac{1}{2}[f(y) + f(x+\delta)] \leqslant \max\{f(x), f(y+\delta)\}$$

for every $x, y+\delta \in I$, $x < y$ and $\delta > 0$.

Definition 4 (See [8]). A mapping $f\colon I \to \mathbb{R}$ is Jensen- or J-quasi-convex if

$$f\Big(\frac{x+y}{2}\Big) \leqslant \max\{f(x), f(y)\} \qquad (JQC)$$

for all $x, y \in I$.

Note that the class $JQC(I)$ of J-quasi-convex functions on I contains the class $J(I)$ of J-convex functions on I, that is, functions satisfying the condition

$$f\Big(\frac{x+y}{2}\Big) \leqslant \frac{f(x)+f(y)}{2} \qquad (J)$$

for all $x, y \in I$.

In [8], Dragomir and Pearce proved the following theorems concerning J-quasi-convex and Wright-quasi-convex functions.

Theorem 1. *Suppose* $a, b \in I \subseteq \mathbb{R}$ *and* $a < b$. *If* $f \in JQC(I) \cap L_1[a, b]$, *then*

$$f\Big(\frac{a+b}{2}\Big) \leqslant \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x + I(a,b) \tag{1.2}$$

where

$$I(a,b) = \frac{1}{2}\int_0^1 |f(ta+(1-t)b) - f((1-t)a+tb)|\,\mathrm{d}t.$$

Theorem 2. *Let* $f\colon I \to \mathbb{R}$ *be a Wright-quasi-convex map on* I *and suppose* $a, b \in I \subseteq \mathbb{R}$ *with* $a < b$ *and* $f \in L_1[a, b]$. *Then one has the inequality*

$$\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \leqslant \max\{f(a), f(b)\}. \tag{1.3}$$

In [8], Dragomir and Pearce also gave the following theorems involving some inclusions.

Theorem 3. *Let* $WQC(I)$ *denote the class of Wright-quasi-convex functions on* $I \subseteq \mathbb{R}$, *then*

$$QC(I) \subset WQC(I) \subset JQC(I). \tag{1.4}$$

Both the inclusions are proper.

Theorem 4. *We have the inlusions*

$$W(I) \subset WQC(I), \quad C(I) \subset QC(I), \quad J(I) \subset JQC(I). \tag{1.5}$$

Each of the inclusions is proper.

For recent results related to quasi-convex functions see the papers [1]–[3], [10]–[18].

In [7], Dragomir defined co-ordinated convex functions and proved the following inequalities.

Let us consider the bidimensional interval $\Delta = [a, b] \times [c, d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$. A function $f\colon \Delta \to \mathbb{R}$ will be called convex on the co-ordinates if the partial mappings

$$f_y\colon [a,b] \to \mathbb{R}, \quad f_y(u) = f(u,y)$$

and

$$f_x\colon [c,d] \to \mathbb{R}, \quad f_x(v) = f(x,v)$$

are convex for all $y \in [c, d]$ and $x \in [a, b]$.

Recall that a mapping $f\colon \Delta \to \mathbb{R}$ is convex on Δ, if the inequality

$$f(\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)w) \leqslant \lambda f(x,y) + (1-\lambda) f(z,w) \tag{1.6}$$

holds for all (x,y), $(z,w) \in \Delta$ and $\lambda \in [0,1]$.

Theorem 5 (See [7], Theorem 1). *Suppose that $f\colon \Delta = [a,b] \times [c,d] \to \mathbb{R}$ is convex on the co-ordinates on Δ. Then one has the inequalities*

$$\begin{aligned} f\Big(\frac{a+b}{2}, \frac{c+d}{2}\Big) &\leqslant \frac{1}{2}\Big[\frac{1}{b-a}\int_a^b f\Big(x, \frac{c+d}{2}\Big)\,\mathrm{d}x + \frac{1}{d-c}\int_c^d f\Big(\frac{a+b}{2}, y\Big)\,\mathrm{d}y\Big] \\ &\leqslant \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,\mathrm{d}y\,\mathrm{d}x \\ &\leqslant \frac{1}{4}\Big[\frac{1}{b-a}\int_a^b f(x,c)\,\mathrm{d}x + \frac{1}{b-a}\int_a^b f(x,d)\,\mathrm{d}x \\ &\quad + \frac{1}{d-c}\int_c^d f(a,y)\,\mathrm{d}y + \frac{1}{d-c}\int_c^d f(b,y)\,\mathrm{d}y\Big] \\ &\leqslant \frac{f(a,c)+f(b,c)+f(a,d)+f(b,d)}{4}. \end{aligned} \tag{1.7}$$

The above inequalities are sharp.

Similar results can be found in [4]–[7], [9], [12]–[16].

This paper is arranged as follows. First, we give some definitions on quasi-convex functions and lemmas based on these definitions. Secondly, we prove several inequalities concerning co-ordinated quasi-convex functions. Also, we discuss inclusions connected with some different classes of co-ordinated convex functions.

2. Definitions and main results

We start with the following definitions and lemmas:

Definition 5. A function $f\colon \Delta = [a,b] \times [c,d] \to \mathbb{R}$ is called a quasi-convex function on the co-ordinates on Δ if the inequality

$$f(\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)w) \leqslant \max\{f(x,y), f(z,w)\}$$

holds for all (x,y), $(z,w) \in \Delta$ and $\lambda \in [0,1]$.

$f\colon \Delta \to \mathbb{R}$ is called co-ordinated quasi-convex on the co-ordinates if the partial mappings

$$f_y\colon [a,b] \to \mathbb{R}, \quad f_y(u) = f(u,y)$$

and

$$f_x\colon [c,d] \to \mathbb{R}, \quad f_x(v) = f(x,v)$$

are convex for all $y \in [c,d]$ and $x \in [a,b]$. We denote by $QC(\Delta)$ the classes of quasi-convex functions on the co-ordinates on Δ.

The following lemma holds.

Lemma 6. *Every quasi-convex mapping $f\colon \Delta \to \mathbb{R}$ is quasi-convex on the co-ordinates.*

Proof. Suppose that $f\colon \Delta = [a,b] \times [c,d] \to \mathbb{R}$ is quasi-convex on Δ. Then the partial mappings

$$f_y\colon [a,b] \to \mathbb{R}, \quad f_y(u) = f(u,y), \quad y \in [c,d]$$

and

$$f_x\colon [c,d] \to \mathbb{R}, \quad f_x(v) = f(x,v), \quad x \in [a,b]$$

are convex on Δ. For $\lambda \in [0,1]$ and $v_1, v_2 \in [c,d]$ one has

$$\begin{aligned} f_x(\lambda v_1 + (1-\lambda)v_2) &= f(x, \lambda v_1 + (1-\lambda)v_2) \\ &= f(\lambda x + (1-\lambda)x, \lambda v_1 + (1-\lambda)v_2) \\ &\leqslant \max\{f(x,v_1), f(x,v_2)\} \\ &= \max\{f_x(v_1), f_x(v_2)\}, \end{aligned}$$

which completes the proof of quasi-convexity of f_x on $[c,d]$. The proof that $f_y\colon [a,b] \to \mathbb{R},\ \ f_y(u) = f(u,y)$ is also quasi-convex on $[a,b]$ for all $y \in [c,d]$, is analogous and we omit the details. □

Definition 6. A function $f\colon \Delta = [a,b] \times [c,d] \to \mathbb{R}$ is called a J-convex function on the co-ordinates on Δ if the inequality

$$f\Big(\frac{x+z}{2}, \frac{y+w}{2}\Big) \leqslant \frac{f(x,y) + f(z,w)}{2}$$

holds for all (x,y), $(z,w) \in \Delta$. We denote by $J(\Delta)$ the class of J-convex functions on the co-ordinates on Δ.

Lemma 7. *Every J-convex mapping $f\colon \Delta \to \mathbb{R}$ is J-convex on the co-ordinates.*

Proof. As concerns the partial mappings, we can write for $v_1, v_2 \in [c,d]$,

$$\begin{aligned} f_x\Big(\frac{v_1+v_2}{2}\Big) &= f\Big(x, \frac{v_1+v_2}{2}\Big) = f\Big(\frac{x+x}{2}, \frac{v_1+v_2}{2}\Big) \\ &\leqslant \frac{f(x,v_1)+f(x,v_2)}{2} = \frac{f_x(v_1)+f_x(v_2)}{2}, \end{aligned}$$

which completes the proof of J-convexity of f_x on $[c,d]$. Similarly, we can prove J-convexity of f_y on $[a,b]$. □

Definition 7. A function $f\colon \Delta = [a,b] \times [c,d] \to \mathbb{R}$ is called a J-quasi-convex function on the co-ordinates on Δ if the inequality

$$f\Big(\frac{x+z}{2}, \frac{y+w}{2}\Big) \leqslant \max\{f(x,y), f(z,w)\}$$

holds for all (x,y), $(z,w) \in \Delta$. We denote by $JQC(\Delta)$ the class of J-quasi-convex functions on the co-ordinates on Δ.

Lemma 8. *Every J-quasi-convex mapping $f\colon \Delta \to \mathbb{R}$ is J-quasi-convex on the co-ordinates.*

Proof. In a way similar to the proof of Lemma 7, we can write for $v_1, v_2 \in [c,d]$

$$\begin{aligned} f_x\Big(\frac{v_1+v_2}{2}\Big) &= f\Big(x, \frac{v_1+v_2}{2}\Big) = f\Big(\frac{x+x}{2}, \frac{v_1+v_2}{2}\Big) \\ &\leqslant \max\{f(x,v_1), f(x,v_2)\} = \max\{f_x(v_1), f_x(v_2)\}, \end{aligned}$$

which completes the proof of J-quasi-convexity of f_x on $[c,d]$. We can also prove J-quasi-convexity of f_y on $[a,b]$. □

Definition 8. A function $f\colon \Delta = [a,b] \times [c,d] \to \mathbb{R}$ is called a Wright-convex function on the co-ordinates on Δ if the inequality

$$f((1-t)a+tb, (1-s)c+sd) + f(ta+(1-t)b, sc+(1-s)d) \leqslant f(a,c) + f(b,d)$$

holds for all (a,c), $(b,d) \in \Delta$ and $t, s \in [0,1]$. We denote by $W(\Delta)$ the class of Wright-convex functions on the co-ordinates on Δ.

Lemma 9. *Every Wright-convex mapping $f\colon \Delta \to \mathbb{R}$ is Wright-convex on the co-ordinates.*

P r o o f. Suppose that $f\colon \Delta \to \mathbb{R}$ is Wright-convex on Δ. Then for the partial mapping, for $v_1, v_2 \in [c,d]$, $x \in [a,b]$, we have

$$\begin{aligned}
&f_x((1-t)v_1 + tv_2) + f_x(tv_1 + (1-t)v_2)\\
&\qquad = f(x,(1-t)v_1 + tv_2) + f(x, tv_1 + (1-t)v_2)\\
&\qquad = f((1-t)x + tx, (1-t)v_1 + tv_2) + f(tx + (1-t)x, tv_1 + (1-t)v_2)\\
&\qquad \leqslant f(x, v_1) + f(x, v_2)\\
&\qquad = f_x(v_1) + f_x(v_2),
\end{aligned}$$

which shows that f_x is Wright-convex on $[c,d]$. Similarly one can see that f_y is Wright-convex on $[a,b]$. $\square$

Definition 9. A function $f\colon \Delta = [a,b] \times [c,d] \to \mathbb{R}$ is called a Wright-quasi-convex function on the co-ordinates on Δ if the inequality

$$\frac{1}{2}[f(tx+(1-t)z, ty+(1-t)w) + f((1-t)x+tz, (1-t)y+tw)] \leqslant \max\{f(x,y), f(z,w)\}$$

holds for all (x,y), $(z,w) \in \Delta$ and $t \in [0,1]$. We denote by $WQC(\Delta)$ the class of Wright-quasi-convex functions on the co-ordinates on Δ.

Lemma 10. *Every Wright-quasi-convex mapping $f\colon \Delta \to \mathbb{R}$ is Wright-quasi-convex on the co-ordinates.*

P r o o f. Suppose that $f\colon \Delta \to \mathbb{R}$ is Wright-quasi-convex on Δ. Then for the partial mapping, for $v_1, v_2 \in [c,d]$, we have

$$\begin{aligned}
&\frac{1}{2}[f_x(tv_1 + (1-t)v_2) + f_x((1-t)v_1 + tv_2)]\\
&\qquad = \frac{1}{2}[f(x, tv_1 + (1-t)v_2) + f(x,(1-t)v_1 + tv_2)]\\
&\qquad = \frac{1}{2}[f(tx + (1-t)x, tv_1 + (1-t)v_2) + f((1-t)x + tx, (1-t)v_1 + tv_2)]\\
&\qquad \leqslant \max\{f(x,v_1), f(x,v_2)\} = \max\{f_x(v_1), f_x(v_2)\},
\end{aligned}$$

which shows that f_x is Wright-quasi-convex on $[c,d]$. Similarly one can see that f_y is Wright-quasi-convex on $[a,b]$. $\square$

Theorem 11. *Suppose that $f\colon \Delta=[a,b]\times[c,d]\to\mathbb{R}$ is J-quasi-convex on the co-ordinates on Δ. If $f_x\in L_1[c,d]$ and $f_y\in L_1[a,b]$, then we have the inequality*

$$\tag{2.1} \begin{aligned} &\frac{1}{2}\Bigl[\frac{1}{b-a}\int_a^b f\Bigl(x,\frac{c+d}{2}\Bigr)\,\mathrm{d}x+\frac{1}{d-c}\int_c^d f\Bigl(\frac{a+b}{2},y\Bigr)\,\mathrm{d}y\Bigr]\\ &\quad\leqslant \frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)\,\mathrm{d}x\,\mathrm{d}y+H(x,y)\end{aligned}$$

where

$$\begin{aligned} H(x,y)&=\frac{1}{4(d-c)}\int_c^d\int_0^1|f(ta+(1-t)b,y)-f((1-t)a+tb,y)|\,\mathrm{d}t\,\mathrm{d}y\\ &\quad+\frac{1}{4(b-a)}\int_a^b\int_0^1|f(x,tc+(1-t)d)-f(x,(1-t)c+td)|\,\mathrm{d}t\,\mathrm{d}x.\end{aligned}$$

Proof. Since $f\colon \Delta\to\mathbb{R}$ is J-quasi-convex on the co-ordinates on Δ, the partial mappings

$$f_y\colon [a,b]\to\mathbb{R},\quad f_y(u)=f(u,y),\quad y\in[c,d]$$

and

$$f_x\colon [c,d]\to\mathbb{R},\quad f_x(v)=f(x,v),\quad x\in[a,b]$$

are J-quasi-convex on Δ. Then by the inequality (1.2), we have

$$f_y\Bigl(\frac{a+b}{2}\Bigr)\leqslant\frac{1}{b-a}\int_a^b f_y(x)\,\mathrm{d}x+\frac{1}{2}\int_0^1|f_y(ta+(1-t)b)-f_y((1-t)a+tb)|\,\mathrm{d}t,$$

that is

$$f\Bigl(\frac{a+b}{2},y\Bigr)\leqslant\frac{1}{b-a}\int_a^b f(x,y)\,\mathrm{d}x+\frac{1}{2}\int_0^1|f(ta+(1-t)b,y)-f((1-t)a+tb,y)|\,\mathrm{d}t.$$

Integrating the resulting inequality with respect to y over $[c,d]$ and dividing both sides of the inequality by $(d-c)$, we get

$$\tag{2.2} \begin{aligned} \frac{1}{d-c}\int_c^d f\Bigl(\frac{a+b}{2},y\Bigr)\,\mathrm{d}y&\leqslant\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)\,\mathrm{d}x\,\mathrm{d}y\\ &\quad+\frac{1}{2(d-c)}\int_c^d\int_0^1|f(ta+(1-t)b,y)-f((1-t)a+tb,y)|\,\mathrm{d}t\,\mathrm{d}y.\end{aligned}$$

By a similar argument, we have

$$\tag{2.3} \begin{aligned} \frac{1}{b-a}\int_a^b f\Bigl(x,\frac{c+d}{2}\Bigr)\,\mathrm{d}x&\leqslant\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,\mathrm{d}y\,\mathrm{d}x\\ &\quad+\frac{1}{2(b-a)}\int_a^b\int_0^1|f(x,tc+(1-t)d)-f(x,(1-t)c+td)|\,\mathrm{d}t\,\mathrm{d}x.\end{aligned}$$

Summing (2.2) and (2.3), we get the required result. □

Theorem 12. *Suppose that $f\colon\ \Delta=[a,b]\times[c,d]\to\mathbb{R}$ is Wright-quasi-convex on the co-ordinates on Δ. If $f_x\in L_1[c,d]$ and $f_y\in L_1[a,b]$, then we have the inequality*

$$\begin{aligned}(2.4)\qquad &\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)\,\mathrm{d}x\,\mathrm{d}y\\ &\leqslant\frac{1}{2}\Big[\max\Big\{\frac{1}{(b-a)}\int_a^b f(x,c)\,\mathrm{d}x,\frac{1}{(b-a)}\int_a^b f(x,d)\,\mathrm{d}x\Big\}\\ &\quad+\max\Big\{\frac{1}{(d-c)}\int_c^d f(a,y)\,\mathrm{d}y,\frac{1}{(d-c)}\int_c^d f(b,y)\,\mathrm{d}y\Big\}\Big].\end{aligned}$$

P r o o f. Since $f\colon\ \Delta\to\mathbb{R}$ is Wright-quasi-convex on the co-ordinates on Δ, the partial mappings

$$f_y\colon\ [a,b]\to\mathbb{R},\quad f_y(u)=f(u,y),\quad y\in[c,d]$$

and

$$f_x\colon\ [c,d]\to\mathbb{R},\quad f_x(v)=f(x,v),\quad x\in[a,b]$$

are Wright-quasi-convex on Δ. Then by the inequality (1.3), we have

$$\frac{1}{b-a}\int_a^b f_y(x)\,\mathrm{d}x\leqslant\max\{f_y(a),f_y(b)\},$$

that is

$$\frac{1}{b-a}\int_a^b f(x,y)\,\mathrm{d}x\leqslant\max\{f(a,y),f(b,y)\}.$$

Dividing both sides of the inequality by $(d-c)$ and integrating with respect to y over $[c,d]$, we get

$$\begin{aligned}(2.5)\qquad &\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)\,\mathrm{d}x\,\mathrm{d}y\\ &\leqslant\max\Big\{\frac{1}{(d-c)}\int_c^d f(a,y)\,\mathrm{d}y,\frac{1}{(d-c)}\int_c^d f(b,y)\,\mathrm{d}y\Big\}.\end{aligned}$$

By a similar argument, we can write

$$\begin{aligned}(2.6)\qquad &\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)\,\mathrm{d}x\,\mathrm{d}y\\ &\leqslant\max\Big\{\frac{1}{(b-a)}\int_a^b f(x,c)\,\mathrm{d}x,\frac{1}{(b-a)}\int_a^b f(x,d)\,\mathrm{d}x\Big\}.\end{aligned}$$

By adding (2.5) and (2.6), we have

$$\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)\,\mathrm{d}x\,\mathrm{d}y$$
$$\leqslant \frac{1}{2}\Big[\max\Big\{\frac{1}{(b-a)}\int_a^b f(x,c)\,\mathrm{d}x, \frac{1}{(b-a)}\int_a^b f(x,d)\,\mathrm{d}x\Big\}$$
$$+\max\Big\{\frac{1}{(d-c)}\int_c^d f(a,y)\,\mathrm{d}y, \frac{1}{(d-c)}\int_c^d f(b,y)\,\mathrm{d}y\Big\}\Big],$$

which completes the proof. □

Theorem 13. *Let $C(\Delta)$, $J(\Delta)$, $W(\Delta)$, $QC(\Delta)$, $JQC(\Delta)$, $WQC(\Delta)$ denote, respectively the classes of co-ordinated convex, co-ordinated J-convex, co-ordinated W-convex, co-ordinated quasi-convex, co-ordinated J-quasi-convex and co-ordinated W-quasi-convex functions on $\Delta=[a,b]\times[c,d]$. Then we have the inclusions*

$$QC(\Delta)\subset WQC(\Delta)\subset JQC(\Delta), \tag{2.7}$$
$$W(\Delta)\subset WQC(\Delta),\quad C(\Delta)\subset J(\Delta),\quad J(\Delta)\subset JQC(\Delta). \tag{2.8}$$

P r o o f. Let $f\in QC(\Delta)$. Then for all (x,y), $(z,w)\in\Delta$ and $t\in[0,1]$, we have

$$f(\lambda x+(1-\lambda)z,\lambda y+(1-\lambda)w)\leqslant \max\{f(x,y),f(z,w)\},$$
$$f((1-\lambda)x+\lambda z,(1-\lambda)y+\lambda w)\leqslant \max\{f(x,y),f(z,w)\}.$$

By adding the inequalities we obtain

$$\frac{1}{2}[f(\lambda x+(1-\lambda)z,\lambda y+(1-\lambda)w)+f((1-\lambda)x+\lambda z,(1-\lambda)y+\lambda w)]$$
$$\leqslant \max\{f(x,y),f(z,w)\} \tag{2.9}$$

that is, $f\in WQC(\Delta)$. In (2.9), if we choose $\lambda=\frac{1}{2}$, we obtain $WQC(\Delta)\subset JQC(\Delta)$. This completes the proof of (2.7).

In order to prove (2.8), taking $f\in W(\Delta)$ and using the definition, we get

$$\frac{1}{2}[f((1-t)a+tb,(1-s)c+sd)+f(ta+(1-t)b,sc+(1-s)d)]\leqslant \frac{f(a,c)+f(b,d)}{2}$$

for all $(a,c),(b,d)\in\Delta$ and $t\in[0,1]$. Using the fact that

$$\frac{f(a,c)+f(b,d)+|f(a,c)-f(b,d)|}{2}=\max\{f(a,c),f(b,d)\}$$

we can write

$$\frac{f(a,c)+f(b,d)}{2} \leqslant \max\{f(a,c), f(b,d)\}$$

for all $(a,c),(b,d) \in \Delta$, thus obtaining $W(\Delta) \subset WQC(\Delta)$.

Taking $f \in C(\Delta)$ and choosing $t = \frac{1}{2}$ in (1.6), we obtain

$$f\Big(\frac{x+z}{2}, \frac{y+w}{2}\Big) \leqslant \frac{f(x,y)+f(z,w)}{2}$$

for all (x,y), $(z,w) \in \Delta$. One can see that $C(\Delta) \subset J(\Delta)$.

Taking $f \in J(\Delta)$, we can write

$$f\Big(\frac{x+z}{2}, \frac{y+w}{2}\Big) \leqslant \frac{f(x,y)+f(z,w)}{2}$$

for all (x,y), $(z,w) \in \Delta$. Using the fact that

$$\frac{f(x,y)+f(z,w)+|f(x,y)-f(z,w)|}{2} = \max\{f(x,y), f(z,w)\}$$

we can write

$$\frac{f(x,y)+f(z,w)}{2} \leqslant \max\{f(x,y), f(z,w)\}.$$

Then obviously, we obtain

$$f\Big(\frac{x+z}{2}, \frac{y+w}{2}\Big) \leqslant \max\{f(x,y), f(z,w)\},$$

which shows that $f \in JQ(\Delta)$. □

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