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ESSENTIAL NORMALITY FOR CERTAIN FINITE LINEAR  
COMBINATIONS OF LINEAR-FRACTIONAL COMPOSITION  
OPERATORS ON THE HARDY SPACE  $H^2$ 

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*Abstract.* In 1999 Nina Zorboska and in 2003 P. S. Bourdon, D. Levi, S. K. Narayan and J. H. Shapiro investigated the essentially normal composition operator  $C_\varphi$ , when  $\varphi$  is a linear-fractional self-map of  $\mathbb{D}$ . In this paper first, we investigate the essential normality problem for the operator  $T_w C_\varphi$  on the Hardy space  $H^2$ , where  $w$  is a bounded measurable function on  $\partial\mathbb{D}$  which is continuous at each point of  $F(\varphi)$ ,  $\varphi \in \mathcal{S}(2)$ , and  $T_w$  is the Toeplitz operator with symbol  $w$ . Then we use these results and characterize the essentially normal finite linear combinations of certain linear-fractional composition operators on  $H^2$ .

*Keywords:* Hardy spaces, essentially normal, composition operator, linear-fractional transformation

*MSC 2010:* 47B33

## 1. INTRODUCTION

Let  $\mathbb{D}$  be the open unit disk in the complex plane  $\mathbb{C}$ ,  $\partial\mathbb{D}$  be its boundary, and  $\text{Hol}(\mathbb{D})$  denotes the space of all holomorphic functions on  $\mathbb{D}$ .

For an analytic function  $f$  on the unit disk and  $0 < r < 1$ , we define the dilated function  $f_r$  by  $f_r(e^{i\theta}) = f(re^{i\theta})$ . It is easy to see that the functions  $f_r$  are continuous on  $\partial\mathbb{D}$  for each  $r$ , hence they are in  $L^p(\partial\mathbb{D}, d\theta/2\pi)$ , where  $d\theta/2\pi$  is the normalized arc length measure on the unit circle.

For  $0 < p < \infty$ , the Hardy space  $H^p(\mathbb{D}) = H^p$  is the set of all analytic functions on the unit disk for which

$$\|f\|_p^p = \sup_{0 < r < 1} \int_0^{2\pi} |f_r(e^{i\theta})|^p \frac{d\theta}{2\pi} < \infty.$$

Also we recall that  $H^\infty(\mathbb{D}) = H^\infty$  is the space of all bounded analytic functions defined on  $\mathbb{D}$ , with the supremum norm  $\|f\|_\infty = \sup_{z \in \mathbb{D}} |f(z)|$ . We know that for  $p \geq 1$ ,  $H^p$  is a Banach space (see, e.g., [8, p. 37]). For more information about the Hardy spaces see, for example, [7] and [8]. For  $\beta \geq 1$ , let  $\mathcal{D}_\beta$  denote the reproducing kernel Hilbert space of functions analytic in the unit disk  $\mathbb{D}$  and having the kernel functions  $K_w(z) = (1 - \bar{w}z)^{-\beta}$ . The Hardy space  $H^2$  is exactly  $\mathcal{D}_1$ .

For each  $\psi \in L^\infty(\partial\mathbb{D})$ , we define the Toeplitz operator  $T_\psi$  on  $H^2$  by  $T_\psi(f) = P(\psi f)$ , where  $P$  denotes the orthogonal projection of  $L^2(\partial\mathbb{D})$  onto  $H^2$ . Since an orthogonal projection has norm 1, clearly  $T_\psi$  is bounded. For any analytic self-map  $\varphi$  of  $\mathbb{D}$ , the composition operator  $C_\varphi$  on  $H^2$  is defined by  $C_\varphi(f) = f \circ \varphi$ . It is well known (see, e.g., [8, p. 29] or [16, Theorem 1]) that the composition operators are bounded on each of the Hardy spaces  $H^p$  ( $0 < p < \infty$ ).

A mapping of the form

$$(1) \quad \varphi(z) = \frac{az + b}{cz + d} \quad (ad - bc \neq 0)$$

is called a linear-fractional transformation. We denote the set of those linear-fractional transformations that take the open unit disk  $\mathbb{D}$  into itself by  $\text{LFT}(\mathbb{D})$ . It is well known that the automorphisms of the unit disk, that is, the one-to-one analytic maps of the disk onto itself, are just the functions  $\varphi(z) = \lambda(a - z)/(1 - \bar{a}z)$ , where  $|\lambda| = 1$  and  $|a| < 1$ .

For bounded operators  $A$  and  $B$  on a Hilbert space, we use the notation  $[A, B] := AB - BA$  for the commutator of  $A$  and  $B$ . Recall that an operator  $A$  is called normal if  $[A, A^*] = 0$  and essentially normal if  $[A, A^*]$  is compact. In 1969, H. J. Schwartz [18] showed that a composition operator on  $H^2$  is normal if and only if it is induced by a dilation  $z \rightarrow az$ , where  $|a| \leq 1$ . In [21] Nina Zorboska has characterized the essentially normal composition operators induced on the Hardy space  $H^2$  by automorphisms of the unit disk. In addition, Zorboska has shown that the composition operators induced on  $H^2$  by linear-fractional transformations fixing no point on the unit circle are not nontrivially essentially normal. P. S. Bourdon, D. Levi, S. K. Narayan, and J. H. Shapiro in [3] have shown that a composition operator induced on  $H^2$  by a linear-fractional self-map of the unit disk is nontrivially essentially normal if and only if it is induced by a parabolic non-automorphism. The essentially normal composition operators on other spaces have been investigated by some authors (see, e.g., [4], [12], and [13]).

If  $\varphi$  and  $\psi$  are linear-fractional self-maps of  $\mathbb{D}$  or  $B_N$ , then  $C_\varphi - C_\psi$  cannot be non-trivially compact; i.e., if the difference is compact, either  $C_\varphi$  and  $C_\psi$  are individually compact or  $\varphi = \psi$ . The fact that a difference of linear-fractional composition operators cannot be non-trivially compact on  $H^2$  or  $A_\alpha^2(\mathbb{D})$  was first obtained by

P. S. Bourdon [2] and J. Moorhouse [14] as a consequence of results on the compactness of a difference of more general composition operators in one variable. Recently there has been a great interest in studying some linear combinations of composition operators; see, for example, [9] and [11].

In this paper, we use the results of T. L. Kriete and J. L. Moorhouse [11] and T. L. Kriete, B. D. MacCluer and J. L. Moorhouse [10] in order to investigate the essential normality problem for certain finite linear combinations of linear-fractional composition operators on  $H^2$ .

## 2. PRELIMINARIES

Here we collect the fundamental facts about some definitions and results which are required in the sequel.

**2.1. Angular derivatives.** Let  $\varphi$  be an analytic self-map of  $\mathbb{D}$ . We say that  $\varphi$  has a finite angular derivative at  $\zeta$  on the unit circle if there is  $\eta$  on the unit circle such that  $(\varphi(z) - \eta)/(z - \zeta)$  has a finite non-tangential limit as  $z \rightarrow \zeta$ . When it exists (as a finite complex number), this limit is denoted by  $\varphi'(\zeta)$ . By the Julia-Carathéodory Theorem (see, e.g., [7, Theorem 2.44] or [19, Chapter 4]),

$$|\varphi'(\zeta)| = d(\zeta) := \liminf_{z \rightarrow \zeta} \frac{1 - |\varphi(z)|}{1 - |z|},$$

where the  $\liminf$  is taken as  $z$  approaches  $\zeta$  unrestrictedly in  $\mathbb{D}$ . Throughout this paper, let  $F(\varphi)$  denote the set of all points in  $\partial\mathbb{D}$  at which  $\varphi$  has a finite angular derivative. A necessary condition for the composition operator  $C_\varphi$  to act compactly on  $H^2$  is that  $F(\varphi)$  is empty; see [20] or [7, Corollary 3.14]. This condition, however, is not sufficient unless  $\varphi$  is of bounded multiplicity (see [7, Corollary 3.21]).

**2.2. Clark measures.** Suppose that  $\varphi$  is an analytic self-map of  $\mathbb{D}$  and  $\alpha$  is a complex number of modulus 1. Since  $\operatorname{Re}((\alpha + \varphi)/(\alpha - \varphi))$  is a positive harmonic function on  $\mathbb{D}$ , there exists a finite positive Borel measure  $\mu_\alpha$  on  $\partial\mathbb{D}$  such that

$$\frac{1 - |\varphi(z)|^2}{|\alpha - \varphi(z)|^2} = \operatorname{Re}\left(\frac{\alpha + \varphi(z)}{\alpha - \varphi(z)}\right) = \int_{\partial\mathbb{D}} P_z d\mu_\alpha$$

for each  $z \in \mathbb{D}$ , where  $P_z(e^{i\theta}) = (1 - |z|^2)/|e^{i\theta} - z|^2$  is the Poisson kernel at  $z$ . The measures  $\mu_\alpha$  are called the Clark measures of  $\varphi$ . There is a unique pair of measures  $\mu_\alpha^{\text{ac}}$  and  $\mu_\alpha^{\text{s}}$  such that  $\mu_\alpha = \mu_\alpha^{\text{ac}} + \mu_\alpha^{\text{s}}$ , where  $\mu_\alpha^{\text{ac}}$  and  $\mu_\alpha^{\text{s}}$  are the absolutely continuous and singular parts with respect to Lebesgue measure, respectively. The

singular part  $\mu_\alpha^s$  is carried by  $\varphi^{-1}(\{\alpha\})$ , the set of those  $\zeta$  in  $\partial\mathbb{D}$  where  $\varphi(\zeta)$  exists and equals  $\alpha$ , and is itself the sum of the pure point measure

$$(2) \quad \mu_\alpha^{pp} = \sum_{\varphi(\zeta)=\alpha} \frac{1}{|\varphi'(\zeta)|} \delta_\zeta,$$

where  $\delta_\zeta$  is the unit point mass measure at  $\zeta$  and a continuous singular measure  $\mu_\alpha^{cs}$ , either of which can vanish. In particular, if  $\varphi$  is a linear-fractional non-automorphism such that  $\varphi(\zeta) = \eta$  for some  $\zeta, \eta \in \partial\mathbb{D}$ , then  $\mu_\alpha^s = 0$  when  $\alpha \neq \eta$  and  $\mu_\eta^s = |\varphi'(\zeta)|^{-1} \delta_\zeta$ . We write  $E(\varphi)$  for the closure in  $\partial\mathbb{D}$  of the union of the closed supports of  $\mu_\alpha^s$  as  $\alpha$  ranges over the unit circle. Therefore, by Equation (2),  $F(\varphi) \subseteq E(\varphi)$ . The measures  $\mu_\alpha$  were introduced as an operator-theoretic tool by D. N. Clark [5] and have been further analyzed by A. B. Aleksandrov [1], A. G. Poltoratski [15] and D. E. Sarason [17].

**2.3. Cowen's adjoint formula.** In [6] Carl Cowen showed that if  $\varphi \in \text{LFT}(\mathbb{D})$  is given by Equation (1), then

$$(3) \quad C_\varphi^* = T_g C_{\sigma_\varphi} T_h^*,$$

where  $\sigma_\varphi(z) := (\bar{a}z - \bar{c})/(-\bar{b}z + \bar{d})$  is a self-map of  $\mathbb{D}$ ,  $g(z) := (-\bar{b}z + \bar{d})^{-1}$ ,  $h(z) := cz + d$  and  $g, h \in H^\infty$ . The map  $\sigma_\varphi$  is called the Krein adjoint of  $\varphi$ ; we will write  $\sigma$  for  $\sigma_\varphi$  except when confusion could arise. If  $\varphi(\zeta) = \eta$  for  $\zeta, \eta \in \partial\mathbb{D}$ , then  $\sigma(\eta) = \zeta$ . Also,  $\varphi$  is an automorphism if and only if  $\sigma$  is, and in this case  $\sigma = \varphi^{-1}$ . For further details see, for example, [3].

We know that if  $\overline{\varphi(\mathbb{D})} \subseteq \mathbb{D}$ , then  $C_\varphi$  is compact (see, e.g., [19]). Let  $\zeta_1, \zeta_2, \eta_1, \eta_2 \in \partial\mathbb{D}$  and  $\zeta_1 \neq \zeta_2$ . Assume that  $\varphi_1, \varphi_2 \in \text{LFT}(\mathbb{D})$  are not automorphisms and that  $\varphi_1(\zeta_1) = \eta_1$  and  $\varphi_2(\zeta_2) = \eta_2$ . Suppose that  $1 \leq i, j \leq 2$  and  $i \neq j$ . We see that  $\varphi_i \circ \sigma_j$  takes  $\partial\mathbb{D}$  into  $\mathbb{D}$ , so  $\|\varphi_i \circ \sigma_j\|_\infty < 1$  and  $C_{\varphi_i \circ \sigma_j}$  is compact on  $H^2$ . Also, it is clear that  $\sigma_j \circ \varphi_i$  takes  $\partial\mathbb{D}$  into  $\mathbb{D}$ , when  $\eta_j \neq \eta_i$ ; therefore, we have  $\|\sigma_j \circ \varphi_i\|_\infty < 1$  and  $C_{\sigma_j \circ \varphi_i}$  is compact on  $H^2$ . We will use these two facts frequently in this paper.

**2.4. Parabolic linear-fractional self-map of  $\mathbb{D}$ .** A map  $\varphi \in \text{LFT}(\mathbb{D})$  whose fixed point set, relative to the Riemann sphere, consists of a single point  $\zeta$  in  $\partial\mathbb{D}$  is termed parabolic. In [19, p. 3] J. H. Shapiro has shown that among the linear-fractional non-automorphisms fixing  $\zeta \in \partial\mathbb{D}$ , the parabolic ones are characterized by  $\varphi'(\zeta) = 1$ ; for further details see [3] and [19].

In the rest of this section, we state some useful definitions and results of [11] that we will need in the sequel.

**2.5. The class  $\mathcal{S}$  and  $\mathcal{S}(2)$ .** For  $\zeta \in F(\varphi)$ , the first-order data of  $\varphi$  at  $\zeta$  is given by the vector  $D_1(\varphi, \zeta) := (\varphi(\zeta), \varphi'(\zeta))$ . In what follows, we look at higher-order data vectors

$$D_k(\varphi, \zeta) := (\varphi(\zeta), \varphi'(\zeta), \varphi''(\zeta), \dots, \varphi^{(k)}(\zeta))$$

at points where the corresponding derivatives make sense.

We say an analytic self-map  $\varphi$  of  $\mathbb{D}$  has an order of contact  $c > 0$  at  $\zeta$  if  $|\varphi(\zeta)| = 1$  and

$$\frac{1 - |\varphi(e^{i\theta})|^2}{|\varphi(\zeta) - \varphi(e^{i\theta})|^c}$$

is essentially bounded above and away from zero as  $e^{i\theta} \rightarrow \zeta$ .

We say an analytic self-map  $\varphi$  of  $\mathbb{D}$  has a  $k$ th-order data at  $\zeta$  in  $F(\varphi)$  if there exist complex numbers  $b_0, b_1, \dots, b_k$  with  $|b_0| = 1$  such that

$$\varphi(z) = b_0 + b_1(z - \zeta) + \dots + b_k(z - \zeta)^k + o(|z - \zeta|^k)$$

as  $z \rightarrow \zeta$  unrestrictedly in  $\mathbb{D}$ . In this case for any  $1 \leq j \leq k$ ,  $j!b_j$  is the non-tangential limit of  $\varphi^{(j)}(z)$  at  $\zeta$  (see, for example, the argument on p. 47 in [17]); we refer to this limit as  $\varphi^{(j)}(\zeta)$ . Note that since  $|b_0| = 1$  and  $\zeta \in F(\varphi)$ ,  $b_1$  is the angular derivative  $\varphi'(\zeta)$ .

We say an analytic self-map  $\varphi$  of  $\mathbb{D}$  has sufficient data at  $\zeta$  in  $\partial\mathbb{D}$  if

- (i)  $\zeta \in F(\varphi)$ ;
- (ii)  $\varphi$  has an order of contact  $2m$  at  $\zeta$  for some natural number  $m$ ;
- (iii)  $\varphi$  has a  $(2m)$ th-order data at  $\zeta$ .

Suppose that  $\varphi$  has a finite angular derivative at  $\zeta$ . Also, let it have an analytic continuation to a neighborhood of  $\zeta$  and  $|\varphi| < 1$  a.e. on  $\partial\mathbb{D}$ . For any  $\alpha$  in  $\partial\mathbb{D}$ , consider the linear-fractional transformation  $\tau_\alpha(z) := i(\alpha - z)/(\alpha + z)$  which takes the unit disk onto the upper half-plane  $\Omega := \{w: \text{Im } w > 0\}$  and  $\alpha$  to 0. Let  $u := \tau_{\varphi(\zeta)} \circ \varphi \circ \tau_\zeta^{-1}$ . Then for  $w$  near zero,  $u(w) = \sum_{n=1}^{\infty} a_n w^n$ . In [11, p. 2930] Kriete et al. have shown that the smallest natural number  $n$  with  $a_n$  non-real must be even. Let  $n = 2m$ . Also, they have proved that  $\varphi$  has an order of contact  $2m$  at  $\zeta$ . In particular, let  $\varphi$  be a non-automorphism linear-fractional self-map of  $\mathbb{D}$  with  $\varphi(\zeta) = \eta$  for some  $\zeta, \eta \in \partial\mathbb{D}$ . Assume that for any  $\alpha \in \partial\mathbb{D}$ , we define the linear-fractional transformation  $S_\alpha(z) := (1 + \bar{\alpha}z)/(1 - \bar{\alpha}z)$  which takes the unit disk onto the right half-plane  $\Pi$  and  $\alpha$  to  $\infty$ . Set  $\phi := S_\eta \circ \varphi \circ S_\zeta^{-1}$ . Since  $\phi(\infty) = \infty$ , the function  $\phi(z) = \lambda z + b$ . Also,  $\varphi = S_\eta^{-1} \circ (\lambda z + b) \circ S_\zeta$  and  $\varphi(\mathbb{D}) \subsetneq \mathbb{D}$ . Therefore,  $\lambda > 0$ ,  $\text{Re } b > 0$  and  $u = \tau_\eta \circ \varphi \circ \tau_\zeta^{-1} = \tau_\eta \circ S_\eta^{-1} \circ (\lambda z + b) \circ S_\zeta \circ \tau_\zeta^{-1}$ . By some

computations,  $S_\zeta \circ \tau_\zeta^{-1}(z) = i/z$  and hence  $\tau_\eta \circ S_\eta^{-1} = i/z$ . Thus,

$$u(z) = (i/z) \circ (\lambda z + b) \circ (i/z) = \sum_{n=0}^{\infty} (-1)^n \frac{b^n}{(i)^n \lambda^{n+1}} z^{n+1}.$$

Therefore,  $\varphi$  has an order of contact 2 at  $\zeta$  and has sufficient data at  $\zeta$ . Let  $\mathcal{S}$  be the class of analytic self-maps  $\varphi$  of  $\mathbb{D}$  for which  $E(\varphi)$  is a finite set (so that  $E(\varphi) = F(\varphi)$ ) and  $\varphi$  has sufficient data at each point of  $F(\varphi)$ . We denote by  $\mathcal{S}(2)$  the set of those  $\varphi$  in  $\mathcal{S}$  which have an order of contact two at each point of  $F(\varphi)$ .

We write  $\mathcal{L}$  for the collection of all non-automorphism linear-fractional self-maps  $\varphi$  of  $\mathbb{D}$  with  $\|\varphi\|_\infty = 1$ . It is obvious that each linear-fractional transformation  $\psi$  is determined by its second-order data  $D_2(\psi, z_0)$  at each point  $z_0$  of analyticity. Now assume that  $\varphi \in \mathcal{S}(2)$  and  $\zeta_0 \in F(\varphi)$ . In [11, p. 2940] Kriete et al. have shown that the unique linear-fractional transformation  $\varphi_0$  with  $D_2(\varphi_0, \zeta_0) = D_2(\varphi, \zeta_0)$  belongs to  $\mathcal{L}$ .

### 3. SOME RESULTS ON ESSENTIAL NORMALITY OF THE OPERATORS $T_w C_\varphi$

The set of all bounded operators and the set of all compact operators from  $H^2$  into itself are denoted by  $B(H^2)$  and  $B_0(H^2)$ , respectively. We will use the notation  $A \equiv B$  to indicate that the difference of two bounded operators  $A$  and  $B$  belongs to  $B_0(H^2)$ . In [10] Kriete et al. have shown that if  $\varphi \in \text{LFT}(\mathbb{D})$  is not an automorphism which satisfies  $\varphi(\zeta) = \eta$  for some  $\zeta, \eta \in \partial\mathbb{D}$ , then

$$(4) \quad C_\varphi^* \equiv |\varphi'(\zeta)|^{-1} C_\sigma.$$

In Theorem 3.1,  $M_w$  denotes the operator on  $L^2 = L^2(\partial\mathbb{D})$  of multiplication by a bounded measurable function  $w$ .

**Theorem 3.1** ([11], Proposition 5.19). *Suppose that  $\varphi \in \mathcal{S}(2)$  with  $F(\varphi) = \{\zeta_1, \dots, \zeta_r\}$ . For  $i = 1, \dots, r$ , let  $\varphi_i$  be the unique linear-fractional transformation with  $D_2(\varphi_i, \zeta_i) = D_2(\varphi, \zeta_i)$ . Also assume that  $w$  is a bounded measurable function on  $\partial\mathbb{D}$  which is continuous at each point of  $F(\varphi)$ . Then*

$$M_w C_\varphi \equiv w(\zeta_1) C_{\varphi_1} + \dots + w(\zeta_r) C_{\varphi_r},$$

where the operators are considered as mapping  $H^2$  to  $L^2$ .

Now we restate Theorem 3.1 in terms of Toeplitz operators.

**Corollary 3.2.** *Suppose that  $\varphi, \varphi_1, \dots, \varphi_r, \zeta_1, \dots, \zeta_r, w$  and  $F(\varphi)$  are as in Theorem 3.1. Then*

$$(5) \quad T_w C_\varphi \equiv w(\zeta_1)C_{\varphi_1} + \dots + w(\zeta_r)C_{\varphi_r},$$

where the operators are considered as mapping  $H^2$  to  $H^2$ .

*Proof.* We know that  $M_w C_\varphi = T_w C_\varphi + H_w C_\varphi$ , where the Hankel operator  $H_w$  is the operator from  $H^2$  into the orthogonal complement of  $H^2$  in  $L^2(\partial\mathbb{D})$  and is defined by  $H_w(g) = (I - P)(wg)$  for each  $g \in H^2$ . By the proof of Corollary 2.2 in [10],  $H_w C_\varphi$  is compact, so the result follows from Theorem 3.1.  $\square$

Let  $\varphi \in \mathcal{S}(2)$  with  $F(\varphi) = \{\zeta_1, \dots, \zeta_r\}$ . For each  $1 \leq i \leq r$ , suppose that  $\sigma_i$  is the Krein adjoint of  $\varphi_i$ , where  $\varphi_i$  is the linear-fractional transformation related to  $\varphi$  and  $\zeta_i$  is as Theorem 3.1. By the preceding corollary

$$(6) \quad (T_w C_\varphi)^* \equiv \overline{w(\zeta_1)}C_{\varphi_1}^* + \dots + \overline{w(\zeta_r)}C_{\varphi_r}^*.$$

Therefore, Corollary 3.2 and Equations (4), (5), and (6) imply that

$$(7) \quad \begin{aligned} (T_w C_\varphi)^* T_w C_\varphi &\equiv (\overline{w(\zeta_1)}C_{\varphi_1}^* + \dots + \overline{w(\zeta_r)}C_{\varphi_r}^*)(w(\zeta_1)C_{\varphi_1} + \dots + w(\zeta_r)C_{\varphi_r}) \\ &\equiv (\overline{w(\zeta_1)}|\varphi'(\zeta_1)|^{-1}C_{\sigma_1} + \dots + \overline{w(\zeta_r)}|\varphi'(\zeta_r)|^{-1}C_{\sigma_r}) \\ &\quad \times (w(\zeta_1)C_{\varphi_1} + \dots + w(\zeta_r)C_{\varphi_r}) \\ &\equiv |w(\zeta_1)|^2|\varphi'(\zeta_1)|^{-1}C_{\varphi_1 \circ \sigma_1} + \dots + |w(\zeta_r)|^2|\varphi'(\zeta_r)|^{-1}C_{\varphi_r \circ \sigma_r}, \end{aligned}$$

where the last equivalence is justified by the fact that  $C_{\varphi_i \circ \sigma_j} \in B_0(H^2)$  for each  $1 \leq i, j \leq r$  and  $i \neq j$ .

**Proposition 3.3.** *Suppose that  $\varphi, \varphi_1, \dots, \varphi_r, \zeta_1, \dots, \zeta_r, w$  and  $F(\varphi)$  are as in Theorem 3.1. If the restriction of  $\varphi$  to  $F(\varphi)$  is a 1-1 function, then*

$$(8) \quad \begin{aligned} [T_w C_\varphi, (T_w C_\varphi)^*] &\equiv |w(\zeta_1)|^2|\varphi'(\zeta_1)|^{-1}(C_{\sigma_1 \circ \varphi_1} - C_{\varphi_1 \circ \sigma_1}) + \dots \\ &\quad + |w(\zeta_r)|^2|\varphi'(\zeta_r)|^{-1}(C_{\sigma_r \circ \varphi_r} - C_{\varphi_r \circ \sigma_r}). \end{aligned}$$

*Proof.* Since the restriction of  $\varphi$  to  $F(\varphi)$  is a 1-1 function,  $C_{\sigma_j \circ \varphi_i} \in B_0(H^2)$  for each  $1 \leq i, j \leq r$  and  $i \neq j$ . Thus, as in the proof of Equation (7), we see that

$$T_w C_\varphi (T_w C_\varphi)^* \equiv |w(\zeta_1)|^2|\varphi'(\zeta_1)|^{-1}C_{\sigma_1 \circ \varphi_1} + \dots + |w(\zeta_r)|^2|\varphi'(\zeta_r)|^{-1}C_{\sigma_r \circ \varphi_r}.$$

The conclusion follows from the above equivalence and Equation (7).  $\square$



We infer from [10, Proposition 3.4] that  $\varphi_i \circ \sigma_i$  and  $\sigma_i \circ \varphi_i$  belong to  $\mathcal{L}$  with the fixed points  $\varphi_i(\zeta_i)$  and  $\zeta_i$ , respectively. Now we present some notation used in [11], then we state a theorem that we will use frequently.

We fix  $\varphi_1, \dots, \varphi_n$  in  $\mathcal{S}$ . Therefore,  $F := F(\varphi_1) \cup \dots \cup F(\varphi_n)$  is a finite set. For  $\zeta \in F$  and  $k = 2, 4, 6, \dots$ , let

$$\mathbb{N}_k(\zeta) := \{j: \zeta \text{ belongs to } F(\varphi_j) \text{ and } \varphi_j \text{ has the order of contact } k \text{ at } \zeta\}.$$

Also we write  $\varepsilon_k(\zeta) := \{D_k(\varphi_j, \zeta): j \in \mathbb{N}_k(\zeta)\}$ .

**Theorem 3.4** ([11], Theorem 5.13). *Suppose that  $\varphi_1, \dots, \varphi_n$  are in  $\mathcal{S}$ . Given complex numbers  $c_1, \dots, c_n$ , the following are equivalent:*

- (i)  $c_1 C_{\varphi_1} + \dots + c_n C_{\varphi_n}$  is compact on  $\mathcal{D}_\beta$ ;
- (ii) for each  $\zeta \in F$ , every even  $k \geq 2$  and every  $d$  in  $\varepsilon_k(\zeta)$ ,

$$\sum_{\substack{j \in \mathbb{N}_k(\zeta) \\ D_k(\varphi_j, \zeta) = d}} c_j = 0.$$

**Proposition 3.5.** *Suppose that  $\varphi, \varphi_1, \dots, \varphi_r, \zeta_1, \dots, \zeta_r, w$  and  $F(\varphi)$  are as in Theorem 3.1. Let the restriction of  $\varphi$  to  $F(\varphi)$  be a 1-1 function. Assume that  $\zeta \in F(\varphi)$  is a fixed point of  $\varphi$  with  $\varphi'(\zeta) \neq 1$ . If  $T_w C_\varphi$  is essentially normal, then  $w(\zeta) = 0$ .*

**Proof.** Without loss of generality, we can assume  $\zeta_1 = \zeta$ . Since the restriction of  $\varphi$  to  $F(\varphi)$  is a 1-1 function, there are only two linear-fractional transformations  $\varphi_1 \circ \sigma_1$  and  $\sigma_1 \circ \varphi_1$  in Equation (8) with the same fixed point at  $\zeta_1$ . By [19, p. 3],  $\varphi_1$  is not a parabolic non-automorphism and Kriete et al. in [10, p. 139] have shown that in this case  $\varphi_1 \circ \sigma_1 \neq \sigma_1 \circ \varphi_1$ . Now apply Theorem 3.4 to  $\zeta = \zeta_1, k = 2$  and  $d = D_2(\varphi_1 \circ \sigma_1, \zeta_1)$ . □

Throughout this paper, let  $\varphi^{[0]}$  be the identity map on  $\mathbb{D}$  and  $\varphi^{[j+1]} := \varphi \circ \varphi^{[j]}$  for each  $j \in \mathbb{N} \cup \{0\}$ . For any  $n \in \mathbb{N}$  and  $\zeta \in F(\varphi)$ , let  $\varphi^{[-n]}(\{\zeta\})$  be the set of all  $z$ , where  $\varphi^{[n]}(z) = \zeta$ . Also, if  $n = 0$ , then  $\varphi^{[-n]}(\{\zeta\}) := \{\zeta\}$ .

**Proposition 3.6.** *Suppose that  $\varphi, \varphi_1, \dots, \varphi_r, \zeta_1, \dots, \zeta_r, w$  and  $F(\varphi)$  are as in Theorem 3.1. Let the restriction of  $\varphi$  to  $F(\varphi)$  be a 1-1 function. Suppose that there are  $\zeta \in F(\varphi)$  and  $\eta \notin F(\varphi)$  with  $\varphi(\zeta) = \eta$ . If  $T_w C_\varphi$  is essentially normal, then  $w(\zeta) = 0$  and, moreover, if for every  $i, 1 \leq i \leq n, \varphi^{[-i]}(\{\zeta\}) \cap F(\varphi) \neq \emptyset$  whenever  $n \in \mathbb{N}$  and  $1 \leq n < r$ , then  $w(z) = 0$  for  $z \in \varphi^{[-i]}(\{\zeta\}) \cap F(\varphi)$ .*

Proof. For convenience, let  $\zeta_1 = \zeta$  and  $\{\zeta_{i+1}\} = \varphi^{[-i]}(\{\zeta_1\}) \cap F(\varphi)$ , where  $0 < i \leq n$ . Since the restriction of  $\varphi$  to  $F(\varphi)$  is a 1-1 function, there is only one linear-fractional transformation  $\varphi_1 \circ \sigma_1$  in Equation (8) which has a finite angular derivative at  $\eta$ . Hence by Theorem 3.4,  $w(\zeta_1) = 0$ , so one has

$$(9) \quad [T_w C_\varphi, (T_w C_\varphi)^*] \equiv |w(\zeta_2)|^2 |\varphi'(\zeta_2)|^{-1} (C_{\sigma_2 \circ \varphi_2} - C_{\varphi_2 \circ \sigma_2}) \\ + \dots + |w(\zeta_r)|^2 |\varphi'(\zeta_r)|^{-1} (C_{\sigma_r \circ \varphi_r} - C_{\varphi_r \circ \sigma_r}) \equiv 0.$$

Since  $\varphi_2 \circ \sigma_2$  is the only linear-fractional transformation in Equation (9) with the fixed point at  $\zeta_1$ , Theorem 3.4 implies that  $w(\zeta_2) = 0$ . Using similar arguments, the result follows.  $\square$

**Proposition 3.7.** *Suppose that  $\varphi, \varphi_1, \dots, \varphi_r, \zeta_1, \dots, \zeta_r, w$  and  $F(\varphi)$  are as in Theorem 3.1. Let the restriction of  $\varphi$  to  $F(\varphi)$  be a 1-1 function. Also assume that there is a smallest integer  $n, 1 < n \leq r$ , such that  $\varphi(\zeta_1) = \zeta_2, \dots, \varphi(\zeta_{n-1}) = \zeta_n$  and  $\varphi(\zeta_n) = \zeta_1$ . If  $T_w C_\varphi$  is essentially normal, then  $\{\varphi_i \circ \sigma_i : 1 \leq i \leq n\} = \{\sigma_i \circ \varphi_i : 1 \leq i \leq n\}$  and for each  $1 \leq i, j \leq n, |w(\zeta_i)|^2 |\varphi'(\zeta_i)|^{-1} = |w(\zeta_j)|^2 |\varphi'(\zeta_j)|^{-1}$  or  $w(\zeta_i) = 0$  for any  $1 \leq i \leq n$ .*

Proof. Without loss of generality, we can assume  $n < r$ . Let  $T_w C_\varphi$  be essentially normal. We infer from Equation (8) that

$$[T_w C_\varphi, (T_w C_\varphi)^*] \\ \equiv (|w(\zeta_1)|^2 |\varphi'(\zeta_1)|^{-1} C_{\sigma_1 \circ \varphi_1} - |w(\zeta_n)|^2 |\varphi'(\zeta_n)|^{-1} C_{\varphi_n \circ \sigma_n}) \\ + (|w(\zeta_2)|^2 |\varphi'(\zeta_2)|^{-1} C_{\sigma_2 \circ \varphi_2} - |w(\zeta_1)|^2 |\varphi'(\zeta_1)|^{-1} C_{\varphi_1 \circ \sigma_1}) \\ + \dots + (|w(\zeta_n)|^2 |\varphi'(\zeta_n)|^{-1} C_{\sigma_n \circ \varphi_n} - |w(\zeta_{n-1})|^2 |\varphi'(\zeta_{n-1})|^{-1} C_{\varphi_{n-1} \circ \sigma_{n-1}}) \\ + |w(\zeta_{n+1})|^2 |\varphi'(\zeta_{n+1})|^{-1} (C_{\sigma_{n+1} \circ \varphi_{n+1}} - C_{\varphi_{n+1} \circ \sigma_{n+1}}) + \dots \\ + |w(\zeta_r)|^2 |\varphi'(\zeta_r)|^{-1} (C_{\sigma_r \circ \varphi_r} - C_{\varphi_r \circ \sigma_r}).$$

It is obvious that  $\varphi_n \circ \sigma_n(\zeta_1) = \sigma_1 \circ \varphi_1(\zeta_1) = \zeta_1, \varphi_1 \circ \sigma_1(\zeta_2) = \sigma_2 \circ \varphi_2(\zeta_2) = \zeta_2, \dots$ , and  $\varphi_{n-1} \circ \sigma_{n-1}(\zeta_n) = \sigma_n \circ \varphi_n(\zeta_n) = \zeta_n$ . Now we define the permutation  $\tau$  on  $\{1, \dots, n\}$  by  $\tau(i) = i - 1$ , when  $1 < i \leq n$  and  $\tau(1) = n$ . If  $\{\varphi_k \circ \sigma_k : 1 \leq k \leq n\} = \{\sigma_k \circ \varphi_k : 1 \leq k \leq n\}$ , then for each  $1 \leq i, j \leq n, |w(\zeta_i)|^2 |\varphi'(\zeta_i)|^{-1} = |w(\zeta_j)|^2 |\varphi'(\zeta_j)|^{-1}$ . This may be seen as follows. Suppose that for some  $1 \leq i, j \leq n, |w(\zeta_i)|^2 |\varphi'(\zeta_i)|^{-1} \neq |w(\zeta_j)|^2 |\varphi'(\zeta_j)|^{-1}$ . Hence there is  $1 \leq j_0 \leq n$ , where  $|w(\zeta_{j_0})|^2 |\varphi'(\zeta_{j_0})|^{-1} \neq |w(\zeta_{\tau(j_0)})|^2 |\varphi'(\zeta_{\tau(j_0)})|^{-1}$ . Since  $\sigma_{j_0} \circ \varphi_{j_0}$  and  $\varphi_{\tau(j_0)} \circ \sigma_{\tau(j_0)}$  are the only two linear-fractional transformations in the above equivalence with the same fixed point at  $\zeta_{j_0}$ , by Theorem 3.4,  $|w(\zeta_{j_0})|^2 |\varphi'(\zeta_{j_0})|^{-1} = |w(\zeta_{\tau(j_0)})|^2 |\varphi'(\zeta_{\tau(j_0)})|^{-1}$ , so it is a contradiction. Let  $w(\zeta_{i_0}) \neq 0$  for some  $1 \leq i_0 \leq n$

and  $\{\varphi_i \circ \sigma_i : 1 \leq i \leq n\} \neq \{\sigma_i \circ \varphi_i : 1 \leq i \leq n\}$ . Then there is  $1 \leq k_0 \leq n$  with  $\sigma_{k_0} \circ \varphi_{k_0} \neq \varphi_{\tau(k_0)} \circ \sigma_{\tau(k_0)}$ . Moreover, as we observed above, there are exactly two linear-fractional transformations  $\sigma_{k_0} \circ \varphi_{k_0}$  and  $\varphi_{\tau(k_0)} \circ \sigma_{\tau(k_0)}$  in the preceding equivalence with the same fixed point at  $\zeta_{k_0}$ . Hence by Theorem 3.4,  $w(\zeta_{k_0}) = w(\zeta_{\tau(k_0)}) = 0$ . Since  $\sigma_{\tau(k_0)} \circ \varphi_{\tau(k_0)}$  and  $\varphi_{\tau^2(k_0)} \circ \sigma_{\tau^2(k_0)}$  are the only two linear-fractional transformations in the preceding equivalence with the same fixed point at  $\zeta_{\tau(k_0)}$  and  $w(\zeta_{\tau(k_0)}) = 0$ , again by Theorem 3.4,  $w(\zeta_{\tau^2(k_0)}) = 0$ . By a similar argument, we see that for each  $1 \leq j \leq n$ ,  $w(\zeta_j) = 0$ , which is a contradiction.  $\square$

For an analytic self-map  $\varphi$  of  $\mathbb{D}$ , let  $\mathbb{P}_\varphi$  denote the set of  $\zeta \in F(\varphi)$ , where  $\varphi(\zeta) = \zeta$  and  $\varphi'(\zeta) = 1$ . It is clear that  $\mathbb{P}_\varphi$  has at most one element (see, e.g., [7, Theorem 2.48]). Let  $\varphi \in \mathcal{S}(2)$  and let  $\varphi_{i_0}$  be the linear-fractional transformation related to  $\varphi$  and  $\zeta_{i_0}$  as in Theorem 3.1 with  $\varphi(\zeta_{i_0}) = \zeta_{i_0}$  and  $\varphi'(\zeta_{i_0}) = 1$ . Hence by Remark 2.6 (a) (i) in [3],  $\varphi_{i_0} \circ \sigma_{i_0} = \sigma_{i_0} \circ \varphi_{i_0}$ , where  $\sigma_{i_0}$  is the Krein adjoint of  $\varphi_{i_0}$ . Therefore, if the restriction of  $\varphi$  to  $F(\varphi)$  is a 1-1 function and  $\mathbb{P}_\varphi$  is a nonempty set, then Equation (8) shows that the member of  $\mathbb{P}_\varphi$  has no effect on essential normality of  $T_w C_\varphi$ .

**Theorem 3.8.** *Suppose that  $\varphi, \varphi_1, \dots, \varphi_r, \zeta_1, \dots, \zeta_r, w$  and  $F(\varphi)$  are as in Theorem 3.1. Let the restriction of  $\varphi$  to  $F(\varphi)$  be a 1-1 function. Then  $T_w C_\varphi$  is essentially normal if and only if for each  $\zeta \in F(\varphi) - \mathbb{P}_\varphi$ ,  $w(\zeta)$  takes one of the following:*

- (i) *If  $\zeta$  is the fixed point of  $\varphi$  and  $\varphi'(\zeta) \neq 1$ , then  $w(\zeta) = 0$ .*
- (ii) *If  $\varphi(\zeta) = \eta$  with  $\eta \notin F(\varphi)$ , then  $w(\zeta) = 0$  and moreover, if for every  $i, 1 \leq i \leq n$ ,  $\varphi^{[-i]}(\{\zeta\}) \cap F(\varphi) \neq \emptyset$  whenever  $n \in \mathbb{N}$  and  $1 \leq n < r$ , then  $w(z) = 0$  for  $z \in \varphi^{[-i]}(\{\zeta\}) \cap F(\varphi)$ .*
- (iii) *Assume that  $w(\zeta)$  is not zero in Statement (i) or (ii), i.e., there is the smallest integer  $n, 1 < n \leq r$ , such that  $\varphi^{[n]}(\zeta) = \zeta$ . For convenience, let  $h_1 = \zeta, h_2 = \varphi(\zeta), \dots, h_n = \varphi^{[n-1]}(\zeta)$ . For each  $1 \leq i \leq n$ , let  $\phi_i$  be the linear-fractional transformation related to  $\varphi$  and  $h_i$  as in Theorem 3.1; also  $\varsigma_i$  be the Krein adjoint of  $\phi_i$ . Then  $\{\phi_i \circ \varsigma_i : 1 \leq i \leq n\} = \{\varsigma_i \circ \phi_i : 1 \leq i \leq n\}$  and for each  $1 \leq i, j \leq n$ ,  $|w(h_i)|^2 |\varphi'(h_i)|^{-1} = |w(h_j)|^2 |\varphi'(h_j)|^{-1}$  or  $w(h_i) = 0$  for any  $1 \leq i \leq n$ .*

*Proof.* Let  $T_w C_\varphi$  be essentially normal. Then by Propositions 3.5 and 3.6, Statements (i) and (ii) hold. Suppose that we cannot obtain the value of  $w(\zeta)$  from Statement (i) or (ii). Since the restriction of  $\varphi$  to  $F(\varphi)$  is a 1-1 function and  $F(\varphi)$  is a finite set, there is a smallest integer  $n, 1 < n \leq r$ , such that  $\varphi^{[n]}(\zeta) = \zeta$ , so by Proposition 3.7, the proof is complete.

Conversely, without loss of generality we can assume that  $\zeta_r \in \mathbb{P}_\varphi$ , there is a smallest natural number  $n, 1 < n < r$ , with  $\varphi(\zeta_1) = \zeta_2, \dots, \varphi(\zeta_{n-1}) = \zeta_n, \varphi(\zeta_n) = \zeta_1$

and for each  $i > n$  and  $i \neq r$ ,  $w(\zeta_i) = 0$ . Thus, Equation (8) implies that

$$\begin{aligned} & [T_w C_\varphi, (T_w C_\varphi)^*] \\ & \equiv (|w(\zeta_1)|^2 |\varphi'(\zeta_1)|^{-1} C_{\sigma_1 \circ \varphi_1} - |w(\zeta_n)|^2 |\varphi'(\zeta_n)|^{-1} C_{\varphi_n \circ \sigma_n}) \\ & \quad + \dots + (|w(\zeta_n)|^2 |\varphi'(\zeta_n)|^{-1} C_{\sigma_n \circ \varphi_n} - |w(\zeta_{n-1})|^2 |\varphi'(\zeta_{n-1})|^{-1} C_{\varphi_{n-1} \circ \sigma_{n-1}}) \\ & \quad + |w(\zeta_r)|^2 |\varphi'(\zeta_r)|^{-1} (C_{\sigma_r \circ \varphi_r} - C_{\varphi_r \circ \sigma_r}). \end{aligned}$$

As we observed before,  $\zeta_r$  has no effect on the essential normality of  $T_w C_\varphi$ . Hence by Theorem 3.4,  $T_w C_\varphi$  is essentially normal.  $\square$

Now for  $\varphi \in \mathcal{S}(2)$ , suppose that the restriction of  $\varphi$  to  $F(\varphi)$  is not a 1-1 function. Let

$$(10) \quad F(\varphi) = \{\zeta_{r_0}, \zeta_{r_0+1}, \dots, \zeta_{r_1-1}, \zeta_{r_1}, \zeta_{r_1+1}, \dots, \zeta_{r_{n-1}-1}, \zeta_{r_{n-1}}, \zeta_{r_{n-1}+1}, \dots, \zeta_{r_n-1}, \zeta_{r_n}, \zeta_{r_n+1}, \dots, \zeta_{r_{n+k}}\}$$

for some  $n, k \in \mathbb{N} \cup \{0\}$  such that

$$(11) \quad \begin{aligned} \varphi(\zeta_{r_0}) &= \varphi(\zeta_{r_0+1}) = \dots = \varphi(\zeta_{r_1-1}), \dots, \varphi(\zeta_{r_{n-1}}) \\ &= \varphi(\zeta_{r_{n-1}+1}) = \dots = \varphi(\zeta_{r_n-1}) \end{aligned}$$

and let the restriction of  $\varphi$  to  $\{\zeta_{r_0}, \zeta_{r_1}, \dots, \zeta_{r_{n-1}}, \zeta_{r_n}, \zeta_{r_{n+1}}, \dots, \zeta_{r_{n+k}}\}$  be a 1-1 function. From now on, unless otherwise stated, let  $\mathbb{A}_i = \{\zeta_{r_i}, \zeta_{r_i+1}, \dots, \zeta_{r_{i+1}-1}\}$  and  $\zeta_{r_{i+1}-1} = \zeta_{r_i+|\mathbb{A}_i|-1}$  for each  $0 \leq i \leq n+k$ ; furthermore, suppose that  $\varphi_{r_i+h}$  is the linear-fractional transformation related to  $\varphi$  and  $\zeta_{r_i+h}$  as in Theorem 3.1, where  $\zeta_{r_i+h} \in F(\varphi)$ ; also assume that  $\sigma_{r_i+h}$  is the Krein adjoint of  $\varphi_{r_i+h}$ . Let  $\zeta_{r_i+h}, \zeta_{r_j+t} \in F(\varphi)$ . It is obvious that  $C_{\varphi_{r_i+h} \circ \sigma_{r_j+t}} \notin B_0(H^2)$  if and only if  $i = j$  and  $t = h$ . Also,  $C_{\sigma_{r_j+t} \circ \varphi_{r_i+h}} \notin B_0(H^2)$  if and only if  $\varphi(\zeta_{r_i+h}) = \varphi(\zeta_{r_j+t})$ . Therefore, by these facts, Equation (4) and after some patient calculations, one obtains

$$(12) \quad \begin{aligned} [T_w C_\varphi, (T_w C_\varphi)^*] & \equiv |w(\zeta_{r_0})|^2 |\varphi'(\zeta_{r_0})|^{-1} C_{\sigma_{r_0} \circ \varphi_{r_0}} \\ & \quad + w(\zeta_{r_0}) \overline{w(\zeta_{r_0+1})} |\varphi'(\zeta_{r_0+1})|^{-1} C_{\sigma_{r_0+1} \circ \varphi_{r_0}} + \dots \\ & \quad + |w(\zeta_{r_1-1})|^2 |\varphi'(\zeta_{r_1-1})|^{-1} C_{\sigma_{r_1-1} \circ \varphi_{r_1-1}} + \dots \\ & \quad + |w(\zeta_{r_n-1})|^2 |\varphi'(\zeta_{r_n-1})|^{-1} C_{\sigma_{r_n-1} \circ \varphi_{r_n-1}} + \dots \\ & \quad + |w(\zeta_{r_n})|^2 |\varphi'(\zeta_{r_n})|^{-1} C_{\sigma_{r_n} \circ \varphi_{r_n}} + \dots \\ & \quad + |w(\zeta_{r_{n+k}})|^2 |\varphi'(\zeta_{r_{n+k}})|^{-1} C_{\sigma_{r_{n+k}} \circ \varphi_{r_{n+k}}} \end{aligned}$$

$$\begin{aligned}
& - (|w(\zeta_{r_0})|^2 |\varphi'(\zeta_{r_0})|^{-1} C_{\varphi_{r_0} \circ \sigma_{r_0}} \\
& + |w(\zeta_{r_0+1})|^2 |\varphi'(\zeta_{r_0+1})|^{-1} C_{\varphi_{r_0+1} \circ \sigma_{r_0+1}} + \dots \\
& + |w(\zeta_{r_n-1})|^2 |\varphi'(\zeta_{r_n-1})|^{-1} C_{\varphi_{r_n-1} \circ \sigma_{r_n-1}} \\
& + |w(\zeta_{r_n})|^2 |\varphi'(\zeta_{r_n})|^{-1} C_{\varphi_{r_n} \circ \sigma_{r_n}} + \dots \\
& + |w(\zeta_{r_n+k})|^2 |\varphi'(\zeta_{r_n+k})|^{-1} C_{\varphi_{r_n+k} \circ \sigma_{r_n+k}}).
\end{aligned}$$

**Proposition 3.9.** *Suppose that  $\varphi$  and  $w$  are as in Theorem 3.1 and  $F(\varphi)$  is as in Equation (10). In accordance with Equation (11), for each  $0 \leq i < n$  we assume that  $\varphi(\zeta_{r_i}) = \varphi(\zeta_{r_{i+1}}) = \dots = \varphi(\zeta_{r_{i+1}-1})$ . If  $T_w C_\varphi$  is essentially normal, then the values of  $w(\zeta_{r_i}), \dots, w(\zeta_{r_{i+1}-1})$  are all zero except at most one of them.*

*Proof.* Without loss of generality, we can assume that  $w(\zeta_{r_i}) \neq 0$  and  $w(\zeta_{r_{i+1}}) \neq 0$ . Let  $B = \{\sigma_{r_i} \circ \varphi_{r_i}, \sigma_{r_{i+1}} \circ \varphi_{r_i}, \dots, \sigma_{r_{i+1}-1} \circ \varphi_{r_i}\}$ . Every linear-fractional transformation in Equation (12) which has a finite angular derivative at  $\zeta_{r_i}$  belongs to  $B$  or

$$\{\varphi_{r_j+h} \circ \sigma_{r_j+h} : 0 \leq j \leq n+k, 0 \leq h \leq |\mathbb{A}_j| - 1 \text{ and } \varphi_{r_j+h}(\zeta_{r_j+h}) = \zeta_{r_i}\}.$$

Now apply Theorem 3.4 to  $k = 2$  and  $d = D_2(\sigma_{r_{i+1}} \circ \varphi_{r_i}, \zeta_{r_i})$ ; hence  $w(\zeta_{r_i})w(\zeta_{r_{i+1}}) = 0$ , which is a contradiction.  $\square$

By the preceding proposition and Equation (12), we can assume that

$$\begin{aligned}
& [T_w C_\varphi, (T_w C_\varphi)^*] \\
& \equiv |w(\zeta_{r_0})|^2 |\varphi'(\zeta_{r_0})|^{-1} (C_{\sigma_{r_0} \circ \varphi_{r_0}} - C_{\varphi_{r_0} \circ \sigma_{r_0}}) + \dots \\
& + |w(\zeta_{r_{n-1}})|^2 |\varphi'(\zeta_{r_{n-1}})|^{-1} (C_{\sigma_{r_{n-1}} \circ \varphi_{r_{n-1}}} - C_{\varphi_{r_{n-1}} \circ \sigma_{r_{n-1}}}) \\
& + |w(\zeta_{r_n})|^2 |\varphi'(\zeta_{r_n})|^{-1} (C_{\sigma_{r_n} \circ \varphi_{r_n}} - C_{\varphi_{r_n} \circ \sigma_{r_n}}) + \dots \\
& + |w(\zeta_{r_{n+k}})|^2 |\varphi'(\zeta_{r_{n+k}})|^{-1} (C_{\sigma_{r_{n+k}} \circ \varphi_{r_{n+k}}} - C_{\varphi_{r_{n+k}} \circ \sigma_{r_{n+k}}}).
\end{aligned}$$

In the next theorem  $\varphi$  and  $w$  are as in Theorem 3.1 and  $F(\varphi)$  is as in Equation (10). In accordance with Equation (11), for each  $0 \leq i < n$  we assume that  $\varphi(\zeta_{r_i}) = \varphi(\zeta_{r_{i+1}}) = \dots = \varphi(\zeta_{r_{i+1}-1})$ . Furthermore,  $G(\varphi)$  in Statements (iii) and (iv) of the theorem is

$$G(\varphi) := \{\zeta : \zeta \in F(\varphi) \text{ and } w(\zeta) \text{ is not zero in Statement (i)}\}.$$

**Theorem 3.10.** *The operator  $T_w C_\varphi$  is essentially normal if and only if for each  $\zeta \in F(\varphi) - \mathbb{P}_\varphi$ ,  $w(\zeta)$  satisfies one of the following conditions:*

- (i) *For each  $0 \leq i < n$ , the values of  $w(\zeta_{r_i}), \dots, w(\zeta_{r_{i+1}-1})$  are all zero except at most one of them.*

- (ii) If  $\zeta$  is the fixed point of  $\varphi$  and  $\varphi'(\zeta) \neq 1$ , then  $w(\zeta) = 0$ .
- (iii) If  $\varphi(\zeta) = \eta$  for  $\eta \notin G(\varphi)$ , then  $w(\zeta) = 0$  and moreover, if for every  $j$ ,  $1 \leq j \leq m$ ,  $\varphi^{[-j]}(\{\zeta\}) \cap G(\varphi) \neq \emptyset$  whenever  $m \in \mathbb{N}$  and  $1 \leq m < |G(\varphi)|$ , then  $w(z) = 0$  for  $z \in \varphi^{[-j]}(\{\zeta\}) \cap G(\varphi)$ .
- (iv) Suppose that  $w(\zeta)$  is not zero in Statement (i) or (ii) or (iii), i.e., there is a smallest integer  $n_0$ ,  $1 < n_0 \leq |G(\varphi)|$ , such that  $\varphi^{[n_0]}(\zeta) = \zeta$ . For convenience, assume that  $h_1 = \zeta, h_2 = \varphi(\zeta), \dots, h_{n_0} = \varphi^{[n_0-1]}(\zeta)$ . For each  $1 \leq i \leq n_0$ , let  $\phi_i$  be the linear-fractional transformation related to  $\varphi$  and  $h_i$  be as in Theorem 3.1; let  $\varsigma_i$  be the Krein adjoint of  $\phi_i$ . Then  $\{\phi_i \circ \varsigma_i : 1 \leq i \leq n_0\} = \{\varsigma_i \circ \phi_i : 1 \leq i \leq n_0\}$  and for every  $1 \leq i, j \leq n_0$ ,  $|w(h_i)|^2 |\varphi'(h_i)|^{-1} = |w(h_j)|^2 |\varphi'(h_j)|^{-1}$  or  $w(h_i) = 0$  for any  $1 \leq i \leq n_0$ .

**Proof.** Let  $T_w C_\varphi$  be essentially normal. Without loss of generality, by Proposition 3.9 we can assume that  $w(\zeta_{r_i+h}) = 0$  when  $h \neq 0$  and  $0 \leq i \leq n-1$ . Thus,  $G(\varphi) \subseteq \{\zeta_{r_0}, \zeta_{r_1}, \dots, \zeta_{r_{n-1}}, \zeta_{r_n}, \dots, \zeta_{r_{n+k}}\}$ . Since the restriction of  $\varphi$  to  $G(\varphi)$  is a 1-1 function, Theorem 3.8 gives the desired conclusion.

Conversely, the conclusion follows from Theorem 3.8. □

For each  $\varphi_i \in \mathcal{L}$ , let  $\sigma_i$  be the Krein adjoint of  $\varphi_i$  and let  $\zeta_i \in F(\varphi_i)$ . In the remainder of this section, we investigate the essential normality problem for certain finite linear combinations of linear-fractional composition operators.

**Proposition 3.11.** *Suppose that  $r, n \in \mathbb{N}$ ,  $1 \leq r \leq n$ , and  $c_1, \dots, c_n \in \mathbb{C}$ . Assume that  $\varphi_1, \dots, \varphi_n \in \mathcal{L}$  are pairwise distinct. Let  $F(\varphi_i) = \{\zeta_i\}$  and  $\zeta \in \bigcap_{i=1}^r F(\varphi_i) - \bigcup_{i=r+1}^n F(\varphi_i)$ . Also for each  $1 \leq j \leq r$ , let  $\varphi_j(\zeta) \notin \{\varphi_i(\zeta) : 1 \leq i \leq r \text{ and } i \neq j\}$ . Furthermore, assume there is at most one integer  $i_0 \in \{1, \dots, r\}$  such that  $\varphi_{i_0}(\zeta) \in \bigcup_{i=1}^n F(\varphi_i)$ . If  $c_1 C_{\varphi_1} + \dots + c_n C_{\varphi_n}$  is essentially normal, then the values of  $c_1, \dots, c_r$  are all zero except at most  $c_{i_0}$ .*

**Proof.** We infer from Equation (4) that

$$\begin{aligned}
 (13) \quad & [c_1 C_{\varphi_1} + \dots + c_n C_{\varphi_n}, (c_1 C_{\varphi_1} + \dots + c_n C_{\varphi_n})^*] \\
 & \equiv \sum_{\varphi_j(\zeta_j) = \varphi_i(\zeta_i)} c_i \bar{c}_j |\varphi'_j(\zeta_j)|^{-1} C_{\sigma_j \circ \varphi_i} - \sum_{\zeta_j = \zeta_i} c_i \bar{c}_j |\varphi'_j(\zeta_j)|^{-1} C_{\varphi_i \circ \sigma_j} \\
 & \equiv \sum_{\varphi_j(\zeta_j) = \varphi_i(\zeta_i)} c_i \bar{c}_j |\varphi'_j(\zeta_j)|^{-1} C_{\sigma_j \circ \varphi_i} - \sum_{1 \leq i, j \leq r} c_i \bar{c}_j |\varphi'_j(\zeta_j)|^{-1} C_{\varphi_i \circ \sigma_j} \\
 & \quad - \sum_{\substack{\zeta_j = \zeta_i \\ i, j > r}} c_i \bar{c}_j |\varphi'_j(\zeta_j)|^{-1} C_{\varphi_i \circ \sigma_j}.
 \end{aligned}$$

For  $j_0 \neq i_0$  and  $1 \leq j_0 \leq r$ , let  $B = \{\varphi_{j_0} \circ \sigma_{j_0}\} \cup \{\varphi_i \circ \sigma_i: r < i \text{ and } \varphi_i(\zeta_i) = \varphi_{j_0}(\zeta_{j_0})\}$ . It is clear that every linear-fractional transformation in the above equivalence which sends  $\varphi_{j_0}(\zeta_{j_0})$  to  $\varphi_{j_0}(\zeta_{j_0})$  belongs to  $B$ . Now apply Theorem 3.4 to  $k = 2$  and  $d = D_2(\varphi_{j_0} \circ \sigma_{j_0}, \varphi_{j_0}(\zeta_{j_0}))$ ; hence there is a finite set  $I$ ,  $I \subseteq \{i: i > r \text{ and } \varphi_i(\zeta_i) = \varphi_{j_0}(\zeta_{j_0})\}$ , such that

$$|c_{j_0}|^2 |\varphi'_{j_0}(\zeta_{j_0})|^{-1} + \sum_{i \in I} |c_i|^2 |\varphi'_i(\zeta_i)|^{-1} = 0.$$

Hence  $c_{j_0} = 0$ , as desired.  $\square$

Let  $n \in \mathbb{N}$ . In the next theorem for each  $1 \leq i \leq n$ ,  $c_i$ ,  $\varphi_i$ ,  $\zeta_i$  and  $F(\varphi_i)$  are as in Proposition 3.11 and  $F := \bigcup_{i=1}^n F(\varphi_i)$ . Also, if for some subset  $\{i_1, \dots, i_m\} \subseteq \{1, \dots, n\}$ ,

$$(14) \quad \bigcap_{l=1}^m F(\varphi_{i_l}) - \bigcup_{\substack{i \neq i_l \\ 1 \leq l \leq m}} F(\varphi_i) \neq \emptyset,$$

then for each  $1 \leq l \leq m$ ,  $\varphi_{i_l}(\zeta_{i_l}) \notin \{\varphi_{i_j}(\zeta_{i_j}): 1 \leq j \leq m \text{ and } j \neq l\}$ ; moreover, there is at most one integer  $j_0 \in \{1, \dots, m\}$  such that  $\varphi_{i_{j_0}}(\zeta_{i_{j_0}}) \in F$ . Furthermore,  $G$  in Statement (iii) of the theorem is

$$G := \{\zeta: \zeta \in F(\varphi_i) \text{ and } c_i \text{ is not zero in Statement (i) for some } 1 \leq i \leq n\}.$$

**Theorem 3.12.** *The operator  $c_1 C_{\varphi_1} + \dots + c_n C_{\varphi_n}$  is essentially normal if and only if for each  $1 \leq j \leq n$  when  $\zeta_j \notin \mathbb{P}_{\varphi_j}$ ,  $c_j$  satisfies one of the following conditions:*

- (i) *Suppose that  $\varphi_{r_1}(\zeta_{r_1}) = \dots = \varphi_{r_k}(\zeta_{r_k})$  for  $1 \leq r_1, \dots, r_k \leq n$  and  $\varphi_i(\zeta_i) \neq \varphi_{r_1}(\zeta_{r_1})$  when  $1 \leq i \leq n$  and  $i \notin \{r_1, \dots, r_k\}$ . Then the values of  $c_{r_1}, \dots, c_{r_k}$  are all zero except at most one of them.*
- (ii) *If  $\zeta_i$  is the fixed point of  $\varphi_i$  and  $\varphi'_i(\zeta_i) \neq 1$ , then  $c_i = 0$ .*
- (iii) *If  $\varphi_r(\zeta_r) \notin G$  when  $1 \leq r \leq n$ , then  $c_r = 0$  and moreover, if for each  $j$ ,  $1 \leq j \leq k$ ,  $\varphi_{r_1}^{-1} \circ \dots \circ \varphi_{r_j}^{-1}(\{\zeta_r\}) \cap G \neq \emptyset$  whenever  $k \in \mathbb{N}$  and  $1 \leq r_1, \dots, r_k \leq n$ , then  $c_{r_1} = \dots = c_{r_k} = 0$ .*
- (iv) *Assume that  $c_i$  is not zero in the preceding statements, i.e., there are distinct integers  $1 \leq r_1, \dots, r_k \leq n$  such that  $\{\zeta_i, \zeta_{r_1}, \dots, \zeta_{r_k}\} \subseteq G$  and  $\varphi_{r_1} \circ \dots \circ \varphi_{r_k} \circ \varphi_i(\zeta_i) = \zeta_i$ . Let  $B = \{i, r_1, \dots, r_k\}$ . Then  $\{\varphi_j \circ \sigma_j: j \in B\} = \{\sigma_j \circ \varphi_j: j \in B\}$  and for every  $j, h \in B$ ,  $|c_j|^2 |\varphi'_j(\zeta_j)|^{-1} = |c_h|^2 |\varphi'_h(\zeta_h)|^{-1}$ , or for each  $j \in B$ ,  $c_j = 0$ .*

Proof. Let  $c_1C_{\varphi_1} + \dots + c_nC_{\varphi_n}$  be essentially normal. Without loss of generality, by Proposition 3.11 and Equation (13), we can assume that there exists an integer  $m$ ,  $1 \leq m \leq n$ , such that for all distinct integers  $1 \leq i, j \leq m$ ,  $F(\varphi_i) \cap F(\varphi_j) = \emptyset$  and

$$\begin{aligned} & [c_1C_{\varphi_1} + \dots + c_nC_{\varphi_n}, (c_1C_{\varphi_1} + \dots + c_nC_{\varphi_n})^*] \\ & \equiv \sum_{\substack{\varphi_j(\zeta_j) = \varphi_i(\zeta_i) \\ 1 \leq i, j \leq m}} c_i \overline{c_j} |\varphi'_j(\zeta_j)|^{-1} C_{\sigma_j \circ \varphi_i} - \sum_{i=1}^m |c_i|^2 |\varphi'_i(\zeta_i)|^{-1} C_{\varphi_i \circ \sigma_i}. \end{aligned}$$

Now let  $A = \{\zeta_i : 1 \leq i \leq m\}$ . We can rewrite

$$\begin{aligned} A = \{ & \zeta_{r_0}, \zeta_{r_0+1}, \dots, \zeta_{r_1-1}, \zeta_{r_1}, \zeta_{r_1+1}, \dots, \zeta_{r_{p-1}-1}, \zeta_{r_{p-1}}, \\ & \zeta_{r_{p-1}+1}, \dots, \zeta_{r_p-1}, \zeta_{r_p}, \zeta_{r_{p+1}}, \dots, \zeta_{r_{p+k}} \} \end{aligned}$$

for some  $p, k \in \mathbb{N} \cup \{0\}$  such that

$$\varphi(\zeta_{r_0}) = \varphi(\zeta_{r_0+1}) = \dots = \varphi(\zeta_{r_1-1}), \dots, \varphi(\zeta_{r_{p-1}}) = \varphi(\zeta_{r_{p-1}+1}) = \dots = \varphi(\zeta_{r_p-1})$$

and for each integer  $i$ ,  $0 \leq i \leq k$ , the value of  $\varphi(\zeta_{r_{i+p}})$  is not equal to  $\varphi(\zeta)$  for each  $\zeta \in A - \{\zeta_{r_{i+p}}\}$ . Also, there exists an integer  $t$ ,  $0 \leq t \leq k$ , such that  $\varphi_{r_{i+p}}(\zeta_{r_{i+p}}) = \zeta_{r_{i+p}}$  and  $\varphi'_{r_{i+p}}(\zeta_{r_{i+p}}) = 1$  for any  $t \leq i \leq k$ . As we observed before, for any  $i$ ,  $t \leq i \leq k$ ,  $\varphi_{r_{i+p}} \circ \sigma_{r_{i+p}} = \sigma_{r_{i+p}} \circ \varphi_{r_{i+p}}$ ; hence  $\zeta_{r_{i+p}}$  has no effect on the essential normality of  $c_1C_{\varphi_1} + \dots + c_nC_{\varphi_n}$ . Therefore, we can see that

$$\begin{aligned} & [c_1C_{\varphi_1} + \dots + c_nC_{\varphi_n}, (c_1C_{\varphi_1} + \dots + c_nC_{\varphi_n})^*] \\ & \equiv |c_{r_0}|^2 |\varphi'_{r_0}(\zeta_{r_0})|^{-1} C_{\sigma_{r_0} \circ \varphi_{r_0}} + c_{r_0} \overline{c_{r_0+1}} |\varphi'_{r_0+1}(\zeta_{r_0+1})|^{-1} C_{\sigma_{r_0+1} \circ \varphi_{r_0}} + \dots \\ & \quad + |c_{r_1-1}|^2 |\varphi'_{r_1-1}(\zeta_{r_1-1})|^{-1} C_{\sigma_{r_1-1} \circ \varphi_{r_1-1}} + \dots \\ & \quad + |c_{r_p-1}|^2 |\varphi'_{r_p-1}(\zeta_{r_p-1})|^{-1} C_{\sigma_{r_p-1} \circ \varphi_{r_p-1}} + \dots \\ & \quad + |c_{r_p}|^2 |\varphi'_{r_p}(\zeta_{r_p})|^{-1} C_{\sigma_{r_p} \circ \varphi_{r_p}} + \dots + |c_{r_{p+t}}|^2 |\varphi'_{r_{p+t}}(\zeta_{r_{p+t}})|^{-1} C_{\sigma_{r_{p+t}} \circ \varphi_{r_{p+t}}} \\ & \quad - (|c_{r_0}|^2 |\varphi'_{r_0}(\zeta_{r_0})|^{-1} C_{\varphi_{r_0} \circ \sigma_{r_0}} + |c_{r_0+1}|^2 |\varphi'_{r_0+1}(\zeta_{r_0+1})|^{-1} C_{\varphi_{r_0+1} \circ \sigma_{r_0+1}} + \dots \\ & \quad + |c_{r_p-1}|^2 |\varphi'_{r_p-1}(\zeta_{r_p-1})|^{-1} C_{\varphi_{r_p-1} \circ \sigma_{r_p-1}} + |c_{r_p}|^2 |\varphi'_{r_p}(\zeta_{r_p})|^{-1} C_{\varphi_{r_p} \circ \sigma_{r_p}} + \dots \\ & \quad + |c_{r_{p+t}}|^2 |\varphi'_{r_{p+t}}(\zeta_{r_{p+t}})|^{-1} C_{\varphi_{r_{p+t}} \circ \sigma_{r_{p+t}}}). \end{aligned}$$

The above equivalence is like Equation (12), so the result follows from a proof similar to that of Theorem 3.10.

Conversely, suppose that for some subset  $\{i_1, \dots, i_m\} \subseteq \{1, \dots, n\}$ , Equation (14) holds. By the hypothesis, there is at most one integer  $j_0$ ,  $1 \leq j_0 \leq m$ , such that  $\varphi_{i_{j_0}}(\zeta_{i_{j_0}}) \in F$ . Since  $G \subseteq F$ , Statement (iii) implies that the values of  $c_{i_1}, \dots, c_{i_m}$



are all zero except at most  $c_{i_{j_0}}$ . Hence without loss of generality we can assume that there is a smallest natural number  $k$ ,  $1 < k < n$ , with  $\varphi_1(\zeta_1) = \zeta_2, \dots, \varphi_{k-1}(\zeta_{k-1}) = \zeta_k$  and  $\varphi_k(\zeta_k) = \zeta_1$ , and for each integer  $i$ ,  $k+1 < i < n$ ,  $c_i = 0$ ; moreover,  $\varphi_{k+1}(\zeta_{k+1}) = \zeta_{k+1}$  and  $\varphi'_{k+1}(\zeta_{k+1}) = 1$ . Thus, Equation (13) implies that

$$\begin{aligned} & [c_1 C_{\varphi_1} + \dots + c_n C_{\varphi_n}, (c_1 C_{\varphi_1} + \dots + c_n C_{\varphi_n})^*] \\ & \equiv \sum_{i=1}^{k+1} |c_i|^2 |\varphi'_i(\zeta_i)|^{-1} C_{\sigma_i \circ \varphi_i} - \sum_{i=1}^{k+1} |c_i|^2 |\varphi'_i(\zeta_i)|^{-1} C_{\varphi_i \circ \sigma_i} \\ & \equiv (|c_1|^2 |\varphi'_1(\zeta_1)|^{-1} C_{\sigma_1 \circ \varphi_1} - |c_k|^2 |\varphi'_k(\zeta_k)|^{-1} C_{\varphi_k \circ \sigma_k}) + \dots \\ & \quad + (|c_k|^2 |\varphi'_k(\zeta_k)|^{-1} C_{\sigma_k \circ \varphi_k} - |c_{k-1}|^2 |\varphi'_{k-1}(\zeta_{k-1})|^{-1} C_{\varphi_{k-1} \circ \sigma_{k-1}}) \\ & \quad + |c_{k+1}|^2 |\varphi'_{k+1}(\zeta_{k+1})|^{-1} (C_{\sigma_{k+1} \circ \varphi_{k+1}} - C_{\varphi_{k+1} \circ \sigma_{k+1}}). \end{aligned}$$

As we mentioned before,  $\zeta_{k+1}$  has no effect on the essential normality of  $c_1 C_{\varphi_1} + \dots + c_n C_{\varphi_n}$ . Hence by Theorem 3.4,  $c_1 C_{\varphi_1} + \dots + c_n C_{\varphi_n}$  is essentially normal.  $\square$

In the following remark, we compare the results which were obtained in [3] with Theorem 3.12 when  $n = 1$ .

**Remark 3.13.** Suppose that  $\varphi \in \text{LFT}(\mathbb{D})$  is not an automorphism and that  $\varphi(\zeta) = \eta$  for some  $\zeta, \eta \in \partial\mathbb{D}$ . Then  $F(\varphi) = \{\zeta\}$  and we have:

- (a) If  $\zeta \neq \eta$ , then by Theorem 3.12,  $C_\varphi$  is not essentially normal (see [3, Theorem 6.1]).
- (b) If  $\zeta = \eta$  and  $\varphi'(\zeta) \neq 1$ , then Theorem 3.12 implies that  $C_\varphi$  is not essentially normal (see [3, Theorem 5.2]).
- (c) If  $\zeta = \eta$  and  $\varphi'(\zeta) = 1$ , then  $\varphi$  is parabolic. We infer from Theorem 3.12 that  $C_\varphi$  is essentially normal (see [3, Theorem 4.1]).

**Remark 3.14.** For  $1 \leq i \leq n$ , let  $\varphi_i$  be a non-automorphism linear-fractional self-map of  $\mathbb{D}$  and  $B = \{i: 1 \leq i \leq n \text{ and } \|\varphi_i\|_\infty = 1\}$ . Assume that for each  $i \in B$ ,  $\varphi_i, \zeta_i$  and  $F(\varphi_i)$  satisfy the hypotheses of Theorem 3.12. Let for any  $i \in B$ ,  $w_i$  be a bounded measurable function on  $\partial\mathbb{D}$  which is continuous at  $\zeta_i$ . Suppose that for  $i \notin B$ ,  $w_i \in L^\infty(\partial\mathbb{D})$ . We know that if  $\|\varphi\|_\infty < 1$ , then  $C_\varphi$  is compact. Therefore, for  $c_1, \dots, c_n \in \mathbb{C}$ , Corollary 2.2 in [10] implies that

$$c_1 T_{w_1} C_{\varphi_1} + \dots + c_n T_{w_n} C_{\varphi_n} \equiv \sum_{i \in B} c_i w_i(\zeta_i) C_{\varphi_i}.$$

Hence by Theorem 3.12 we can characterize the essentially normal finite linear combinations of these operators on  $H^2$ .

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