Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 62 (2012), No. 4, 1003–1009

Persistent URL: `http://dml.cz/dmlcz/143041`

Czechoslovak Mathematical Journal, 62 (137) (2012), 1003–1009

CONVEX DOMINATION IN THE COMPOSITION AND CARTESIAN PRODUCT OF GRAPHS

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(Received August 4, 2011)

Abstract. In this paper we characterize the convex dominating sets in the composition and Cartesian product of two connected graphs. The concepts of clique dominating set and clique domination number of a graph are defined. It is shown that the convex domination number of a composition $G[H]$ of two non-complete connected graphs G and H is equal to the clique domination number of G. The convex domination number of the Cartesian product of two connected graphs is related to the convex domination numbers of the graphs involved.

Keywords: convex dominating set, convex domination number, clique dominating set, composition, Cartesian product

MSC 2010: 05C69

1. INTRODUCTION

Let $G = (V(G), E(G))$ be a connected simple graph. For any two vertices x and y of G, the symbol $d_G(x, y)$ is the length of a shortest path connecting vertices x and y in G. Any x-y path of length $d_G(x, y)$ is called an *x-y geodesic*. A set $C \subseteq V(G)$ is *convex* in G if, for every two vertices $x, y \in C$, the vertex set of every x-y geodesic is contained in C. The concept of convexity in graphs is discussed in the book by Buckley and Harary [1]. This concept was also investigated in [2] and [3]. In [2], the authors characterized the convex sets in graphs resulting from some binary operations such as the join, composition, and Cartesian product of graphs.

A subset S of $V(G)$ is a *dominating set* in G if for every $v \in V(G) \setminus S$, there exists $x \in S$ such that $xv \in E(G)$. If S is both convex and a dominating set, then it is a *convex dominating set*. If S is a clique (the induced graph $\langle S \rangle$ is complete) and a dominating set, then S is called a *clique dominating set* in G. The *domination number* (resp. *convex domination number* and *clique domination number*) $\gamma(G)$

(resp. $\gamma_{\rm con}(G)$ and $\gamma_{\rm cl}(G)$) of G is the smallest cardinality of a dominating (resp. convex dominating and clique dominating) set in G. A dominating set S in G is called a *minimum dominating set* if the cardinality of S is equal to $\gamma(G)$. Minimum convex dominating and minimum clique dominating sets are defined similarly. Various types of domination in graphs and some corresponding results may be found in [4] and [5]. Convex domination is studied and investigated in [6].

In this paper we characterize the convex dominating sets in the composition and Cartesian product of two connected graphs. As quick consequences, the convex domination numbers of the composition and Cartesian product of two connected graphs are determined.

2. Convex domination in the composition of two connected graphs

The *composition* $G[H]$ of two graphs G and H is the graph with $V(G[H]) = V(G) \times V(H)$ and $(u,u')(v,v') \in E(G[H])$ if and only if either $uv \in E(G)$ or $u = v$ and $u'v' \in E(H)$.

The first result is due to Canoy and Garces [2].

Theorem 2.1. *Let G be a connected graph and K_n the complete graph of order n. A subset $C = \bigcup\limits_{x\in S} (\{x\} \times T_x)$ of $V(G[K_n])$, where $S \subseteq V(G)$ and $T_x \subseteq V(K_n)$ for every $x \in S$, is convex in $G[K_n]$ if and only if S is convex in G.*

The next result characterizes the convex dominating sets in $G[K_n]$.

Theorem 2.2. *Let G be a connected graph and K_n the complete graph of order $n \geqslant 2$. A subset $C = \bigcup\limits_{x\in S} (\{x\} \times T_x)$ of $V(G[K_n])$ is a convex dominating set in $G[K_n]$ if and only if S is a convex dominating set in G.*

P r o o f. Suppose C is a convex dominating set in $G[K_n]$. By Theorem 2.1, S is a convex set in G. Let $x \in V(G) \setminus S$ and let $a \in V(K_n)$. Then $(x,a) \notin C$. Since C is a dominating set, there exists $(z,b) \in C$ such that $(x,a)(z,b) \in E(G[K_n])$. It follows that $z \in S$ and $xz \in E(G)$. Thus S is a dominating set in G.

For the converse, suppose that S is a convex dominating set in G. By Theorem 2.1, C is a convex set in $G[K_n]$. Let $(x,y) \in V(G[K_n]) \setminus C$. Consider the following cases:

Case 1. Suppose $x \in S$. Then $y \notin T_x$. Pick $p \in T_x$. Then $(x,p) \in C$ and $(x,y)(x,p) \in E(G[K_n])$.

Case 2. Suppose $x \notin S$. By assumption, there exists $z \in S$ such that $xz \in E(G)$. Choose $q \in T_z$. Then $(z,q) \in C$ and $(x,y)(z,q) \in E(G[K_n])$.

Accordingly, C is a dominating set in $G[K_n]$. □

Corollary 2.3. *Let G be a connected graph and K_n the complete graph of order $n \geqslant 1$. Then $\gamma_{\mathrm{con}}(G[K_n]) = \gamma_{\mathrm{con}}(G)$.*

Proof. If $n = 1$, then $G[K_n] \cong G$. Hence $\gamma_{\mathrm{con}}(G[K_n]) = \gamma_{\mathrm{con}}(G)$ for $n = 1$. So suppose $n \geqslant 2$. Let $C = \bigcup_{x \in S} (\{x\} \times T_x)$ be a minimum convex dominating set in $G[K_n]$. Then S is a convex dominating set in G by Theorem 2.2. It follows that

$$\gamma_{\mathrm{con}}(G[K_n]) = |C| \geqslant |S| \geqslant \gamma_{\mathrm{con}}(G).$$

Now, let S be a minimum convex dominating set in G. For each $x \in S$, let $T_x = \{a\}$, where $a \in V(K_n)$. Then $C = \bigcup_{x \in S} (\{x\} \times T_x)$ is a convex dominating set in $G[K_n]$ by Theorem 2.2. Therefore,

$$\gamma_{\mathrm{con}}(G) = |S| = |C| \geqslant \gamma_{\mathrm{con}}(G[K_n]).$$

Therefore, $\gamma_{\mathrm{con}}(G[K_n]) = \gamma_{\mathrm{con}}(G)$. □

The next result is found in [1].

Theorem 2.4. *Let G and H be connected non-complete graphs. A subset $C = \bigcup_{x \in S} (\{x\} \times T_x)$ of $V(G[H])$ is convex in $G[H]$ if and only if S is a clique in G and T_x is a clique in H for each $x \in S$.*

Theorem 2.5. *Let G and H be connected non-complete graphs with $\gamma_{\mathrm{cl}}(G) \geqslant 2$. A subset $C = \bigcup_{x \in S} (\{x\} \times T_x)$ of $V(G[H])$ is a convex dominating set in $G[H]$ if and only if S is a clique dominating set in G and T_x is a clique in H for each $x \in S$.*

Proof. Suppose C is a convex dominating set in $G[H]$. By Theorem 2.4, S is a clique in G and T_x is a clique in H for each $x \in S$. Let $y \in V(G) \setminus S$ and choose $a \in V(H)$. Then $(y, a) \notin C$. Since C is a dominating set, there exists $(z, b) \in C$ such that $(y, a)(z, b) \in E(G[H])$. Clearly, $z \in S$ and $yz \in E(G)$. This shows that S is a (convex) dominating set in G.

For the converse, suppose that S is a clique dominating set in G and T_x is a clique in H for each $x \in S$. Then $C = \bigcup_{x \in S} (\{x\} \times T_x)$ of $V(G[H])$ is a convex set in $G[H]$ by Theorem 2.4. Now let $(x, y) \in V(G[H]) \setminus C$ and consider the following cases:

Case 1. Suppose $x \in S$. Then $y \notin T_x$. Since $\gamma_{\mathrm{cl}}(G) \geqslant 2$, we may choose $z \in S$ such that $x \neq z$. Since S is a clique, $xz \in E(G)$. If $c \in T_z$, then $(z, c) \in C$ and $(x, y)(z, c) \in E(G[H])$.

Case 2. Suppose $x \notin S$. By assumption, there exists $w \in S$ such that $xw \in E(G)$. Choose $d \in T_w$. Then $(w, d) \in C$ and $(x, y)(w, d) \in E(G[H])$.

Accordingly, C is a (convex) dominating set in $G[H]$. □

Corollary 2.6. *Let G and H be connected non-complete graphs with $\gamma_{\rm cl}(G) \geqslant 2$. Then $\gamma_{\rm con}(G[H]) = \gamma_{\rm cl}(G)$.*

P r o o f. Let $C = \bigcup_{x\in S} (\{x\} \times T_x)$ be a minimum convex dominating set in $G[H]$. Then S is a clique dominating set in G and T_x is a clique in H for each $x \in S$ by Theorem 2.5. It follows that

$$\gamma_{\rm con}(G[H]) = |C| \geqslant |S| \geqslant \gamma_{\rm cl}(G).$$

Now, let S^* be a minimum clique dominating set in G. For each $x \in S^*$, let $T^*_x = \{a\}$, where $a \in V(H)$. Then $C^* = \bigcup_{x\in S^*} (\{x\} \times T^*_x)$ is a convex dominating set in $G[H]$ by Theorem 2.5. Therefore,

$$\gamma_{\rm cl}(G) = |S^*| = |C^*| \geqslant \gamma_{\rm con}(G[H]).$$

Therefore, $\gamma_{\rm con}(G[K_n]) = \gamma_{\rm cl}(G)$. □

Theorem 2.7. *Let G and H be connected non-complete graphs with $\gamma_{\rm cl}(G) = \gamma(G) = 1$. A subset $C = \bigcup_{x\in S} (\{x\} \times T_x)$ of $V(G[H])$ is a convex dominating set in $G[H]$ if and only if*

(i) *S is a clique dominating set with $|S| \geqslant 2$ and T_x is a clique in H for each $x \in S$, or*

(ii) *S is a (clique) dominating set with $|S| = 1$ and T_x is a clique dominating set in H for each $x \in S$.*

P r o o f. Suppose C is a convex dominating set in $G[H]$. Then S is a clique in G and T_x is a clique in H for each $x \in S$ by Theorem 2.4. Let $y \in V(G) \setminus S$ and choose $a \in V(H)$. Then $(y, a) \notin C$. Hence there exists $(z, b) \in C$ such that $(y, a)(z, b) \in E(G[H])$. This implies that $z \in S$ and $zy \in E(G)$. Thus S is a dominating set in G. If $|S| \geqslant 2$, then we are done. So suppose $|S| = 1$. Let $x \in S$ and let $a \in V(H) \setminus T_x$. Then $(x, a) \notin C$. Since C is a dominating set, there exists $(w, b) \in C$ such that $(x, a)(w, b) \in E(G[H])$. It follows that $x = w$. Thus, $b \in T_x$ and $ab \in E(H)$. This shows that T_x is a dominating set in H for every $x \in S$.

For the converse, suppose first that (i) holds. Then it is routine to show that C is a convex dominating set in $G[H]$). Next, suppose that (ii) holds. Then C is a convex set in $G[H]$. Let $(u, v) \in V(G[H]) \setminus C$ and consider the following cases:

Case 1. Suppose $u \in S$. Then $v \notin T_u$. Since T_x is a dominating set in H, there exists $q \in T_u$ such that $qv \in E(H)$. Hence $(u, q) \in C$ and $(u, v)(u, q) \in E(G[H])$.

Case 2. Suppose $u \notin S$. By assumption, there exists $w \in S$ such that $uw \in E(G)$. Choose $a \in T_w$. Then $(w, a) \in C$ and $(u, v)(w, a) \in E(G[H])$.

Accordingly, C is a (convex) dominating set in $G[H]$. □

Corollary 2.8. *Let G and H be connected non-complete graphs with $\gamma_{cl}(G) = 1$. Then $\gamma_{con}(G[H]) = 1$ if $\gamma(H) = 1$ and $\gamma_{con}(G[H]) = 2$ if $\gamma(H) \neq 1$.*

Proof. Suppose $\gamma(H) = 1$. Let $S_1 = \{x\}$ and $S_2 = \{a\}$ be dominating sets of G and H, respectively. Then $C = \{(x, a)\}$ is clearly a convex dominating set of $G[H]$. Thus $\gamma_{con}(G[H]) = |C| = 1$.

Next, suppose that $\gamma(H) \geqslant 2$. Suppose $D = \{(u, v)\}$ is a dominating set of $G[H]$. If $c \in V(H) \setminus \{v\}$, then $(u, c) \notin D$; hence $(u, c)(u, v) \in E(G[H])$. It follows that $cv \in E(H)$. This implies that $T = \{v\}$ is a dominating set in H, contrary to our assumption that $\gamma(H) \neq 1$. Therefore, $\gamma_{con}(G[H]) \geqslant 2$. Now, let $S' = \{z\}$ be a (convex) dominating set in G. Choose $w \in V(G) \setminus \{z\}$ such that $zw \in E(G)$ and set $S = \{z, w\}$. Pick $a \in V(H)$ and set $T_z = T_w = \{a\}$. Then $C = \{(z, a), (w, a)\}$ is a convex dominating set by Theorem 2.7(i). Therefore $\gamma_{con}(G[H]) = |C| = 2$. □

3. Convex domination in the Cartesian product of two connected graphs

The *Cartesian product* $G \times H$ of two graphs G and H is the graph with $V(G \times H) = V(G) \times V(H)$ and $(u, u')(v, v') \in E(G \times H)$ if and only if either $uv \in E(G)$ and $u' = v'$ or $u = v$ and $u'v' \in E(H)$.

The following result in [2] characterizes the convex sets in the Cartesian product of two connected graphs.

Theorem 3.1. *Let G and H be connected graphs. A subset C of $V(G \times H)$ is convex if and only if $C = C_1 \times C_2$, where C_1 and C_2 are convex sets in G and H, respectively.*

Lemma 3.2. *Let G and H be connected graphs. If a subset $C = C_1 \times C_2$ of $V(G \times H)$ is a dominating set in $G \times H$, then C_1 and C_2 are dominating sets in G and H, respectively.*

Proof. Suppose C_1 is not a dominating set in G. Then there exists $x \in V(G) \setminus C_1$ such that $xy \notin E(G)$ for all $y \in C_1$. Pick $z \in V(H)$. Then $(x, z) \in V(G \times H) \setminus (C_1 \times C_2)$. Since $x \notin C_1$, $(x, q) \notin C_1 \times C_2$ for all $q \in C_2$. Also, since $xy \notin E(G)$ for all $y \in C_1$, it follows that $(x, z)(y, z) \notin E(G \times H)$ for all $y \in C_1$. This implies that $C_1 \times C_2$ is not a dominating set in $G \times H$, contrary to our assumption. Therefore, C_1 is a dominating set in G. Similarly, C_2 is a dominating set in H. □

Theorem 3.3. *Let G and H be connected graphs. A subset C of $V(G \times H)$ is a convex dominating set in $G \times H$ if and only if $C = C_1 \times C_2$ and*

(i) *C_1 is a convex dominating set in G and $C_2 = V(H)$, or*

(ii) *C_2 is a convex dominating set in H and $C_1 = V(G)$.*

Proof. Suppose that C is a convex dominating set in $G \times H$. Then, by Theorem 3.1 $C = C_1 \times C_2$, where C_1 and C_2 are convex sets in G and H, respectively. By Lemma 3.2, C_1 and C_2 are dominating sets in G and H, respectively. Now, suppose that $C_1 \neq V(G)$ and $C_2 \neq V(H)$. Pick $x \in V(G) \setminus C_1$ and $y \in V(H) \setminus C_2$. Then (x,p), $(q,y) \notin C$ for all $p \in C_2$ and for any $q \in C_1$. This implies that there exists no $(u,v) \in C$ such that $(x,y)(u,v) \in E(G \times H)$. Therefore C is not a dominating set in $G \times H$, contrary to our assumption. Accordingly, $C_1 = V(G)$ or $C_2 = V(H)$.

For the converse, suppose that (i) holds. Then $C = C_1 \times C_2$ is convex by Theorem 3.1. Let $(z,w) \in V(G \times H) \setminus (C_1 \times C_2)$. Then $z \notin C_1$. Let $u \in C_1$ such that $uz \in E(G)$. Then $(u,w) \in C_1 \times C_2$ and $(z,w)(u,w) \in E(G \times H)$. Therefore $C = C_1 \times C_2$ is a dominating set in $G \times H$. Using a similar argument, we can show that C is a convex dominating set in $G \times H$ if condition (ii) holds. □

Corollary 3.4. *Let G and H be connected graphs of orders m and n, respectively. Then $\gamma_{\text{con}}(G \times H) = \min\{m\gamma_{\text{con}}(H), n\gamma_{\text{con}}(G)\}$.*

Proof. Let $C = C_1 \times C_2$ be a minimum convex dominating set in $G \times H$. By Theorem 3.3, C_1 is a convex dominating set in G and $C_2 = V(H)$ or C_2 is a convex dominating set in H and $C_1 = V(G)$. Thus $\gamma_{\text{con}}(G \times H) = |C_1||C_2| \geqslant \min\{m\gamma_{\text{con}}(H), n\gamma_{\text{con}}(G)\}$.

Suppose that C_1 is a minimum convex dominating set in G. Then $C = C_1 \times V(H)$ is a convex dominating set in $G \times H$ by Theorem 3.3. Thus $|C| = n\gamma_{\text{con}}(G) \geqslant \gamma_{\text{con}}(G \times H)$. If C_2 is a minimum convex dominating set in H, then $C = V(G) \times C_2$ is a convex dominating set in $G \times H$ by Theorem 3.3. It follows that $|C| = m\gamma_{\text{con}}(H) \geqslant \gamma_{\text{con}}(G \times H)$. Therefore $\gamma_{\text{con}}(G \times H) \leqslant \min\{m\gamma_{\text{con}}(H), n\gamma_{\text{con}}(G)\}$. This proves the desired equality. □

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