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SMOOTHNESS FOR THE COLLISION LOCAL TIME OF TWO MULTIDIMENSIONAL BIFRACTIONAL BROWNIAN MOTIONS

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Abstract. Let $B^{H_i,K_i} = \{B_t^{H_i,K_i},\ t \geqslant 0\}$, $i = 1,2$ be two independent, d-dimensional bifractional Brownian motions with respective indices $H_i \in (0,1)$ and $K_i \in (0,1]$. Assume $d \geqslant 2$. One of the main motivations of this paper is to investigate smoothness of the collision local time

$$l_T = \int_0^T \delta(B_s^{H_1,K_1} - B_s^{H_2,K_2})\,\mathrm{d}s, \qquad T > 0,$$

where δ denotes the Dirac delta function. By an elementary method we show that l_T is smooth in the sense of Meyer-Watanabe if and only if $\min\{H_1K_1, H_2K_2\} < 1/(d+2)$.

Keywords: bifractional Brownian motion, collision local time, intersection local time, chaos expansion

MSC 2010: 60G15, 60G18, 60J55

1. Introduction

We consider two independent bifractional Brownian motions B^{H_1,K_1} and B^{H_2,K_2} on $\mathbb{R}^d, d \geqslant 2$, with respective indices $H_i \in (0,1)$ and $K_i \in (0,1]$, $i = 1,2$. This means that we have two d-dimensional independent centered Gaussian processes $B^{H_1,K_1} = \{B_t^{H_1,K_1}, t \geqslant 0\}$ and $B^{H_2,K_2} = \{B_t^{H_2,K_2}, t \geqslant 0\}$ with covariance structure given by

$$E[B_t^{H_1,K_1,i} B_s^{H_1,K_1,j}] = \delta_{ij} R^{H_1,K_1}(t,s);$$
$$E[B_t^{H_2,K_2,i} B_s^{H_2,K_2,j}] = \delta_{ij} R^{H_2,K_2}(t,s)$$

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where $i, j = 1, \ldots, d$, $s, t \geqslant 0$ and

$$R^{H_l,K_l}(t,s) = \frac{1}{2^{K_l}}[(t^{2H_l} + s^{2H_l})^{K_l} - |t-s|^{2H_lK_l}], \quad l = 1, 2.$$

Bifractional Brownian motion is H_lK_l-self similar, and satisfies the estimates (see Houdré-Villa [4])

$$2^{-K_l}|t-s|^{2H_lK_l} \leqslant E[(B_t^{H_l,K_l} - B_s^{H_l,K_l})^2] \leqslant 2^{1-K_l}|t-s|^{2H_lK_l}. \tag{1.1}$$

Thus, Kolmogorov's continuity criterion implies that the bifractional Brownian motion is Hölder continuous of order δ strictly less than H_lK_l. This process was first introduced by Houdré-Villa [4]. B^{H_l,K_l} is neither a Markov process nor a semimartingale unless $H_l = \frac{1}{2}$ and $K_l = 1$. So many of the powerful techniques from stochastic analysis are not available when dealing with B^{H_l,K_l}. More works on bifractional Brownian motion can be found in Es-sebaiy-Tudor [3], Kruk *et al.* [8], Lei-Nualart [9], Russo-Tudor [13], Tudor-Xiao [15], Yan *et al.* [17], [18] and the references therein.

Clearly, if $K_l = 1$, the process B^{H_l,K_l} is the classical fractional Brownian motion. In recent years the fractional Brownian motion has become an object of intense study, due to its interesting properties and its applications in various scientific areas including telecommunications, turbulence, image processing and finance. Recall that the fractional Brownian motion (fBm) with Hurst index $H \in (0,1)$ is a mean zero Gaussian process $B^H = \{B_t^H, t \geqslant 0\}$ such that

$$R_H(t,s) = E[B_t^H B_s^H] = \frac{1}{2}[t^{2H} + s^{2H} - |t-s|^{2H}]$$

for all $t, s \geqslant 0$. For $H = 1/2$, B^H coincides with the standard Brownian motion B. B^H is neither a semimartingale nor a Markov process unless $H = 1/2$. Some surveys and complete literature could be found in Hu [6], Mishura [10], Nualart [12]. On the other hand, many authors have proposed to use more general self-similar Gaussian processes and random fields as stochastic models. Such applications have raised many interesting theoretical questions about self-similar Gaussian processes and fields in general. Therefore, some generalizations of the fBm were introduced. However, in contrast to the extensive studies on fBm, there has been little systematic investigation on other self-similar Gaussian processes. The main reason for this is the complexity of dependence structures for self-similar Gaussian processes which do not have stationary increments.

Recently, Jiang-Wang [7] (see also Yan *et al.* [17]) considered the collision local time of two independent, 1-dimensional bifractional Brownian motions $B^{H_i,K_i} =$

$\{B_t^{H_i,K_i},\ t \geqslant 0\}$, $i = 1,2$ with respective indices $H_i \in (0,1), K_i \in (0,1]$. The so-called collision local time is formally defined as

$$l_T = \int_0^T \delta(B_s^{H_1,K_1} - B_s^{H_2,K_2})\,\mathrm{d}s, \qquad T \geqslant 0,$$

where δ denotes the Dirac delta function. It is a measure of the amount of time that the trajectories of the two processes, B^{H_1,K_1} and B^{H_2,K_2}, collide on the time interval $[0,T]$. They showed that the random variable l_T exists in L^2 for all $T \geqslant 0$, and it is smooth in the sense of Meyer-Watanabe if $\min\{H_1K_1, H_2K_2\} < 1/3$. Moreover, Shen-Yan [14] showed the condition is also necessary, which motivates the following question:

- ▷ What are the necessary and sufficient conditions for smoothness of l_T with $d \geqslant 2$?

In this paper we consider this and a related problem. One of our main results is the following.

Theorem 1.1. *Let l_T, $T \geqslant 0$ be the collision local time process of two independent, d-dimensional bifractional Brownian motions $B^{H_i,K_i} = \{B_t^{H_i,K_i},\ t \geqslant 0\}$, $i = 1,2$ with respective indices $H_i \in (0,1)$, $K_i \in (0,1]$. Then for every $T > 0$, the random variable l_T is smooth in the sense of Meyer-Watanabe if and only if* $\min\{H_1K_1, H_2K_2\} < 1/(d+2)$.

The paper is organized as follows. In Section 2, we recall some facts for the chaos expansion. The proof of Theorem 1.1 will be given in Section 3. In Section 4, as a related problem we study the intersection local time of two independent, d-dimensional bifractional Brownian motions $B^{H,K}$ and $\widetilde{B}^{H,K}$ with the same indices $H \in (0,1)$, $K \in (0,1]$, which is formally defined as

$$I(B^{H,K}, \widetilde{B}^{H,K}) = \int_0^T\!\!\int_0^T \delta(B_t^{H,K} - \widetilde{B}_s^{H,K})\,\mathrm{d}s\,\mathrm{d}t;$$

we show that it exists in L^2 if and only if $HK < 2/d$ (this result is in accordance with the paper Nualart *et al.* [11]), and it is smooth in the sense of Meyer-Watanabe if and only if $HK < 2/(d+2)$.

2. Preliminaries

In this section, we first recall the chaos expansion, which is an orthogonal decomposition of $L^2(\Omega, P)$. We refer to Hu [5], Nualart [12] and the references therein for more details. Let $X = \{X_t, t \in [0,T]\}$ be a d-dimensional Gaussian process defined on the probality space $(\Omega, \mathscr{F}, P)$ with mean zero. If $p_n(x_1, \dots, x_k)$ is a polynomial of degree n of k variables $x_1, \dots, x_k$, then we call $p_n(X_{t_1}^{i_1}, \dots, X_{t_k}^{i_k})$ a polynomial functional of X with $t_1, \dots, t_k \in [0,T]$ and $1 \leqslant i_1, \dots, i_k \leqslant d$. Let $\mathscr{P}_n$ be the completion with respect to the $L^2(\Omega, P)$ norm of the set $\{p_m(X_{t_1}^{i_1}, \dots, X_{t_k}^{i_k})\colon\ 0 \leqslant m \leqslant n, t \in [0,T]\}$. Clearly $\mathscr{P}_n$ is a subspace of $L^2(\Omega, P)$. If $\mathscr{C}_n$ denotes the orthogonal complement of $\mathscr{P}_{n-1}$ in $\mathscr{P}_n$, then $L^2(\Omega, P)$ is actually the direct sum of $\mathscr{C}_n$, i.e.,

$$L^2(\Omega, P) = \bigoplus_{n=0}^{\infty} \mathscr{C}_n. \tag{2.1}$$

Namely, for any functional $F \in L^2(\Omega, P)$ there are F_n in $\mathscr{C}_n$, $n = 0, 1, 2, \dots$, such that

$$F = \sum_{n=0}^{\infty} F_n. \tag{2.2}$$

The decomposition (2.2) is called the *chaos expansion* of F and F_n is called the n-th chaos of F. Clearly, we have

$$E(|F|^2) = \sum_{n=0}^{\infty} E(|F_n|^2). \tag{2.3}$$

Recall that the Meyer-Watanabe test function space $\mathscr{U}$ (see Watanabe [16]) is defined as

$$\mathscr{U} := \left\{ F \in L^2(\Omega, P)\colon\ F = \sum_{n=0}^{\infty} F_n \text{ and } \sum_{n=0}^{\infty} nE(|F_n|^2) < \infty \right\},$$

and $F \in L^2(\Omega, P)$ is said to be smooth if $F \in \mathscr{U}$.

Now, for $F \in L^2(\Omega, P)$, we define an operator Υ_κ with $\kappa \in [0,1]$ by

$$\Upsilon_\kappa F := \sum_{n=0}^{\infty} \kappa^n F_n. \tag{2.4}$$

Set $\Theta(\kappa) := \Upsilon_{\sqrt{\kappa}} F$, then $\Theta(1) = F$. Define $\Phi_\Theta(\kappa) := \frac{\mathrm{d}}{\mathrm{d}\kappa}(\|\Theta(\kappa)\|^2)$, where $\|F\|^2 := E(|F|^2)$ for $F \in L^2(\Omega, P)$. We have

$$\Phi_\Theta(\kappa) = \sum_{n=1}^{\infty} n\kappa^{n-1} E(|F_n|^2). \tag{2.5}$$

Note that $\|\Theta(\kappa)\|^2 = E(|\Theta(\kappa)|^2) = \sum\limits_{n=1}^{\infty} E(\kappa^n |F_n|^2)$.

Proposition 2.1. *Let $F \in L^2(\Omega, P)$. Then $F \in \mathscr{U}$ if and only if $\Phi_\Theta(1) < \infty$.*

Consider two d-dimensional independent bifractional Brownian motions $B^{H_i,K_i} = \{B_t^{H_i,K_i},\ t \geqslant 0\}$, $i = 1, 2$, with respective indices $H_i \in (0,1)$, $K_i \in (0,1]$. Let $H_n(x)$, $x \in \mathbb{R}$ be the Hermite polynomial of degree n. That is,

$$H_n(x) = (-1)^n \frac{1}{n!} \mathrm{e}^{x^2/2} \frac{\partial^n}{\partial x^n} \mathrm{e}^{-x^2/2}. \tag{2.6}$$

Then

$$\mathrm{e}^{tx - t^2/2} = \sum_{n=0}^{\infty} t^n H_n(x)$$

for all $t \in \mathscr{C}$ and $x \in \mathbb{R}$, which implies that

$$\begin{aligned} \exp(\mathrm{i}u\langle \xi, B_t^{H_1,K_1} - B_t^{H_2,K_2}\rangle + \frac{1}{2}u^2 \langle \xi, \mathrm{Var}(B_t^{H_1,K_1,1} - B_t^{H_2,K_2,2})\xi\rangle \\ = \sum_{n=0}^{\infty} (\mathrm{i}u)^n \sigma^n(t,\xi) H_n\Big(\frac{\langle \xi, B_t^{H_1,K_1} - B_t^{H_2,K_2}\rangle}{\sigma(t,\xi)}\Big), \end{aligned}$$

where $\mathrm{i} = \sqrt{-1}$ and $\sigma(t,\xi) = \sqrt{\mathrm{Var}(B_t^{H_1,K_1,1} - B_t^{H_2,K_2,2})|\xi|^2}$ for $\xi \in \mathbb{R}^d$. Because of the orthogonality of $\{H_n(x), x \in \mathbb{R}\}_{n \in \mathbb{Z}_+}$, we see that

$$(\mathrm{i}u)^n \sigma^n(t,\xi) H_n\Big(\frac{\langle \xi, B_t^{H_1,K_1} - B_t^{H_2,K_2}\rangle}{\sigma(t,\xi)}\Big)$$

is the n-th chaos of $\exp(\mathrm{i}u\langle \xi, B_t^{H_1,K_1} - B_t^{H_2,K_2}\rangle + \frac{1}{2}u^2|\xi|^2 \mathrm{Var}(B_t^{H_1,K_1,1} - B_t^{H_2,K_2,2}))$ for all $t \geqslant 0$. Similarly, we can prove the same results if we use $B_t^{H,K} - \widetilde{B}_s^{H,K}$ instead of $B_t^{H_1,K_1} - B_t^{H_2,K_2}$.

3. Existence and smoothness of the collision local time

In this section we consider the existence and smoothness of the collision local time process. Our main object is to prove Theorem 1.1 by using the idea of An-Yan [1] and Chen-Yan [2]. For simplicity throughout this paper we let C stand for a positive constant depending only on the subscripts and whose value may be different in different appearances. Let $B^{H_i,K_i} = \{B_t^{H_i,K_i},\ t \geqslant 0\}$, $i = 1, 2$, be two independent, d-dimensional bifractional Brownian motions with respective indices $H_i \in (0,1)$, $K_i \in (0,1]$. The so-called collision local time of B^{H_1,K_1} and B^{H_2,K_2} is formally defined as

$$l_T = \int_0^T \delta(B_s^{H_1,K_1} - B_s^{H_2,K_2})\,\mathrm{d}s, \qquad T \geqslant 0, \tag{3.1}$$

where δ is the Dirac delta function. In order to give a rigorous meaning to l_T we approximate the Dirac delta function by the heat kernel

$$p_\varepsilon(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}\langle \xi, x\rangle} \mathrm{e}^{-\varepsilon|\xi|^2/2} \, \mathrm{d}\xi. \tag{3.2}$$

For $\varepsilon > 0$ we define

$$\begin{aligned} l_{\varepsilon,T} &= \int_0^T p_\varepsilon(B_s^{H_1,K_1} - B_s^{H_2,K_2}) \, \mathrm{d}s \\ &= \frac{1}{(2\pi)^d} \int_0^T \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}\langle \xi, B_s^{H_1,K_1} - B_s^{H_2,K_2}\rangle} \cdot \mathrm{e}^{-\varepsilon|\xi|^2/2} \, \mathrm{d}\xi \, \mathrm{d}s. \end{aligned} \tag{3.3}$$

First, we will prove the following theorem.

Theorem 3.1 (Existence of the collision local time)**.** *Let $H_i \in (0,1)$, $K_i \in (0,1]$. Assume $d \geqslant 2$. Then $l_{\varepsilon,T}$ converges in L^2, as $\varepsilon \to 0$ if and only if $H_1K_1 \wedge H_2K_2 < 1/d$. Moreover, if the limit is denoted by l_T, then $l_T \in L^2(\Omega, P)$.*

Before proving Theorem 3.1, we need some preparations. Denote

$$\lambda_t = \mathrm{Var}(B_t^{H_1,K_1,1} - B_t^{H_2,K_2,2})$$

and

$$\varrho_{s,t} = E[(B_t^{H_1,K_1,1} - B_t^{H_2,K_2,2})(B_s^{H_1,K_1,1} - B_s^{H_2,K_2,2})]$$

for $s,t \geqslant 0$. Then it is easy to obtain

$$E[l_{\varepsilon,T}] = \frac{1}{(2\pi)^{d/2}} \int_0^T (\lambda_s + \varepsilon)^{-d/2} \, \mathrm{d}s \tag{3.4}$$

and

$$E[l_{\varepsilon,T}^2] = \frac{1}{(2\pi)^d} \int_{[0,T]^2} [(\lambda_s + \varepsilon)(\lambda_t + \varepsilon) - \varrho_{s,t}^2]^{-d/2} \, \mathrm{d}s \, \mathrm{d}t. \tag{3.5}$$

By symmetry one may assume $0 \leqslant s \leqslant t \leqslant T$, and we set $s = xt$, $0 \leqslant x \leqslant 1$. Thus we can rewrite λ_s and $\varrho_{s,t}$ as

$$\lambda_s = (xt)^{2H_1K_1} + (xt)^{2H_2K_2} \tag{3.6}$$

and

$$\begin{aligned}
(3.7)\qquad \varrho_{s,t} &= \frac{1}{2^{K_1}}[(t^{2H_1}+(xt)^{2H_1})^{K_1}-(t-xt)^{2H_1K_1}] \\
&\quad + \frac{1}{2^{K_2}}[(t^{2H_2}+(xt)^{2H_2})^{K_2}-(t-xt)^{2H_2K_2}] \\
&= \frac{t^{2H_1K_1}}{2^{K_1}}[(1+x^{2H_1})^{K_1}-(1-x)^{2H_1K_1}] \\
&\quad + \frac{t^{2H_2K_2}}{2^{K_2}}[(1+x^{2H_2})^{K_2}-(1-x)^{2H_2K_2}].
\end{aligned}$$

It follows that

$$(3.8)\qquad \lambda_s\lambda_t - \varrho_{s,t}^2 = \frac{t^{4H_1K_1}}{2^{2K_1}}f_1(x) + \frac{t^{4H_2K_2}}{2^{2K_2}}f_2(x) + \frac{t^{2H_1K_1+2H_2K_2}}{2^{K_1+K_2}}g(x),$$

where

$$f_i(x) := 2^{2K_i}x^{2H_iK_i} + 2(1+x^{2H_i})^{K_i}(1-x)^{2H_iK_i} - (1+x^{2H_i})^{2K_i} - (1-x)^{4H_iK_i}$$

for $i=1,2$, and

$$\begin{aligned}
g(x) &= 2^{K_1+K_2}(x^{2H_1K_1}+x^{2H_2K_2}) - 2(1+x^{2H_1})^{K_1}(1+x^{2H_2})^{K_2} \\
&\quad - 2(1-x)^{2H_1K_1+2H_2K_2} + 2(1+x^{2H_1})^{K_1}(1-x)^{2H_2K_2} \\
&\quad + 2(1+x^{2H_2})^{K_2}(1-x)^{2H_1K_1}.
\end{aligned}$$

In order to prove Theorem 3.1 we need to estimate $f_i(x)$, $i=1,2$, and $g(x)$. For simplicity we assume that the notation $F \asymp G$ means that there are positive constants C_1 and C_2 such that

$$C_1G(x) \leqslant F(x) \leqslant C_2G(x)$$

in the common domain for F and G. For $a,b \in \mathbb{R}$, $a \wedge b := \min\{a,b\}$ and $a \vee b := \max\{a,b\}$.

Lemma 3.1. *Let $0 < H_i < 1$, $0 < K_i \leqslant 1$, for $i=1,2$. Then we have*

$$(3.9)\qquad f_i(x) \asymp x^{2H_iK_i}(1-x)^{2H_iK_i},$$

$$(3.10)\qquad g(x) \asymp x^{2H_1K_1}(1-x)^{2H_2K_2} + x^{2H_2K_2}(1-x)^{2H_1K_1}$$

for all $x \in [0,1]$.

Clearly, the estimates (3.9) and (3.10) can be proved by using the asymptotic property of functions

$$\frac{f_i(x)}{x^{2H_iK_i}(1-x)^{2H_iK_i}}, i=1,2; \qquad \frac{g(x)}{x^{2H_1K_1}(1-x)^{2H_2K_2}+x^{2H_2K_2}(1-x)^{2H_1K_1}}$$

as $x \to 0$ and $x \to 1$, respectively.

P r o o f of Theorem 3.1. A slight extension of (3.5) yields

$$E[l_{\varepsilon,T} l_{\eta,T}] = \frac{1}{(2\pi)^d} \int_{[0,T]^2} [(\lambda_s + \varepsilon)(\lambda_t + \eta) - \varrho_{s,t}^2]^{-d/2} \,\mathrm{d}s\,\mathrm{d}t.$$

Consequently, a necessary and sufficient condition for the convergence in $L^2(\Omega, P)$ of $l_{\varepsilon,T}$ is that

$$\Lambda_T \equiv \int_{[0,T]^2} (\lambda_s \lambda_t - \varrho_{s,t}^2)^{-d/2} \,\mathrm{d}s\,\mathrm{d}t < \infty.$$

Thus, it is sufficient to prove that

$$\Lambda_T \equiv \int_{[0,T]^2} (\lambda_s \lambda_t - \varrho_{s,t}^2)^{-d/2} \,\mathrm{d}s\,\mathrm{d}t < \infty$$

if and only if $H_1K_1 \wedge H_2K_2 < 1/d$. It follows from Lemma 3.1 that

$$\begin{aligned}
&\lambda_s \lambda_t - \varrho_{s,t}^2 \\
&\quad = \frac{t^{4H_1K_1}}{2^{2K_1}} f_1(x) + \frac{t^{4H_2K_2}}{2^{2K_2}} f_2(x) + \frac{t^{2H_1K_1+2H_2K_2}}{2^{K_1+K_2}} g(x) \\
&\quad \asymp (x^{2H_1K_1} t^{2H_1K_1} + x^{2H_2K_2} t^{2H_2K_2})[(1-x)^{2H_1K_1} t^{2H_1K_1} + (1-x)^{2H_2K_2} t^{2H_2K_2}] \\
&\quad \asymp (s^{2H_1K_1} + s^{2H_2K_2})[(t-s)^{2H_1K_1} + (t-s)^{2H_2K_2}]
\end{aligned}$$

for all $0 \leqslant s \leqslant t$ and $x = s/t$. We have

$$\begin{aligned}
&\int_0^T \int_0^T (\lambda_t \lambda_s - \varrho_{s,t}^2)^{-d/2} \,\mathrm{d}s\,\mathrm{d}t \\
&\asymp C_{H_1,K_1,H_2,K_2} \int_0^T \mathrm{d}t \int_0^t [s^{2H_1K_1} + s^{2H_2K_2}]^{-d/2} [(t-s)^{2H_1K_1} + (t-s)^{2H_2K_2}]^{-d/2} \,\mathrm{d}s \\
&\asymp C_{T,H_1,H_2,K_1,K_2} \int_0^T \mathrm{d}t \int_0^t \frac{1}{s^{d(H_1K_1 \wedge H_2K_2)} (t-s)^{d(H_1K_1 \wedge H_2K_2)}} \,\mathrm{d}s.
\end{aligned}$$

It follows that

$$\int_0^T \int_0^T (\lambda_t \lambda_s - \varrho_{s,t}^2)^{-d/2} \,\mathrm{d}s\,\mathrm{d}t < \infty \tag{3.11}$$

if and only if $H_1K_1 \wedge H_2K_2 < 1/d$. □

The following proposition is important for the proof of Theorem 1.1.

Proposition 3.1. *Let $\lambda_t, \varrho_{s,t}$ denote as above. Then for $T \geqslant 0$, $l_T \in \mathscr{U}$ if and only if*

$$\int_0^T\int_0^T \varrho_{s,t}^2(\lambda_t\lambda_s - \varrho_{s,t}^2)^{-d/2-1}\,\mathrm{d}s\,\mathrm{d}t < \infty. \tag{3.12}$$

In order to prove Proposition 3.1, we need some preliminaries. Let X, Y be two random variables with joint Gaussian distribution such that $E(X) = E(Y) = 0$ and $E(X^2) = E(Y^2) = 1$. Then for all $n, m \geqslant 0$ we have (see, for example, Nualart [12])

$$E(H_n(X)H_m(Y)) = \begin{cases} 0, & m \neq n, \\ \dfrac{1}{n!}[E(XY)]^n, & m = n. \end{cases} \tag{3.13}$$

Lemma 3.2 (Chen-Yan [2]). *Suppose $d \geqslant 1$. For any $x \in [-1, 1)$ we have*

$$\sum_{n=1}^{\infty}\sum_{\substack{k_1,\dots,k_d=0\\ k_1+\dots+k_d=n}}^{n} \frac{2n(2k_1-1)!!\cdot\ldots\cdot(2k_d-1)!!}{(2k_1)!!\cdot\ldots\cdot(2k_d)!!}x^n \asymp x(1-x)^{-(d/2+1)}.$$

It follows from $\varrho_{s,t}^2 \leqslant \lambda_s\lambda_t$ that

$$\begin{aligned}\frac{\varrho_{s,t}^2}{(\lambda_s\lambda_t - \varrho_{s,t}^2)^{d/2+1}} &= \frac{\varrho_{s,t}^2}{\lambda_s\lambda_t}\Big(1 - \frac{\varrho_{s,t}^2}{\lambda_s\lambda_t}\Big)^{-(d/2+1)}\Big(\frac{1}{\lambda_s\lambda_t}\Big)^{d/2} \\ &\asymp \sum_{n=1}^{\infty}\sum_{\substack{k_1,\dots,k_d=0\\ k_1+\dots+k_d=n}}^{n} \frac{2n(2k_1-1)!!\cdot\ldots\cdot(2k_d-1)!!}{(2k_1)!!\cdot\ldots\cdot(2k_d)!!}\frac{\varrho_{s,t}^{2n}}{(\lambda_s\lambda_t)^{n+d/2}}.\end{aligned}$$

P r o o f of Proposition 3.1. For $\varepsilon > 0, T \geqslant 0$ we denote

$$\Theta_\varepsilon(u, T, l_{\varepsilon,T}) := E(|\Upsilon_{\sqrt{u}}l_{\varepsilon,T}|^2)$$

and $\Theta(u, T, l_T) := E(|\Upsilon_{\sqrt{u}}l_T|^2)$. Thus, by Proposition 2.1 we have to prove that (3.12) holds if and only if $\Phi_\Theta(1) < \infty$. Clearly, we have

$$\begin{aligned} l_{\varepsilon,T} &= \int_0^T p_\varepsilon(B_t^{H_1,K_1} - B_t^{H_2,K_2})\,\mathrm{d}t \\ &= \frac{1}{(2\pi)^d}\int_0^T\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}\langle\xi, B_t^{H_1,K_1} - B_t^{H_2,K_2}\rangle}\mathrm{e}^{-\varepsilon|\xi|^2/2}\,\mathrm{d}\xi\,\mathrm{d}t \\ &= \frac{1}{(2\pi)^d}\int_0^T\int_{\mathbb{R}^d} \mathrm{e}^{-\frac{1}{2}(\lambda_t+\varepsilon)|\xi|^2} \\ &\quad\times\sum_{n=0}^{\infty}\mathrm{i}^n\sigma^n(t,\xi)H_n\Big(\frac{\langle\xi, B_t^{H_1,K_1} - B_t^{H_2,K_2}\rangle}{\sigma(t,\xi)}\Big)\,\mathrm{d}\xi\,\mathrm{d}t \equiv \sum_{n=0}^{\infty}F_n.\end{aligned}$$

Thus, by (3.13) and Lemma 3.2 we have

$$\begin{aligned}
\Phi_{\Theta_\varepsilon}(1) &= \sum_{n=0}^{\infty} nE(|F_n|^2) \\
&= \sum_{n=0}^{\infty} \frac{n}{(2\pi)^{2d}} E\Big[\int_{[0,T]^2}\int_{\mathbb{R}^{2d}} \mathrm{e}^{-\frac{1}{2}((\lambda_t+\varepsilon)|\xi|^2+(\lambda_s+\varepsilon)|\eta|^2)}\sigma^n(t,\xi)\sigma^n(s,\eta) \\
&\quad \times H_n\Big(\frac{\langle \xi, B_t^{H_1,K_1}-B_t^{H_2,K_2}\rangle}{\sigma(t,\xi)}\Big) H_n\Big(\frac{\langle \eta, B_s^{H_1,K_1}-B_s^{H_2,K_2}\rangle}{\sigma(s,\eta)}\Big)\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}s\,\mathrm{d}t\Big] \\
&= \sum_{n=1}^{\infty} \frac{1}{(2\pi)^{2d}(n-1)!}\int_{[0,T]^2} \varrho_{s,t}^n\,\mathrm{d}s\,\mathrm{d}t \\
&\quad \times \int_{\mathbb{R}^{2d}} \mathrm{e}^{-\frac{1}{2}((\lambda_t+\varepsilon)|\xi|^2+(\lambda_s+\varepsilon)|\eta|^2)}\langle\xi,\eta\rangle^n\,\mathrm{d}\xi\,\mathrm{d}\eta \\
&= \sum_{n=1}^{\infty} \frac{1}{(2\pi)^{2d}(2n-1)!}\int_{[0,T]^2} \varrho_{s,t}^{2n}\,\mathrm{d}s\,\mathrm{d}t \\
&\quad \times \int_{\mathbb{R}^{2d}} \mathrm{e}^{-\frac{1}{2}((\lambda_t+\varepsilon)|\xi|^2+(\lambda_s+\varepsilon)|\eta|^2)}\langle\xi,\eta\rangle^{2n}\,\mathrm{d}\xi\,\mathrm{d}\eta \\
&= \frac{1}{(2\pi)^d}\sum_{n=1}^{\infty}\sum_{\substack{k_1,\dots,k_d=0\\ k_1+\dots+k_d=n}}^{n} \frac{2n(2k_1-1)!!\cdot\ldots\cdot(2k_d-1)!!}{(2k_1)!!\cdot\ldots\cdot(2k_d)!!} \\
&\quad \times \int_{[0,T]^2} \frac{\varrho_{s,t}^{2n}}{((\lambda_t+\varepsilon)(\lambda_s+\varepsilon))^{n+d/2}}\,\mathrm{d}s\,\mathrm{d}t \\
&\asymp \int_{[0,T]^2} \varrho_{s,t}^2((\lambda_t+\varepsilon)(\lambda_s+\varepsilon)-\varrho_{s,t}^2)^{-d/2-1}\,\mathrm{d}s\,\mathrm{d}t,
\end{aligned}$$

where we have used the following fact:

$$\begin{aligned}
\int_{\mathbb{R}} \xi^{2k}\mathrm{e}^{-\frac{1}{2}(\lambda_t+\varepsilon)\xi^2}\,\mathrm{d}\xi &= 2\int_0^{\infty} \xi^{2k}\mathrm{e}^{-\frac{1}{2}(\lambda_t+\varepsilon)\xi^2}\,\mathrm{d}\xi \\
&= 2^{k+\frac{1}{2}}\Gamma\Big(k+\frac{1}{2}\Big)(\lambda_t+\varepsilon)^{-(k+\frac{1}{2})} = \sqrt{2\pi}(2k-1)!!\,(\lambda_t+\varepsilon)^{-(k+\frac{1}{2})}.
\end{aligned}$$

It follows that

$$\lim_{\varepsilon\to 0}\Phi_{\Theta_\varepsilon}(1) \asymp \int_0^T\int_0^T \varrho_{s,t}^2(\lambda_t\lambda_s-\varrho_{s,t}^2)^{-d/2-1}\,\mathrm{d}s\,\mathrm{d}t$$

for all $T \geqslant 0$. This completes the proof. □

Now we are ready to prove Theorem 1.1.

P r o o f of Theorem 1.1. By Proposition 3.1, it is sufficient to prove that

$$\int_0^T\int_0^T \varrho_{s,t}^2(\lambda_t\lambda_s-\varrho_{s,t}^2)^{-d/2-1}\,\mathrm{d}s\,\mathrm{d}t<\infty$$

if and only if $\min\{H_1K_1,H_2K_2\}<1/(d+2)$. Without loss of generality we may assume $s\leqslant t$ and $s=xt$, where $x\in[0,1]$. It follows from Lemma 3.1 that

$$\lambda_s\lambda_t-\varrho_{s,t}^2\asymp(s^{2H_1K_1}+s^{2H_2K_2})[(t-s)^{2H_1K_1}+(t-s)^{2H_2K_2}]$$

for all $0\leqslant s\leqslant t$ and $x=s/t$.

First, we give the proof of the sufficient condition. Since

$$\begin{aligned}\varrho_{s,t}&=\frac{t^{2H_1K_1}}{2^{K_1}}[(1+x^{2H_1})^{K_1}-(1-x)^{2H_1K_1}]\\&\quad+\frac{t^{2H_2K_2}}{2^{K_2}}[(1+x^{2H_2})^{K_2}-(1-x)^{2H_2K_2}]\leqslant T^{2H_1K_1}+T^{2H_2K_2},\end{aligned}$$

we have

$$\begin{aligned}&\int_0^T\int_0^T \varrho_{s,t}^2(\lambda_t\lambda_s-\varrho_{s,t}^2)^{-(d+2)/2}\,\mathrm{d}s\,\mathrm{d}t\\&\leqslant C_{H_1,K_1,H_2,K_2}\int_0^T\mathrm{d}t\int_0^t\frac{(T^{2H_1K_1}+T^{2H_2K_2})^2\,\mathrm{d}s}{[s^{2H_1K_1}+s^{2H_2K_2}]^{(d+2)/2}[(t-s)^{2H_1K_1}+(t-s)^{2H_2K_2}]^{(d+2)/2}}\\&\leqslant C_{T,H_1,H_2,K_1,K_2}\int_0^T\mathrm{d}t\int_0^t\frac{1}{s^{(d+2)(H_1K_1\wedge H_2K_2)}(t-s)^{(d+2)(H_1K_1\wedge H_2K_2)}}\,\mathrm{d}s<\infty\end{aligned}$$

if $H_1K_1\wedge H_2K_2<1/(d+2)$.

Now we give the proof of the necessary condition. We split the proof into two cases.

Case I. We claim that

$$(1+x^{2H_i})^{K_i}-(1-x)^{2H_iK_i}\geqslant K_i2^{K_i-1}x^{2H_i}$$

for $0<2H_iK_i<1$, $H_i\in(0,1)$, $K_i\in(0,1]$. In fact, by differentiation the expression

$$(1+x^{2H_i})^{K_i}-(1-x)^{2H_iK_i}-K_i2^{K_i-1}x^{2H_i}$$

is non-negative for all $0\leqslant x\leqslant 1$. It follows that

$$\begin{aligned}\varrho_{s,t}&=\frac{t^{2H_1K_1}}{2^{K_1}}[(1+x^{2H_1})^{K_1}-(1-x)^{2H_1K_1}]+\frac{t^{2H_2K_2}}{2^{K_2}}[(1+x^{2H_2})^{K_2}-(1-x)^{2H_2K_2}]\\&\geqslant\frac{t^{2H_1K_1}}{2^{K_1}}K_12^{K_1-1}x^{2H_1}+\frac{t^{2H_2K_2}}{2^{K_2}}K_22^{K_2-1}x^{2H_2}\\&=\frac{K_1}{2}t^{2H_1K_1}x^{2H_1}+\frac{K_2}{2}t^{2H_2K_2}x^{2H_2}\\&\geqslant\min\Big\{\frac{K_1}{2},\frac{K_2}{2}\Big\}(t^{2H_1K_1}x^{2H_1}+t^{2H_2K_2}x^{2H_2}).\end{aligned}$$

This yields for $T > 0$

$$
\begin{aligned}
&\int_0^T\int_0^T \varrho_{s,t}^2(\lambda_t\lambda_s - \varrho_{s,t}^2)^{-(d+2)/2}\,\mathrm{d}s\,\mathrm{d}t \\
&\geqslant C_{K_1,K_2}\int_0^T \mathrm{d}t\int_0^t \frac{(t^{2H_1K_1}x^{2H_1}+t^{2H_2K_2}x^{2H_2})^2}{[s^{2H_1K_1}+s^{2H_2K_2}]^{(d+2)/2}[(t-s)^{2H_1K_1}+(t-s)^{2H_2K_2}]^{(d+2)/2}}\,\mathrm{d}s \\
&= C_{K_1,K_2}\int_0^T \mathrm{d}t\int_0^t \frac{(t^{2H_1K_1-2H_1}s^{2H_1}+t^{2H_2K_2-2H_2}s^{2H_2})^2}{[s^{2H_1K_1}+s^{2H_2K_2}]^{(d+2)/2}[(t-s)^{2H_1K_1}+(t-s)^{2H_2K_2}]^{(d+2)/2}}\,\mathrm{d}s \\
&\geqslant C_{T,H_1,H_2,K_1,K_2}\int_0^T \mathrm{d}t\int_0^t \frac{s^{4(H_1\wedge H_2)}}{s^{(d+2)(H_1K_1\wedge H_2K_2)}(t-s)^{(d+2)(H_1K_1\wedge H_2K_2)}}\,\mathrm{d}s.
\end{aligned}
$$

Case II. We claim that

$$
(1+x^{2H_i})^{K_i} - (1-x)^{2H_iK_i} \geqslant (1+x^{2H_i})^{K_i} - 1 + x^{2H_iK_i} \geqslant x^{2H_iK_i}
$$

for $1 < 2H_iK_i < 2, H_i \in (0,1), K_i \in (0,1]$. It follows that

$$
\varrho_{s,t} \geqslant \frac{t^{2H_1K_1}}{2^{K_1}}x^{2H_1K_1} + \frac{t^{2H_2K_2}}{2^{K_2}}x^{2H_2K_2} \geqslant \min\left\{\frac{K_1}{2},\frac{K_2}{2}\right\}(s^{2H_1K_1}+s^{2H_2K_2}).
$$

This yields for $T > 0$

$$
\begin{aligned}
&\int_0^T\int_0^T \varrho_{s,t}^2(\lambda_t\lambda_s - \varrho_{s,t}^2)^{-(d+2)/2}\,\mathrm{d}s\,\mathrm{d}t \\
&\geqslant C_{H_1,H_2,K_1,K_2}\int_0^T \mathrm{d}t\int_0^t \frac{(s^{2H_1K_1}+s^{2H_2K_2})^2\,\mathrm{d}s}{[s^{2H_1K_1}+s^{2H_2K_2}]^{(d+2)/2}[(t-s)^{2H_1K_1}+(t-s)^{2H_2K_2}]^{(d+2)/2}} \\
&= C_{H_1,H_2,K_1,K_2}\int_0^T \mathrm{d}t\int_0^t \frac{\mathrm{d}s}{[s^{2H_1K_1}+s^{2H_2K_2}]^{(d-2)/2}[(t-s)^{2H_1K_1}+(t-s)^{2H_2K_2}]^{(d+2)/2}} \\
&\geqslant C_{T,H_1,H_2,K_1,K_2}\int_0^T \mathrm{d}t\int_0^t \frac{1}{s^{(d-2)(H_1K_1\wedge H_2K_2)}(t-s)^{(d+2)(H_1K_1\wedge H_2K_2)}}\,\mathrm{d}s.
\end{aligned}
$$

It follows that

$$
\int_0^T\int_0^T \varrho_{s,t}^2(\lambda_t\lambda_s - \varrho_{s,t}^2)^{-\frac{d}{2}-1}\,\mathrm{d}s\,\mathrm{d}t < \infty \tag{3.14}
$$

if and only if $\min\{H_1K_1, H_2K_2\} < 1/(d+2)$. □

4. Existence and smoothness of the intersection local time

In this section we study the intersection local time of two independent, d-dimensional bifractional Brownian motions $B^{H,K}$ and $\widetilde{B}^{H,K}$ with the same indices $H \in (0,1)$, $K \in (0,1]$, which is formally defined as

$$I(B^{H,K}, \widetilde{B}^{H,K}) = \int_0^T \int_0^T \delta(B_t^{H,K} - \widetilde{B}_s^{H,K})\,\mathrm{d}s\,\mathrm{d}t;$$

it is a measure of the amount of time that the trajectories of the two processes $B^{H,K}$ and $\widetilde{B}^{H,K}$ intersect on the time interval $[0,T]$. Nualart *et al.* [11] consider intersection local time for two independent, d-dimensional fractional Brownian motions. They prove that the intersection local time exists in L^2 if and only if $Hd < 2$. The object of study in this section will be the smoothness of the intersection local time of $B^{H,K}$ and $\widetilde{B}^{H,K}$. We show that $I(B^{H,K}, \widetilde{B}^{H,K})$ is smooth in the sense of Meyer-Watanabe if and only if $HK < 2/(d+2)$. Our method used here is essentially due to An-Yan [1] and Chen-Yan [2].

As we pointed out, the definition is only formal, in order to give a rigorous meaning to $I(B^{H,K}, \widetilde{B}^{H,K})$ we approximate the Dirac delta function by the heat kernel

$$p_\varepsilon(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}\langle \xi, x\rangle} \mathrm{e}^{-\varepsilon|\xi|^2/2}\,\mathrm{d}\xi. \tag{4.1}$$

For $\varepsilon > 0$ we define

$$\begin{aligned} I_\varepsilon(B^{H,K}, \widetilde{B}^{H,K}) &= \int_0^T \int_0^T p_\varepsilon(B_t^{H,K} - \widetilde{B}_s^{H,K})\,\mathrm{d}s\,\mathrm{d}t \\ &= \frac{1}{(2\pi)^d} \int_0^T \int_0^T \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}\langle \xi, B_t^{H,K} - \widetilde{B}_s^{H,K}\rangle} \mathrm{e}^{-\varepsilon|\xi|^2/2}\,\mathrm{d}\xi\,\mathrm{d}s\,\mathrm{d}t. \end{aligned} \tag{4.2}$$

First, we consider the existence of the intersection local time process.

Theorem 4.1 (Existence of the intersection local time). *Let $H \in (0,1)$, $K \in (0,1]$. Assume $d \geqslant 2$. Then $I_\varepsilon(B^{H,K}, \widetilde{B}^{H,K})$ converges in L^2 as $\varepsilon \to 0$ if and only if $HKd < 2$. Moreover, if the limit is denoted by $I(B^{H,K}, \widetilde{B}^{H,K})$, then $I(B^{H,K}, \widetilde{B}^{H,K}) \in L^2(\Omega, P)$.*

Denote

$$
\begin{aligned}
a_{s,t} &\equiv \operatorname{Var}(B_t^{H,K,1} - \widetilde{B}_s^{H,K,2}) = t^{2HK} + s^{2HK}, \\
a_{u,v} &\equiv \operatorname{Var}(B_v^{H,K,1} - \widetilde{B}_u^{H,K,2}) = v^{2HK} + u^{2HK}, \\
\varrho_{s,t,u,v} &= E[(B_t^{H,K,1} - \widetilde{B}_s^{H,K,2})(B_v^{H,K,1} - \widetilde{B}_u^{H,K,2})] \\
&= \frac{1}{2^K}[(t^{2H} + v^{2H})^K - |t-v|^{2HK}] \\
&\quad + \frac{1}{2^K}[(s^{2H} + u^{2H})^K - |s-u|^{2HK}]
\end{aligned} \tag{4.3}
$$

for all $s, t, u, v \geqslant 0$. By Nualart *et al.* [11], we have

$$
E[I_\varepsilon(B^{H,K}, \widetilde{B}^{H,K})] = \frac{1}{(2\pi)^{d/2}} \int_0^T \int_0^T (a_{s,t} + \varepsilon)^{-d/2} \, \mathrm{d}s \, \mathrm{d}t, \tag{4.4}
$$

$$
\begin{aligned}
&E[I_\varepsilon^2(B^{H,K}, \widetilde{B}^{H,K})] \\
&\quad = \frac{1}{(2\pi)^d} \int_{[0,T]^4} ((a_{s,t} + \varepsilon)(a_{u,v} + \varepsilon) - \varrho_{s,t,u,v}^2)^{-d/2} \, \mathrm{d}s \, \mathrm{d}t \, \mathrm{d}u \, \mathrm{d}v.
\end{aligned} \tag{4.5}
$$

Without loss of generality we may assume $v \leqslant t$, $u \leqslant s$ and $v = xt$, $u = ys$ with $x, y \in [0,1]$. Then we can rewrite $a_{u,v}$ and $\varrho_{s,t,u,v}$ as

$$
\begin{aligned}
a_{u,v} &= x^{2HK} t^{2HK} + y^{2HK} s^{2HK}, \\
\varrho_{s,t,u,v} &= \frac{1}{2^K} t^{2HK} [(1 + x^{2H})^K - (1-x)^{2HK}] \\
&\quad + \frac{1}{2^K} s^{2HK} [(1 + y^{2H})^K - (1-y)^{2HK}].
\end{aligned} \tag{4.6}
$$

It follows that

$$
a_{s,t} a_{u,v} - \varrho_{s,t,u,v}^2 = \frac{t^{4HK}}{2^{2K}} f(x) + \frac{s^{4HK}}{2^{2K}} f(y) + \frac{t^{2HK} s^{2HK}}{2^{2K}} g(x,y), \tag{4.7}
$$

where

$$
f(x) := 2^{2K} x^{2HK} - [(1 + x^{2H})^K - (1-x)^{2HK}]^2
$$

and

$$
\begin{aligned}
g(x,y) &= 2^{2K}(x^{2HK} + y^{2HK}) \\
&\quad - 2[(1 + x^{2H})^K - (1-x)^{2HK}][(1 + y^{2H})^K - (1-y)^{2HK}].
\end{aligned} \tag{4.8}
$$

Thus, by Lemma 3.1 we get

$$
f(x) \asymp x^{2HK} (1-x)^{2HK} \tag{4.9}
$$

and

$$g(x,y) \asymp x^{2HK}(1-y)^{2HK} + y^{2HK}(1-x)^{2HK} \tag{4.10}$$

for all $x, y \in [0,1]$.

P r o o f of Theorem 4.1. A slight extension of (4.5) yields

$$\begin{aligned}
&E[I_\varepsilon(B^{H,K}, \widetilde{B}^{H,K}) I_\eta(B^{H,K}, \widetilde{B}^{H,K})] \\
&\qquad = \frac{1}{(2\pi)^d} \int_{[0,T]^4} ((a_{s,t}+\varepsilon)(a_{u,v}+\eta) - \varrho^2_{s,t,u,v})^{-d/2}\,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}u\,\mathrm{d}v.
\end{aligned}$$

Consequently, a necessary and sufficient condition for the convergence in $L^2(\Omega, P)$ of $I_\varepsilon(B^{H,K}, \widetilde{B}^{H,K})$ is that

$$\Lambda_T \equiv \int_{[0,T]^4} (a_{s,t}a_{u,v} - \varrho^2_{s,t,u,v})^{-d/2}\,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}u\,\mathrm{d}v < \infty.$$

Thus, it is sufficient to prove that

$$\Lambda_T \equiv \int_{[0,T]^4} (a_{s,t}a_{u,v} - \varrho^2_{s,t,u,v})^{-d/2}\,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}u\,\mathrm{d}v < \infty$$

if and only if $HKd < 2$. By symmetry we have

$$\Lambda_T = 4\int_0^T\int_0^t\int_0^T\int_0^s (a_{s,t}a_{u,v} - \varrho^2_{s,t,u,v})^{-d/2}\,\mathrm{d}u\,\mathrm{d}s\,\mathrm{d}v\,\mathrm{d}t.$$

By (4.9) and (4.10) we have

$$\begin{aligned}
a_{s,t}a_{u,v} - \varrho^2_{s,t,u,v} &= \frac{t^{4HK}}{2^{2K}} f(x) + \frac{s^{4HK}}{2^{2K}} f(y) + \frac{t^{2HK}s^{2HK}}{2^{2K}} g(x,y) \\
&\asymp [t^{4HK}x^{2HK}(1-x)^{2HK} + s^{4HK}y^{2HK}(1-y)^{2HK} \\
&\quad + t^{2HK}s^{2HK}(x^{2HK}(1-y)^{2HK} + y^{2HK}(1-x)^{2HK})] \\
&\asymp [x^{2HK}t^{2HK} + y^{2HK}s^{2HK}][(1-x)^{2HK}t^{2HK} + (1-y)^{2HK}s^{2HK}] \\
&\asymp [v^{2HK} + u^{2HK}][(t-v)^{2HK} + (s-u)^{2HK}]
\end{aligned}$$

for all $0 \leqslant v < t$, $0 \leqslant u < s$ and $x = v/t$, $y = u/s$. This yields for all $H \in (0,1)$, $K \in (0,1]$ and $T > 0$

$$\begin{aligned}
\Lambda_T &\leqslant C\int_0^T \mathrm{d}t \int_0^t (v^{HK}(t-v)^{HK})^{-d/2}\,\mathrm{d}v \int_0^T \mathrm{d}s \int_0^s (u^{HK}(s-u)^{HK})^{-d/2}\,\mathrm{d}u \\
&= C\biggl(\int_0^T t^{1-HKd}\,\mathrm{d}t \int_0^1 x^{-HKd/2}(1-x)^{-HKd/2}\biggr)^2 < \infty,
\end{aligned}$$

if $HKd < 2$. On the other hand, making a change to spherical coordinates, as the integrand in A_T is always positive, we have

$$\begin{aligned}\Lambda_T &\geqslant \int_{D_T} [(v^{2HK}+u^{2HK})((t-v)^{2HK}+(s-u)^{2HK})]^{-d/2}\,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}u\,\mathrm{d}v\\ &= \int_0^T r^{3-2HKd}\,\mathrm{d}r\int_\Theta \varphi(\theta)\,\mathrm{d}\theta\end{aligned}$$

where

$$D_T := \{(s,t,u,v)\in\mathbb{R}_+^4\colon\ s^2+t^2+u^2+v^2\leqslant T^2\}.$$

Note that the angular integral is different from zero thanks to the positivity of the integrand. It follows that

$$\int_0^T\int_0^T\int_0^T\int_0^T (a_{s,t}a_{u,v}-\varrho^2_{s,t,u,v})^{-d/2}\,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}u\,\mathrm{d}v<\infty \tag{4.11}$$

if and only if $HKd < 2$. □

Next we establish the smoothness of the random variable $I(B^{H,K},\widetilde{B}^{H,K})$ under some restrictions on parameters.

Theorem 4.2. *Suppose that $d\geqslant 2$. Let $I(B^{H,K},\widetilde{B}^{H,K})$ be the intersection local time of two independent, d-dimensional bifractional Brownian motions $B^{H,K}$ and $\widetilde{B}^{H,K}$ with $H\in(0,1)$, $K\in(0,1]$. Then for every $T>0$, the random variable $I(B^{H,K},\widetilde{B}^{H,K})$ is smooth in the sense of Meyer-Watanabe if and only if $HK<2/(d+2)$.*

In order to prove Theorem 4.2, we need the following proposition.

Proposition 4.1. *Let $a_{s,t}$, $a_{u,v}$, $\varrho_{s,t,u,v}$ be as above. For all $T\geqslant 0$, $I(B^{H,K},\widetilde{B}^{H,K})\in\mathscr{U}$ if and only if*

$$\int_{[0,T]^4}\varrho^2_{s,t,u,v}(a_{s,t}a_{u,v}-\varrho^2_{s,t,u,v})^{-d/2-1}\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}s\,\mathrm{d}t<\infty. \tag{4.12}$$

P r o o f. The proposition could be proved along the lines of the proof of Proposition 3.1. For the sake of completeness, we give the main arguments of the proof. For $\varepsilon>0$, $T\geqslant 0$ we denote

$$\Theta_\varepsilon(\kappa) := E(|\Upsilon_{\sqrt{\kappa}}I_\varepsilon(B^{H,K},\widetilde{B}^{H,K})|^2)$$

and $\Theta(\kappa) := E(|\Upsilon_{\sqrt{\kappa}} I(B^{H,K}, \widetilde{B}^{H,K})|^2)$. Thus, by Proposition 2.1 it suffices to prove (4.12) if and only if $\Phi_\Theta(1) < \infty$. Notice that

$$\begin{aligned}
I_\varepsilon(B^{H,K}, \widetilde{B}^{H,K}) &= \int_0^T\int_0^T p_\varepsilon(B_t^{H,K} - \widetilde{B}_s^{H,K})\,\mathrm{d}s\,\mathrm{d}t \\
&= \frac{1}{(2\pi)^d}\int_0^T\int_0^T\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}\langle\xi, B_t^{H,K} - \widetilde{B}_s^{H,K}\rangle}\mathrm{e}^{-\varepsilon|\xi|^2/2}\,\mathrm{d}\xi\,\mathrm{d}s\,\mathrm{d}t \\
&= \frac{1}{(2\pi)^d}\int_0^T\int_0^T\int_{\mathbb{R}^d} \mathrm{e}^{-\frac{1}{2}(a_{s,t}+\varepsilon)|\xi|^2} \\
&\quad \times \sum_{n=0}^{\infty} \mathrm{i}^n\sigma^n(t,s,\xi)H_n\Big(\frac{\langle\xi, B_t^{H,K} - \widetilde{B}_s^{H,K}\rangle}{\sigma(t,s,\xi)}\Big)\,\mathrm{d}\xi\,\mathrm{d}s\,\mathrm{d}t \equiv \sum_{n=0}^{\infty} F_n.
\end{aligned}$$

Thus, by (3.13) and Lemma 3.2 we have

$$\begin{aligned}
\Phi_{\Theta_\varepsilon}(1) &= \sum_{n=0}^{\infty} nE(|F_n|^2) \\
&= \sum_{n=0}^{\infty}\frac{n}{(2\pi)^{2d}}E\Big[\int_{[0,T]^4}\int_{\mathbb{R}^{2d}} \mathrm{e}^{-\frac{1}{2}((a_{s,t}+\varepsilon)|\xi|^2 + (a_{u,v}+\varepsilon)|\eta|^2)}\sigma^n(t,s,\xi)\sigma^n(u,v,\eta) \\
&\quad \times H_n\Big(\frac{\langle\xi, B_t^{H,K} - \widetilde{B}_s^{H,K}\rangle}{\sigma(t,s,\xi)}\Big)H_n\Big(\frac{\langle\eta, B_u^{H,K} - \widetilde{B}_v^{H,K}\rangle}{\sigma(u,v,\eta)}\Big)\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}s\,\mathrm{d}t\Big] \\
&= \sum_{n=1}^{\infty}\frac{1}{(2\pi)^{2d}(n-1)!}\int_{[0,T]^4}\varrho_{s,t,u,v}^n\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}s\,\mathrm{d}t \\
&\quad \times \int_{\mathbb{R}^{2d}} \mathrm{e}^{-\frac{1}{2}((a_{s,t}+\varepsilon)|\xi|^2 + (a_{u,v}+\varepsilon)|\eta|^2)}\langle\xi,\eta\rangle^n\,\mathrm{d}\xi\,\mathrm{d}\eta \\
&= \sum_{n=1}^{\infty}\frac{1}{(2\pi)^{2d}(2n-1)!}\int_{[0,T]^4}\varrho_{s,t,u,v}^{2n}\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}s\,\mathrm{d}t \\
&\quad \times \int_{\mathbb{R}^{2d}} \mathrm{e}^{-\frac{1}{2}((a_{s,t}+\varepsilon)|\xi|^2 + (a_{u,v}+\varepsilon)|\eta|^2)}\langle\xi,\eta\rangle^{2n}\,\mathrm{d}\xi\,\mathrm{d}\eta \\
&= \frac{1}{(2\pi)^d}\sum_{n=1}^{\infty}\sum_{\substack{k_1,\ldots,k_d=0\\ k_1+\ldots+k_d=n}}^{n}\frac{2n(2k_1-1)!!\cdot\ldots\cdot(2k_d-1)!!}{(2k_1)!!\cdot\ldots\cdot(2k_d)!!} \\
&\quad \times \int_{[0,T]^4}\frac{\varrho_{s,t,u,v}^{2n}}{((a_{s,t}+\varepsilon)(a_{u,v}+\varepsilon))^{n+d/2}}\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}s\,\mathrm{d}t \\
&\asymp \int_{[0,T]^4}\varrho_{s,t,u,v}^2((a_{s,t}+\varepsilon)(a_{u,v}+\varepsilon) - \varrho_{s,t,u,v}^2)^{-d/2-1}\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}s\,\mathrm{d}t.
\end{aligned}$$

It follows that

$$\lim_{\varepsilon\to 0}\Phi_{\Theta_\varepsilon}(1)\asymp\int_{[0,T]^4}\varrho^2_{s,t,u,v}(a_{s,t}a_{u,v}-\varrho^2_{s,t,u,v})^{-d/2-1}\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}s\,\mathrm{d}t$$

for all $T\geqslant 0$. This completes the proof. □

Now we are ready to prove Theorem 4.2.

P r o o f of Theorem 4.2. By Proposition 2.1 and Proposition 4.1 it suffices to show that

$$\int_{[0,T]^4}\varrho^2_{s,t,u,v}(a_{s,t}a_{u,v}-\varrho^2_{s,t,u,v})^{-d/2-1}\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}s\,\mathrm{d}t<\infty \tag{4.13}$$

if and only if $HK<2/(d+2)$. By (4.9) and (4.10) we have

$$a_{s,t}a_{u,v}-\varrho^2_{s,t,u,v}\asymp[x^{2HK}t^{2HK}+y^{2HK}s^{2HK}][(1-x)^{2HK}t^{2HK}+(1-y)^{2HK}s^{2HK}].$$

First, we give the proof of the necessary condition.

When $HK>\ 1/2$, we have

$$(1+x^{2H})^K-(1-x)^{2HK}\geqslant(1+x^{2H})^K-1+x^{2HK}\geqslant x^{2HK}$$

for all $x\in(0,1)$, which leads to

$$\varrho_{s,t,u,v}\geqslant\frac{1}{2^K}(t^{2HK}x^{2HK}+s^{2HK}y^{2HK}).$$

It follows that

$$\begin{aligned}
&\int_0^T\int_0^T\int_0^T\int_0^T(a_{s,t}a_{u,v}-\varrho^2_{s,t,u,v})^{-\frac{d}{2}-1}\varrho^2_{s,t,u,v}\,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}u\,\mathrm{d}v\\
&\quad\geqslant C_{T,H,K}\int_0^T\int_0^1\int_0^T\int_0^1\frac{(t^{2HK}x^{2HK}+s^{2HK}y^{2HK})st}{((1-x)^{2HK}t^{2HK}+(1-y)^{2HK}s^{2HK})^{1+\frac{d}{2}}}\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t\\
&\quad\geqslant C_{T,H,K}\int_0^1\int_0^1\int_0^1\int_0^1\frac{(t^{2HK}x^{2HK}+s^{2HK}y^{2HK})st}{((1-x)^{2HK}t^{2HK}+(1-y)^{2HK}s^{2HK})^{1+\frac{d}{2}}}\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t\\
&\quad\geqslant C_{T,H,K}\int_0^1\mathrm{d}y\int_0^y\mathrm{d}x\int_0^x\mathrm{d}t\int_0^t\mathrm{d}s\,\frac{s^{2HK+1}x^{2HK}}{t^{2HK(1+d/2)-1}(1-x)^{2HK(1+d/2)}}\\
&\quad\geqslant C_{T,H,K}\int_0^1\mathrm{d}y\int_0^y\frac{x^{4-HK(d-2)}}{(1-x)^{2HK(1+d/2)}}\,\mathrm{d}x\\
&\quad= C_{T,H,K}\int_0^1 x^{4-HK(d-2)}(1-x)^{1-2HK(1+d/2)}\,\mathrm{d}x,
\end{aligned}$$

which implies that $HK<2/(d+2)$ if the convergence (4.13) holds.

When $HK < \frac{1}{2}$, we have

$$(1+x^{2H})^K - (1-x)^{2HK} \geqslant K2^{K-1}x^{2H}$$

for $x \in (0,1)$, which leads to

$$\varrho_{s,t,u,v} \geqslant \frac{K}{2}(t^{2HK}x^{2H} + s^{2HK}y^{2H}).$$

It follows that

$$\begin{aligned}
&\int_0^T\int_0^T\int_0^T\int_0^T (a_{s,t}a_{u,v} - \varrho^2_{s,t,u,v})^{-(d+2)/2}\varrho^2_{s,t,u,v}\,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}u\,\mathrm{d}v \\
&\geqslant C_{T,H,K}\int_0^1\int_0^1\int_0^1\int_0^1 \\
&\quad\times \frac{(t^{2HK}x^{2H} + s^{2HK}y^{2H})^2 st\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t}{(x^{2HK}t^{2HK} + s^{2HK}y^{2HK})^{(d+2)/2}((1-x)^{2HK}t^{2HK} + (1-y)^{2HK}s^{2HK})^{(d+2)/2}} \\
&\geqslant C_{T,H,K}\int_0^1\int_0^1\int_0^1\int_0^1 \frac{(t^{2H}x^{2H} + s^{2H}y^{2H})^2 st}{((1-x)^{2HK}t^{2HK} + (1-y)^{2HK}s^{2HK})^{(d+2)/2}}\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t \\
&\geqslant C_{T,H,K}\int_0^1 \mathrm{d}y\int_0^y \mathrm{d}x\int_0^x \mathrm{d}t\int_0^t \mathrm{d}s\,\frac{s^{8H+1}}{t^{HK(d+2)-1}(1-x)^{HK(d+2)}} \\
&\geqslant C_{T,H,K}\int_0^1 \mathrm{d}y\int_0^y \frac{x^{8H+4-HK(d+2)}}{(1-x)^{HK(d+2)}}\,\mathrm{d}x = C_{T,H,K}T^{4-2HK}\int_0^1 \frac{x^{8H+4-HK(d+2)}}{(1-x)^{HK(d+2)-1}}\,\mathrm{d}x,
\end{aligned}$$

which implies that $HK < 2/(d+2)$ if the convergence (4.13) holds.

Now we give the proof of the sufficient condition. Notice that

$$\begin{aligned}
\varrho_{s,t,u,v} &= \frac{1}{2^K}t^{2HK}[(1+x^{2H})^K - (1-x)^{2HK}] \\
&\quad + \frac{1}{2^K}s^{2HK}[(1+y^{2H})^K - (1-y)^{2HK}] \leqslant 2T^{2HK}
\end{aligned}$$

for all $x, y \in (0,1)$ and $s, t \in (0,T)$. It follows that

$$\begin{aligned}
&\int_0^T\int_0^T\int_0^T\int_0^T (a_{s,t}a_{u,v} - \varrho^2_{s,t,u,v})^{-(d+2)/2}\varrho^2_{s,t,u,v}\,\mathrm{d}u\,\mathrm{d}s\,\mathrm{d}v\,\mathrm{d}t \\
&\quad\leqslant C_{H,K}\int_0^T\int_0^t\int_0^T\int_0^s \frac{T^{4HK}\,\mathrm{d}u\,\mathrm{d}s\,\mathrm{d}v\,\mathrm{d}t}{[v^{2HK} + u^{2HK})((t-v)^{2HK} + (s-u)^{2HK})]^{(d+2)/2}} \\
&\quad\leqslant C_{T,H,K}\int_0^T\int_0^t\int_0^T\int_0^s (u^{HK}v^{HK}(s-u)^{HK}(t-v)^{HK})^{-d/2-1}\,\mathrm{d}u\,\mathrm{d}s\,\mathrm{d}v\,\mathrm{d}t \\
&\quad= C_{T,H,K}\bigg(\int_0^T\int_0^s u^{-HK-HKd/2}(s-u)^{-HK-HKd/2}\,\mathrm{d}u\,\mathrm{d}s\bigg)^2 \\
&\quad= C_{T,H,K}\bigg(\int_0^T s^{1-2HK-HKd}\,\mathrm{d}s\int_0^1 y^{-HK-HKd/2}(1-y)^{-HK-HKd/2}\,\mathrm{d}y\bigg)^2 < \infty
\end{aligned}$$

if $HK < 2/(d+2)$. Thus, the proof is completed. □

Remark 4.1. Let $B^{H,K}$ be a bifractional Brownian motion and let W be a Brownian motion independent of $B^{H,K}$. Define the process $X^{H,K}$ as

$$X_t^{H,K} = \int_0^\infty (1 - \mathrm{e}^{-\theta t^{2H}}) \theta^{-(1+K)/2} \, \mathrm{d}W_\theta. \tag{4.14}$$

Then $X^{H,K}$ is a centered Gaussian process, and Lei-Nualart [9] showed that the following decomposition holds:

$$C_1 X_t^{H,K} + B_t^{H,K} \overset{d}{=} C_2 B_t^{HK}, \tag{4.15}$$

where $\overset{d}{=}$ means the equality in distributions, B^{HK} is a fractional Brownian motion with Hurst index HK and

$$C_1 = \sqrt{\frac{2^{-K}K}{\Gamma(1-K)}}, \qquad C_2 = 2^{(1-K)/2}.$$

Thus, if we could show that the collision local times of X^{H_1,K_1} and X^{H_2,K_2} and the intersection local times of $X^{H,K}$ and $\widetilde{X}^{H,K}$ are smooth in the sense of Meyer-Watanabe, then the main results in this paper could be proved briefly.

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References

[1] *L. An, L. Yan*: Smoothness for the collision local time of fractional Brownian motion. Preprint, 2010.
[2] *C. Chen, L. Yan*: Remarks on the intersection local time of fractional Brownian motions. Stat. Probab. Lett. *81* (2011), 1003–1012.
[3] *K. Es-Sebaiy, C. A. Tudor*: Multidimensional bifractional Brownian motion: Itô and Tanaka formulas. Stoch. Dyn. *7* (2007), 365–388.
[4] *Ch. Houdré, J. Villa*: An example of infinite dimensional quasi-helix. Stochastic models. Seventh symposium on probability and stochastic processes, June 23–28, 2002, Mexico City, Mexico. Selected papers. Providence, RI: American Mathematical Society (AMS), Contemp. Math. *336* (2003), 195–201.
[5] *Y. Hu*: Self-intersection local time of fractional Brownian motion - via chaos expansion. J. Math, Kyoto Univ. *41* (2001), 233–250.
[6] *Y. Hu*: Integral transformations and anticipative calculus for fractional Brownian motions. Mem. Am. Math. Soc. *825* (2005).
[7] *Y. Jiang, Y. Wang*: Self-intersection local times and collision local times of bifractional Brownian motions. Sci. China, Ser. A *52* (2009), 1905–1919.
[8] *I. Kruk, F. Russo, C. A. Tudor*: Wiener integrals, Malliavin calculus and covariance measure structure. J. Funct. Anal. *249* (2007), 92–142.
[9] *P. Lei, D. Nualart*: A decomposition of the bifractional Brownian motion and some applications. Stat. Probab. Lett. *79* (2009), 619–624.

[10] *Y. Mishura*: Stochastic Calculus for Fractional Brownian Motions and Related Processes. Lecture Notes in Mathematics 1929. Springer, Berlin, 2008.
[11] *D. Nualart, S. Ortiz-Latorre*: Intersection local time for two independent fractional Brownian motions. J. Theor. Probab. *20* (2007), 759–767.
[12] *D. Nualart*: The Malliavin Calculus and Related Topics. 2nd ed. Probability and Its Applications. Springer, Berlin, 2006.
[13] *F. Russo, C. A. Tudor*: On bifractional Brownian motion. Stochastic Processes Appl. *116* (2006), 830–856.
[14] *G. Shen, L. Yan*: Smoothness for the collision local times of bifractional Brownian motions. Sci. China, Math. *54* (2011), 1859–1873.
[15] *C. A. Tudor, Y. Xiao*: Sample path properties of bifractional Brownian motion. Bernoulli *13* (2007), 1023–1052.
[16] *S. Watanabe*: Lectures on Stochastic Differential Equations and Malliavin Calculus. Lectures on Mathematics and Physics. Mathematics, 73. Tata Institute of Fundamental Research. Springer, Berlin, 1984.
[17] *L. Yan, J. Liu, C. Chen*: On the collision local time of bifractional Brownian motions. Stoch. Dyn. *9* (2009), 479–491.
[18] *L. Yan, B. Gao, J. Liu*: The Bouleau-Yor identity for a bi-fractional Brownian motion. To appear in Stochastics 2012.

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