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EQUATION $f(p(x)) = q(f(x))$ FOR GIVEN REAL FUNCTIONS p, q

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Abstract. We investigate functional equations $f(p(x)) = q(f(x))$ where p and q are given real functions defined on the set $\mathbb{R}$ of all real numbers. For these investigations, we can use methods for constructions of homomorphisms of mono-unary algebras. Our considerations will be confined to functions p, q which are strictly increasing and continuous on $\mathbb{R}$. In this case, there is a simple characterization for the existence of a solution of the above equation. First, we give such a characterization. Further, we present a construction of any solution of this equation if some exists. This construction is demonstrated in detail and discussed by means of an example.

Keywords: homomorphism of mono-unary algebras, functional equation, strictly increasing continuous real functions

MSC 2010: 08A60, 65Q20, 97I70

A group of mathematicians around Jan Chvalina have investigated functional equations $f(p(x)) = q(f(x))$ for a couple of given real functions p and q—see papers [1]–[9]. They have demonstrated the importance of such functional equations e.g. in works [14]–[16]. A large collection of functions p, q they considered were strictly increasing continuous functions ([8], [9]).

Therefore, we concentrate our considerations of the above equations on p and q which are strictly increasing continuous real functions.

More exactly, let $\mathbb{R}$ be the set of all real numbers. We investigate the equation

$$f(p(x)) = q(f(x)) \tag{1}$$

where $p, q\colon \mathbb{R} \to \mathbb{R}$ are given real functions and $f\colon \mathbb{R} \to \mathbb{R}$ a searched real one—as a so called solution f of this equation.

If we consider mono-unary algebras $(\mathbb{R}, p)$ and $(\mathbb{R}, q)$ where $p, q\colon \mathbb{R} \to \mathbb{R}$ are real functions, then the form of equation (1) implies the following. A function $f\colon \mathbb{R} \to \mathbb{R}$

is a solution of (1) if and only if f is a homomorphism of $(\mathbb{R}, p)$ into $(\mathbb{R}, q)$. Hence we can look for solutions of the functional equation (1) using the methods of mono-unary algebras.

1. Existence of solutions

As usual, under a mono-unary algebra we will understand a pair (A, o) where A is a set and o a unary operation on A. Let $\mathbb{N}$ denote the set of natural numbers (or non negative integers) and let $\mathbb{Z}$ denote the set of all integers. For any $n \in \mathbb{N}$, o^n denotes the n-th iteration of o and further, for any $x \in A$, $o^{-1}(x)$ denotes the set of all origins of x for the mapping o; if o is injective and $n \in \mathbb{N}$, $n > 0$ is arbitrary, then o^{-1} denotes the inverse mapping of o and we denote $o^{-n} = (o^{-1})^n$.

We will use basic concepts for mono-unary algebras, as a *connected* mono-unary algebra (i.e. (A, o) such that, for any $x, y \in A$, there are $m, n \in \mathbb{N}$ with $o^m(x) = o^n(y)$) and a *homomorphism* of (A, o) into a mono-unary algebra (A', o') (i.e. a mapping $h\colon A \to A'$ with $h(o(x)) = o'(h(x))$ for any $x \in A$). For a basis of the theory of mono-unary algebras—see for example [10] or [17].

By $\mathbb{C}_n$ we will denote a cycle with n elements, i.e. a mono-unary algebra $(\{a_1, a_2, \ldots, a_n\}, o)$ such that $o(a_i) = a_{i+1}$ for $i = 1, 2, \ldots, n-1$ and $o(a_n) = a_1$ hold. Further, we will use mono-unary algebras $(\mathbb{N}, \nu_0)$ and $(\mathbb{Z}, \nu)$ where ν is a unary operation on $\mathbb{Z}$ defined by $\nu(x) = x + 1$ for any $x \in \mathbb{Z}$ and $\nu_0 = \nu \cap \mathbb{N}^2$ holds. To simplify matters, we will denote both operations ν_0 and ν by the same symbol ν.

Furthermore, if A is a set, then $\operatorname{card} A$ will denote the cardinal number of A. And finally, the symbol $\simeq$ denotes the existence of an isomorphism between two algebras.

Lemma 1. *Let (A, o) be a connected mono-unary algebra. Then o is injective if and only if*

$$(A, o) \simeq \begin{cases} \mathbb{C}_{\operatorname{card} A} & \text{for } A \text{ finite,} \\ (\mathbb{Z}, \nu) & \text{for } A \text{ infinite and } o \text{ surjective,} \\ (\mathbb{N}, \nu) & \text{otherwise} \end{cases}$$

is satisfied.

Proof. If (A, o) has one of the described forms, then for any $x \in A$, $\operatorname{card} o^{-1}(x) \leqslant 1$ holds and so o is injective.

On the other hand, let o be injective.

First, let $\operatorname{card} A < \aleph_0$. Then there is a cycle $Z \subseteq A$. If we had $Z \neq A$, then there would exist $x \in A - Z$ such that $o(x) \in Z$. Thus, we would have $\operatorname{card} o^{-1}(o(x)) > 1$ which would be a contradiction. Therefore, $Z = A$.

Further, let $\operatorname{card} A \geqslant \aleph_0$ hold. If there is $a \in A$ with $\operatorname{card} o^{-1}(a) = 0$ (o is not surjective), then the mapping $\alpha\colon A \to \mathbb{N}$ such that $\alpha(o^n(a)) = n$ for any $n \in N$ is an isomorphism of (A, o) and $(\mathbb{N}, \nu)$. Otherwise, if $\operatorname{card} o^{-1}(x) = 1$ for any $x \in A$ and $a \in A$ is arbitrary, then the mapping $\beta\colon A \to \mathbb{Z}$ such that $\beta(o^n(a)) = n$ for any $n \in \mathbb{Z}$ is an isomorphism of (A, o) and $(\mathbb{Z}, \nu)$. $\square$

Starting from now, if we speak about components of a mono-unary algebra, then we always mean *connected components* of this algebra.

Corollary 2. *Let (A, o) be a mono-unary algebra where o is injective. Then the following assertions are valid.*

a) *The components of (A, o) are either cycles or they are isomorphic to $(\mathbb{Z}, \nu)$ or to $(\mathbb{N}, \nu)$.*

b) *If o is surjective on A, then the components of (A, o) are either cycles or they are isomorphic to $(\mathbb{Z}, \nu)$.*

Now, we want to consider a special case of mono-unary algebras. They are mono-unary algebras on the set of real numbers $\mathbb{R}$, i.e. algebras of the kind $(\mathbb{R}, p)$ where p is a function $p\colon \mathbb{R} \to \mathbb{R}$.

At the same time, we will suppose that p is injective and, moreover, we will choose the case that p is strictly increasing. Strictly increasing (stricly decreasing) functions or sequences will be called increasing (decreasing) in short.

First, we want to mention some properties of functions that are generated by a given function p. (As above, if $p\colon \mathbb{R} \to \mathbb{R}$ is a function, then p^{-1} denotes the inverse function to p, if some exists and further, if $n \in N$ is arbitrary and $(p^{-1})^n$ exists, then we denote $p^{-n} = (p^{-1})^n$.) The following fact is well-known.

Lemma 3. *Let $p\colon \mathbb{R} \to \mathbb{R}$ be an increasing function. Then the function p^{-1} exists and is increasing.*

Further, for any increasing function p on $\mathbb{R}$, the iterations of p have some simple properties. At the same time, we can form "iterations" of p with negative powers as well. For this, if $\alpha \in \mathbb{N} \cup \{\omega_0\}$ is arbitrary (where ω_0 is the ordinal type of $\mathbb{N}$), then we denote $W(\alpha) = \{n \in \mathbb{N};\ n < \alpha\}$ as usual.

Let $x \in \mathbb{R}$ be arbitrary. Furthermore, let α be the greatest ordinal number such that $p^{-n}(x)$ is defined for any $n \in W(\alpha)$. Then α is a finite ordinal number or ω_0. We can investigate the sequence $(p^{-n}(x))_{n \in W(\alpha)}$.

Lemma 4. *Let $p\colon \mathbb{R} \to \mathbb{R}$ be an increasing function. Let $x \in \mathbb{R}$ be arbitrary. Then the sequence*

a) *$(p^n(x))_{n\in\mathbb{N}}$ is increasing and $(p^{-n}(x))_{n\in W(\alpha)}$ is decreasing if and only if $p(x) > x$;*

b) *$(p^n(x))_{n\in\mathbb{N}}$ is decreasing and $(p^{-n}(x))_{n\in W(\alpha)}$ is increasing if and only if $p(x) < x$.*

Proof. We prove a). The necessity of the condition is clear. On the other hand, let $p(x) > x$ hold. Then, by induction, $p^{n+1}(x) > p^n(x)$ for any $n \in \mathbb{N}$ because p is increasing. Further, for $\alpha = 1$, we have nothing more to prove. Let $\alpha > 1$. Then $p^{-1}(x)$ is defined and since $p^{-1}(p(x))$ is defined too, the condition $p(x) > x$ implies $x = p^{-1}(p(x)) > p^{-1}(x)$ because p^{-1} is increasing. Then $p^{-(n+1)}(x) < p^{-n}(x)$ for any $n \in \mathbb{N}$ with $n+1 \in W(\alpha)$ by induction.

The assertion b) is dual to a). □

Corollary 5. *Let $p\colon \mathbb{R} \to \mathbb{R}$ be an increasing function. Then the following assertions are valid.*

a) *$(\mathbb{R}, p)$ is a mono-unary algebra whose components are isomorphic to $\mathbb{C}_1$ or to $(\mathbb{Z}, \nu)$ or to $(\mathbb{N}, \nu)$.*

b) *If $x \in \mathbb{R}$ is arbitrary such that $p(x) > x$ ($p(x) < x$) holds, then the component containing x is an increasing (decreasing) sequence of real numbers.*

Indeed, the assertion a) is a consequence of Lemma 1 where the cycles cannot have more than just one element by Lemma 4. Furthermore, b) is a direct consequence of assertion a) and Lemma 4.

Another well-known fact for increasing functions is the following lemma.

Lemma 6. *Let $p\colon \mathbb{R} \to \mathbb{R}$ be an increasing function. Then $\lim\limits_{x\to-\infty} p(x)$ and $\lim\limits_{x\to\infty} p(x)$ exist.*

We denote

$$p(-\infty) = \lim_{x\to-\infty} p(x)$$

and

$$p(\infty) = \lim_{x\to\infty} p(x)$$

for any increasing function $p\colon \mathbb{R} \to \mathbb{R}$.

In what follows, we confine our considerations to *continuous* increasing functions. For these functions, the following facts are well-known.

Lemma 7. *Let $p\colon \mathbb{R} \to \mathbb{R}$ be an increasing continuous function. Then $p(\mathbb{R}) = (p(-\infty), p(\infty))$ holds and thus, p is surjective on $\mathbb{R}$ if and only if $p(-\infty) = -\infty$ and $p(\infty) = \infty$.*

Lemma 8. *Let $p\colon \mathbb{R} \to \mathbb{R}$ be an increasing continuous function and let $a, b \in \mathbb{R} \cup \{-\infty, \infty\}$, $a < b$ be arbitrary. Then $p((a,b)) = (p(a), p(b))$ holds.*

Proof. If $x \in (a,b)$ is arbitrary, then $a < x < b$ and so $p(a) < p(x) < p(a)$ because p is increasing. Therefore, $p(x) \in (p(a), p(b))$ holds and, altogether, we have $p((a,b)) \subseteq (p(a), p(b))$.

On the other hand, let $y \in (p(a), p(b))$ be arbitrary. Then there is $x \in \mathbb{R}$ such that $p(x) = y$ because p is continuous on $\mathbb{R}$. But, since p is increasing, $x \leqslant a$ would imply $y = p(x) \leqslant p(a)$ which would be a contradiction and, similarly, $x \geqslant b$ would lead to a contradiction too. This implies $x \in (a,b)$ and so we have $y = p(x) \in p((a,b))$. Hence, altogether, we obtain $(p(a), p(b)) \subseteq p((a,b))$. $\square$

Now, the assertion in Lemma 7 can be formulated for intervals in $\mathbb{R}$ generally.

Lemma 9. *Let $p\colon \mathbb{R} \to \mathbb{R}$ be an increasing continuous function. Let $a, b \in \mathbb{R} \cup \{-\infty, \infty\}, a < b$ be arbitrary. Then $p((a,b)) = (a,b)$ holds if and only if $p(a) = a$ and $p(b) = b$.*

Proof. The condition is sufficient by Lemma 8.

On the other hand, the condition is necessary. Indeed, let $p((a,b)) = (a,b)$ be satisfied. Then $p(x) < b$ for any $x \in (a,b)$. Since p is continuous in b, we have $\lim\limits_{x \to b_-} p(x) = p(b)$ which implies $p(b) \leqslant b$. Moreover, $p(x) < p(b)$ holds for any $x \in (a,b)$ because p is increasing. Now, if we had $p(b) < b$, then we would have $p(x) < p(b) < b$ for any $x \in (a,b)$ which would be a contradiction to the surjectivity of p on (a,b). Hence $p(b) = b$. Analogously, we can show that $p(a) = a$ holds. $\square$

In algebraic setting, we obtain the following assertion.

Corollary 10. *Let $p\colon \mathbb{R} \to \mathbb{R}$ be an increasing continuous function. Let $a, b \in \mathbb{R} \cup \{-\infty, \infty\}$, $a < b$ be arbitrary. Then $((a,b), p|(a,b))$ is a subalgebra of $(\mathbb{R}, p)$ with components isomorphic to $(\mathbb{Z}, \nu)$ or to $\mathbb{C}_1$ if and only if $p(a) = a$ and $p(b) = b$ hold.*

Indeed, this assertion is a consequence of Lemma 9 and Corollary 2 b) because of surjectivity of p on the interval (a,b).

Example 11. Let $p(x) = x^3$ for any $x \in \mathbb{R}$. Then p is increasing and continuous on $\mathbb{R}$ and $(\mathbb{R}, p)$ is a mono-unary algebra such that $p(a) = a$ for $a \in \{-1, 0, 1\}$. Its subalgebras $(\{a\}, p|\{a\})$ for $a \in \{-1, 0, 1\}$ are isomorphic to $\mathbb{C}_1$ and further, the components of the subalgebra $((a,b), p|(a,b))$ for any $(a,b) \in \{(-\infty, -1), (-1, 0), (0, 1), (1, \infty)\}$ are isomorphic to $(\mathbb{Z}, \nu)$.

Further, for any increasing continuous function p on $\mathbb{R}$, the iterations of p have some other simple properties.

Corollary 12. *Let $p\colon \mathbb{R} \to \mathbb{R}$ be a continuous increasing function. Let $x \in \mathbb{R}$ be arbitrary. Then the following assertions hold.*

a) $\lim\limits_{n\to\infty} p^n(x) = a$ *exists for some* $a \in \mathbb{R} \cup \{-\infty, \infty\}$ *and* $p(a) = a$ *holds.*

b) *Let* $p^{-n}(x)$ *be defined for any* $n \in \mathbb{N}$. *Then* $\lim\limits_{n\to\infty} p^{-n}(x) = a$ *exists for some* $a \in \mathbb{R} \cup \{-\infty, \infty\}$ *and* $p(a) = a$ *holds.*

Proof. a) If $p(x) = x$, then $p^n(x) = x$ for any $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} p^n(x) = x = p(x)$ holds which is the assertion of a). Let, on the other hand, $p(x) > x$ hold. Then the sequence $(p^n(x))_{n\in\mathbb{N}}$ is increasing by 4 a) and thus $\lim\limits_{n\to\infty} p^n(x) = a$ exists for some $a \in \mathbb{R} \cup \{\infty\}$.

Let $a = \infty$. Then, by definition of $p(\infty)$, $p(\infty) = \lim\limits_{t\to\infty} p(t) = \lim\limits_{n\to\infty} p^n(x) = \infty$ and we have finished again. Hence now, let $a \in \mathbb{R}$. Since p is continuous in a we obtain $\lim\limits_{n\to\infty} p(p^n(x)) = p(a)$, which implies $p(a) = a$ because $\lim\limits_{n\to\infty} p^{n+1}(x) = a$ holds.

Analogously, we can prove a) for $p(x) < x$ using 4 b), considering a decreasing sequence $(p^n(x))_{n\in\mathbb{N}}$ and $a = \lim\limits_{n\to\infty}$ with $a \in \mathbb{R} \cup \{-\infty\}$.

Finally, the assertion b) is dual to a). $\square$

For a characterization of the existence of homomorphisms of the investigated mono-unary algebras on $\mathbb{R}$, we define the following mappings. Let ∞_1, ∞_2 be two arbitrary symbols (used in paper [12] for the following special definition); please, do not confuse ∞_1, ∞_2 with symbols $-\infty$ and ∞ for extension of set $\mathbb{R}$. Further, let ω be symbol for the ordinal number of $\mathbb{N}$ which we use instead of ω_0 to be briefer.

Let $p\colon \mathbb{R} \to \mathbb{R}$ be an increasing continuous function, $\mathbb{A} = (A, p|A)$ a component of the algebra $(\mathbb{R}, p)$. Then we put

$$\chi\mathbb{A} = \begin{cases} \infty_2 & \text{if } \mathbb{A} \simeq \mathbb{C}_1, \\ \infty_1 & \text{if } \mathbb{A} \simeq (\mathbb{Z}, \nu), \\ \omega & \text{if } \mathbb{A} \simeq (\mathbb{N}, \nu). \end{cases}$$

Let $\{\mathbb{A}_k;\ k \in K\}$ be the system of all components of the algebra $(\mathbb{R}, p)$. Then we define

$$\operatorname{char}(\mathbb{R}, p) = \{\chi\mathbb{A}_k;\ k \in K\}$$

and call it the *characteristic* of p.

For the sake of brevity, we will write $\operatorname{char}(p)$ instead of $\operatorname{char}(\mathbb{R}, p)$.

We see that $\operatorname{char}(p) \subseteq \{\omega, \infty_1, \infty_2\}$ for any mono-unary algebra $(\mathbb{R}, p)$ where p is increasing and continuous.

Now, for any given function on $\mathbb{R}$, we define the following set which is very important for the computation of its characteristic.

Let $p\colon \mathbb{R} \to \mathbb{R}$ be an increasing continuous function. Then we define

$$I_p = \{x \in \mathbb{R} \cup \{-\infty, \infty\};\ p(x) = x\}.$$

Lemma 13. *Let $p\colon \mathbb{R} \to \mathbb{R}$ be an increasing continuous function. Then the following assertions hold.*

a) *Let $a, b \in \mathbb{R}$, $a < b$ be such that the inequlities $p(a) > a$ and $p(b) < b$ (or $p(a) < a$ and $p(b) > b$) are satisfied. Then $I_p \cap (a, b) \neq \emptyset$ holds.*

b) *$I_p \neq \emptyset$ holds.*

Proof. a) Let $p(a) > a$ and $p(b) < b$ hold. Define $q(x) = x - p(x)$ for any $x \in \mathbb{R}$. Then q is a continuous function on R. Moreover, $q(a) = a - p(a) < 0$ and $q(b) = b - p(b) > 0$. Thus, there is $c \in (a, b)$ such that $q(c) = 0$. Hence $p(c) = c$ and $c \in I_p$.

The other assertion with the opposite inequalities is dual to the former.

b) If $p(\infty) = \infty$ then $\infty \in I_p$.

Now, let $p(\infty) = b < \infty$. If we had $p(-\infty) = -\infty$, then we would have $-\infty \in I_p$ and we could finish again.

Hence, let $p(-\infty) = a > -\infty$. By Lemma 7, $a < p(x) < b$ for any $x \in \mathbb{R}$. Thus, $p(a) > a$ and $p(b) < b$ and we obtain $I_p \cap (a, b) \neq \emptyset$ by a) of this lemma. Hence $I_p \neq \emptyset$ holds. $\square$

Lemma 14. *Let $p\colon \mathbb{R} \to \mathbb{R}$ be an increasing continuous function. Then*

a) *$\infty_2 \in \operatorname{char}(p)$ if and only if $I_p \cap \mathbb{R} \neq \emptyset$,*

b) *$\infty_1 \in \operatorname{char}(p)$ if and only if there are $a, b, x \in \mathbb{R} \cup \{-\infty, \infty\}$ with $a < x < b$ such that $a, b \in I_p$ and $x \notin I_p$ hold,*

c) *$\omega \in \operatorname{char}(p)$ if and only if $\{-\infty, \infty\} \not\subseteq I_p$.*

Proof. a) $\infty_2 \in \operatorname{char}(p)$ holds if and only if there is $x \in \mathbb{R}$ with $p(x) = x$, i.e. $I_p \cap \mathbb{R} \neq \emptyset$ is satisfied.

b) For the necessity of the condition, let $\infty_1 \in \operatorname{char}(p)$ hold. Then there is $x \in \mathbb{R}$ such that the set $\{p^n(x);\ n \in \mathbb{Z}\}$ is infinite. This implies that $x \notin I_p$. Further, $\lim\limits_{n\to-\infty} p^n(x) = a$ and $\lim\limits_{n\to\infty} p^n(x) = b$ exist for some $a, b \in \mathbb{R} \cup \{-\infty, \infty\}$ and $p(a) = a$ and $p(b) = b$ hold by Corollary 10, i.e. $a, b \in I_p$. Finally, by Lemma 4, either $a < x < b$ or $b < x < a$ holds.

For the sufficiency, let the condition be satisfied. Then $\infty_1 \in \operatorname{char}(p)$ by Corollary 10.

c) $\omega \in \operatorname{char}(p)$ holds exactly in the case that there is $x \in \mathbb{R}$ such that $p^{-1}(x)$ is not defined, i.e. p is not surjective. But, by Lemma 7, this is satisfied if and only if $-\infty < p(-\infty)$ or $p(\infty) < \infty$, i.e. $\{-\infty, \infty\} \nsubseteq I_p$ holds. $\square$

Now, we can compute $\operatorname{char}(p)$ for any increasing continuous function on $\mathbb{R}$.

Lemma 15. *Let $p\colon \mathbb{R} \to \mathbb{R}$ be an increasing continuous function.*

a) *If $\{-\infty, \infty\} \subseteq I_p$, then*

$$\operatorname{char}(p) = \begin{cases} \{\infty_2\} & \textit{if } I_p \cap \mathbb{R} = \mathbb{R}, \\ \{\infty_1\} & \textit{if } I_p \cap \mathbb{R} = \emptyset, \\ \{\infty_1, \infty_2\} & \textit{otherwise.} \end{cases}$$

b) *If $\{-\infty, \infty\} \nsubseteq I_p$, then*

$$\operatorname{char}(p) = \begin{cases} \{\omega\} & \textit{if } I_p \cap \mathbb{R} = \emptyset, \\ \{\omega, \infty_2\} & \textit{if } I_p \cap \{-\infty, \infty\} = \emptyset \textit{ and } \operatorname{card}(I_p \cap \mathbb{R}) = 1, \\ \{\omega, \infty_1, \infty_2\} & \textit{otherwise.} \end{cases}$$

P r o o f. a) $\omega \notin \operatorname{char}(p)$ holds by Lemma 14 c).

Now, if $I_p \cap \mathbb{R} = \mathbb{R}$, then $\infty_2 \in \operatorname{char}(p)$ by Lemma 14 a) and $\infty_1 \notin \operatorname{char}(p)$ by Lemma 14 b) because the condition in b) requires the existence of $x \in \mathbb{R}$ with $x \notin I_p$. Hence $\operatorname{char}(p) = \{\infty_2\}$.

Further, if $I_p \cap \mathbb{R} = \emptyset$, then $\infty_2 \notin \operatorname{char}(p)$ by Lemma 14 a) and $\infty_1 \in \operatorname{char}(p)$ by Lemma 14 b) because $-\infty, \infty \in I_p$ and $x \notin I_p$ holds for some (arbitrary) $x \in \mathbb{R}$.

Finally, if $\mathbb{R} \neq I_p \cap \mathbb{R} \neq \emptyset$, then $\operatorname{char}(p) = \{\infty_1, \infty_2\}$ by Lemma 14 a) and b) because the conditions of both the assertions are satisfied.

b) $\omega \in \operatorname{char}(p)$ holds by Lemma 14 c).

Further, the conditions $\{-\infty, \infty\} \nsubseteq I_p$ and $\operatorname{card}(I_p \cap \{-\infty, \infty\}) \leqslant 1$ are equal.

First moreover, let $\operatorname{card}(I_p \cap (\mathbb{R} \cup \{-\infty, \infty\})) \leqslant 1$ be satisfied and let $x \in \mathbb{R}$ be arbitrary.

If $p(x) = x$, then $x \in I_p$ which implies $\infty_2 \in \operatorname{char}(p)$.

Now, let $p(x) \neq x$ hold. We generate the set $A = \{p^n(x);\ n \in \mathbb{N}\} \cup \{p^{-n}(x);\ n \in W(\alpha)\}$ where α is the greatest ordinal number such that $p^{-n}(x)$ is defined for any $n \in W(\alpha)$. Put $a = \lim\limits_{n\to\infty} p^n(x)$. Then $a \in I_p$ by Corollary 12 a). But this implies that α is finite because if we had $b = \lim\limits_{n\to\infty} p^{-n}(x)$, then we would obtain $b \in I_p$ by Corollary 12 b) and since $b \neq a$ holds we would have a contradiction to the above assumption. Therefore, the component $(A, A|p)$ of $(\mathbb{R}, p)$ is isomorphic to $(\mathbb{N}, \nu)$.

Altogether in this case, any component of $(\mathbb{R}, p)$ is either a cycle (at most one) or it is isomorphic to $(\mathbb{N}, \nu)$. Hence, $\operatorname{char}(p) = \{\omega\}$ holds for $I_p \cap \mathbb{R} = \emptyset$ and $\operatorname{char}(p) = \{\omega, \infty_2\}$ holds for $\operatorname{card}(I_p \cap \mathbb{R}) = 1$ and, consequently, $I_p \cap \{-\infty, \infty\} = \emptyset$.

Secondly, let, on the contrary, $\operatorname{card}(I_p \cap (\mathbb{R} \cup \{-\infty, \infty\})) > 1$ be satisfied. Since $\operatorname{card}(I_p \cap \{-\infty, \infty\}) \leqslant 1$ we have $I_p \cap \mathbb{R} \neq \emptyset$ and so $\infty_2 \in \operatorname{char}(p)$.

Further, there are components of $(\mathbb{R}, p)$ which are isomorphic to $(\mathbb{Z}, \nu)$ by Corollary 10 (because $\operatorname{card}(I_p \cap \mathbb{R}) = 1$ implies $\operatorname{card}(I_p \cap \{-\infty, \infty\}) = 1$) and so $\infty_1 \in \operatorname{char}(p)$ as well. Altogether, $\operatorname{char}(p) = \{\omega, \infty_1, \infty_2\}$ in this case. $\square$

We want to notice that the set $\{\omega, \infty_1\}$ is the only nonempty subset of the set $\{\omega, \infty_1, \infty_2\}$ which does not occur as a value of $\operatorname{char}(p)$.

The two formulas of Lemma 15 for computing $\operatorname{char}(p)$ of a given function p can be joined together in a single one. In this formula, we can observe the "interaction" of both parts of the set I_p, namely of part $I_p \cap \mathbb{R}$ and part $I_p \cap \{-\infty, \infty\}$.

Theorem 16. *Let $p\colon \mathbb{R} \to \mathbb{R}$ be an increasing continuous function. Denote $I_p^{(\mathbb{R})} = I_p \cap \mathbb{R}$ and $I_p^{(\infty)} = I_p \cap \{-\infty, \infty\}$. Then*

$$\operatorname{char}(p) = \begin{cases} \{\infty_2\} & \text{if } I_p^{(\mathbb{R})} = \mathbb{R}, \\ \{\infty_1\} & \text{if } I_p^{(\mathbb{R})} = \emptyset \text{ and } I_p^{(\infty)} = \{-\infty, \infty\}, \\ \{\omega\} & \text{if } I_p^{(\mathbb{R})} = \emptyset \text{ and } I_p^{(\infty)} \neq \{-\infty, \infty\}, \\ \{\infty_1, \infty_2\} & \text{if } \emptyset \neq I_p^{(\mathbb{R})} \neq \mathbb{R} \text{ and } I_p^{(\infty)} = \{-\infty, \infty\}, \\ \{\omega, \infty_2\} & \text{if } \operatorname{card} I_p^{(\mathbb{R})} = 1 \text{ and } I_p^{(\infty)} = \emptyset, \\ \{\omega, \infty_1, \infty_2\} & \text{otherwise.} \end{cases}$$

Examples 17. (More detailed computations for some of the following functions can be found e.g. in [8].)

a) Let $p(x) = x$ for any $x \in \mathbb{R}$. Then $I_p \cap \mathbb{R} = \mathbb{R}$ which, by Lemma 15 a), implies $\operatorname{char}(p) = \{\infty_2\}$; it is the only function with this characteristic.

b) Let $p \in \{\{x + c;\ x \in \mathbb{R}\}, c \in \mathbb{R} - \{0\}\} \cup \{\{x + a^x;\ x \in \mathbb{R}\},\ a \in \mathbb{R},\ a > 1\}$. Then $I_p = \{-\infty, \infty\}$ which, by Lemma 15 a), implies $\operatorname{char}(p) = \{\infty_1\}$.

c) Let $p \in \{\{x^3;\ x \in \mathbb{R}\}, \{x + \sin x;\ x \in \mathbb{R}\}\}$. Then $I_p = \{-1, 0, 1, -\infty, \infty\}$ or $I_p = \{k\pi;\ k \in \mathbb{Z}\} \cup \{-\infty, \infty\}$ which, by Lemma 15 a), implies $\operatorname{char}(p) = \{\infty_1, \infty_2\}$.

d) Let $p \in \{\{a^x;\ x \in \mathbb{R}\}, a \in \mathbb{R}, a > 1\}$. Then $I_p = \{\infty\}$ which, by Lemma 15 b), implies $\operatorname{char}(p) = \{\omega\}$.

e) Let $p \in \{\{\arctan x;\ x \in \mathbb{R}\}\} \cup \{\{ax/(a + |x|);\ x \in \mathbb{R}\},\ a \in \mathbb{R},\ a > 0\} \cup \{\{ax/\sqrt{a^2 + x^2};\ x \in \mathbb{R}\}, a \in \mathbb{R}, a > 0\}$. Then $I_p = \{0\}$ which, by Lemma 15 b), implies $\operatorname{char}(p) = \{\omega, \infty_2\}$.

f) Let $p \in \{\{a^x - 1;\ x \in \mathbb{R}\};\ a \in \mathbb{R},\ a > 1\}$. Then $I_p = \{0, \infty\}$ which, by Lemma 15 b), implies $\operatorname{char}(p) = \{\omega, \infty_1, \infty_2\}$.

For a simple characterization of the existence of a homomorphism of a mono-unary algebra $(\mathbb{R}, p)$ into another one $(\mathbb{R}, q)$, we use the following relations.

We define the relation $\leqslant$ on the set $\{\omega, \infty_1, \infty_2\}$ of the symbols used above so that $\leqslant$ is the reflexive and transitive closure of the relation $< = \{(\omega, \infty_1), (\infty_1, \infty_2)\}$. See Figure 1 where we show the relation $<$. (The symbols ∞_1 and ∞_2 are abbreviated, respectively by 1 and 2 there.) Then $\leqslant$ is an ordering.

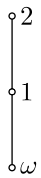

Figure 1

Further, we define the relation ϱ on the set $\Theta = \{\{\omega\}, \{\infty_1\}, \{\infty_2\}, \{\omega, \infty_2\}, \{\infty_1, \infty_2\}, \{\omega, \infty_1, \infty_2\}\}$ of all possible values of a characteristic used above as follows.

If $\theta, \theta' \in \Theta$ are arbitrary, then $\theta \; \varrho \; \theta'$ holds if and only if, for any $\alpha \in \theta$, there is $\alpha' \in \theta'$ with $\alpha \leqslant \alpha'$.

We see that the relation ϱ is a quasi-ordering on Θ. It is represented in Figure 2 where the reflexive arrows are omitted and a simple line without arrows means arrows in both directions. (The symbols ∞_1 and ∞_2 are abbreviated by 1 and 2 again.)

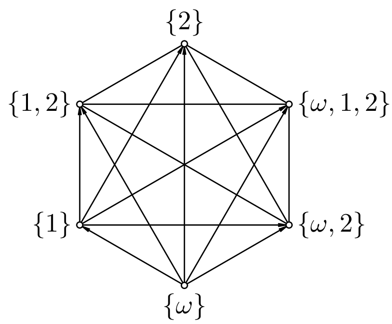

Figure 2

Now, we come back to the original question about the existence of solutions of the functional equation (1). The following assertion is a consequence of investigations in [12] (cf. [11], too); or—without [12]—it follows easily by a direct consideration of the facts.

Theorem 18. *Let $p, q\colon \mathbb{R} \to \mathbb{R}$ be two increasing continuous functions. Then the following assertions are equivalent.*

a) *There is a homomorphism of $(\mathbb{R}, p)$ into $(\mathbb{R}, q)$.*

b) *Equation $f(p(x)) = q(f(x))$ has a solution.*

c) $\operatorname{char}(p) \varrho \operatorname{char}(q)$ *is satisfied.*

Examples 19. From Examples 17, we choose a function in each of the groups a) through f) as their representative: x, $x+1$, x^3, e^x, $\arctan x$, $\mathrm{e}^x - 1$. Then we obtain the following values of their characteristics:

$\operatorname{char}(x) = \{\infty_2\}$,
$\operatorname{char}(x+1) = \{\infty_1\}$,
$\operatorname{char}(x^3) = \{\infty_1, \infty_2\}$,
$\operatorname{char}(\mathrm{e}^x) = \{\omega\}$,
$\operatorname{char}(\arctan x) = \{\omega, \infty_2\}$,
$\operatorname{char}(\mathrm{e}^x - 1) = \{\omega, \infty_1, \infty_2\}$.

The following table shows the solvability of functional equation (1), i.e. $f(p(x)) = q(f(x))$, for given functions p and q where "the functions p" are in the rows and "the functions q" are in the columns. Symbol '+' means that equation (1) is solvable, symbol '−' that it is not. Symbol '(+)' means that (1) has a solution but it is one function only which is constant.

p \ q	x	$x+1$	x^3	e^x	$\arctan x$	$\mathrm{e}^x - 1$
x	+	−	+	−	(+)	+
$x+1$	+	+	+	−	(+)	+
x^3	+	−	+	−	(+)	+
e^x	+	+	+	+	+	+
$\arctan x$	+	−	+	−	+	+
$\mathrm{e}^x - 1$	+	−	+	−	+	+

2. Construction of solutions

Now, we want to discuss possibilities of a construction of solutions of functional equation (1), i.e. $f(p(x)) = q(f(x))$, for given functions p and q if some exist.

First, we will extend the concept of a characteristic function χ that we have defined for components of a mono-unary algebra $(\mathbb{R}, p)$.

Let $p\colon \mathbb{R} \to \mathbb{R}$ be an increasing continuous function. For simplicity, we will confine our considerations to the case that the set I_p (of all $x \in \mathbb{R} \cup \{-\infty, \infty\}$ with $p(x) = x$) is *finite*.

Now, if $a \in I_p \cap \mathbb{R}$ is arbitrary, then the mono-unary algebra $(\{a\}, p|\{a\})$ is a cycle $\mathbb{C}_1$.

Further, let $a, b \in I_p \cup \{-\infty, \infty\}, a < b$ being arbitrary such that $I_p \cap (a, b) = \emptyset$.

In the case that $a, b \in I_p$ holds, any component of the mono-unary algebra $((a, b)), p|(a, b))$ is isomorphic to $(\mathbb{Z}, \nu)$ by Corollary 10.

In the opposite case that either $a = -\infty \notin I_p$ or $b = \infty \notin I_p$ hold (both is not possible by Lemma 13 b)), any component of the mono-unary algebra $((a, b)), p|(a, b))$ is isomorphic to $(\mathbb{N}, \nu)$ by Lemma 1 and Corollary 10 as well.

The set I_p determines the following decomposition of the set $\mathbb{R}$ that we denote by $\mathbb{R}_p$:

$$\mathbb{R}_p = \{\{a\};\ a \in I_p \cap \mathbb{R}\} \cup \{(a, b);\ a, b \in I_p \cup \{-\infty, \infty\}, a < b, I_p \cap (a, b) = \emptyset\}.$$

Then, by the above considerations, we can define a function $\chi_p^{(i)}\colon \mathbb{R}_p \to \{\omega, \infty_1, \infty_2\}$ (on the set of blocks of $\mathbb{R}_p$) in the following way.

If $a \in I_p \cap \mathbb{R}$ is arbitrary, then we put

$$\chi_p^{(i)}(\{a\}) = \infty_2.$$

If $a, b \in I_p \cup \{-\infty, \infty\}$, $a < b$ are arbitrary such that $(a, b) \in \mathbb{R}_p$ holds, then we define

$$\chi_p^{(i)}((a, b)) = \begin{cases} \infty_1 & \text{if } a, b \in I_p, \\ \omega & \text{if either } a = -\infty \notin I_p \text{ or } b = \infty \notin I_p. \end{cases}$$

The function $\chi_p^{(i)}$ can be called the *interval characteristic function* of p where the elements of $I_p \cap \mathbb{R}$ are intervals with one element only. For any interval $J \in \mathbb{R}_p$, $\chi_p^{(i)}$ characterizes all components of the corresponding mono-unary algebra $(J, J|p)$. Namely, by the definition of the function χ of a component of the mono-unary algebra $(\mathbb{R}, p)$, the identity $\chi_p^{(i)}(J) = \chi(C)$ is satisfied for any component C of $(J, J|p)$, which can be easily shown by the above considerations.

On the other hand, for a construction of solutions of equation (1) for given functions p and q, we use the following assertion that is a consequence of Lemma 8.

Corollary 20. *Let $p\colon \mathbb{R} \to \mathbb{R}$ be an increasing continuous function. If $x \in \mathbb{R}$ is arbitrary such that $p(x) > x$ holds and $n \in \mathbb{N}$ is arbitrary, then the following assertions hold:*

a) $p^n([x, p(x))) = [p^n(x), p^{n+1}(x))$;

b) *if $p^{-n}(x)$ exists, then* $p^n([p^{-n}(x), p^{-n+1}(x))) = [x, p(x))$.

Indeed, the assertions a) and b) for open intervals are satisfied by induction using Lemma 8. Since p is continuous on $\mathbb{R}$, we obtain the assertions for the "half closed" intervals as well.

Obviously, a dual assertion to Corollary 20 can be formulated for an arbitrary $x \in \mathbb{R}$ with $p(x) < x$.

As a further consequence, we will use the following assertion as well.

Corollary 21. *Let $p\colon \mathbb{R} \to \mathbb{R}$ be an increasing continuous function. Let $a, b \in \mathbb{R} \cup \{-\infty, \infty\}$, $a < b$ being arbitrary such that $I_p \cap (a, b) = \emptyset$ holds. Then the following assertions hold.*

a) *Let $a, b \in I_p$ hold. If $x \in (a, b)$ is arbitrary such that $p(x) > x$, then $\{[p^m(x), p^{m+1}(x));\ m \in \mathbb{Z}\}$ is a decomposition of the set (a, b).*

b) *Let $a = -\infty$, $-\infty \notin I_p$ and $b \in I_p$ (or $b = \infty, \infty \notin I_p$ and $a \in I_p$) hold. Then $\{p^n((-\infty, p(-\infty)]);\ n \in \mathbb{N}\}$ ($\{p^n([p(\infty), \infty));\ n \in \mathbb{N}\}$, respectively) is a decomposition of the set (a, b).*

Indeed, the assertion a) is a consequence of Corollary 20 and Corollary 12 because the condition $I_p \cap (a, b) = \emptyset$ implies $\lim\limits_{n\to\infty} p^{-n}(x) = a$ and $\lim\limits_{n\to\infty} p^n(x) = b$. Similarly, the assertion b) is a consequence of Corollary 20 and Corollary 12 as well.

Clearly, a dual assertion to Corollary 21 can be formulated for any $x \in \mathbb{R}$ with $p(x) < x$.

Finally, we must use the relation ϱ for the existence of a solution of equation (1) but not only this relation, we will use the ordering $\leqslant$ of the ordered set $(\{\omega, \infty_1, \infty_2\}, \leqslant)$ directly; the ordering $\leqslant$ was defined as a basis for ϱ (see the definitions behind Examples 17).

Now, we can formulate a construction of any solution of the functional equation $f(p(x)) = q(f(x))$, if some exist.

Construction 22. *Let $p, q\colon \mathbb{R} \to \mathbb{R}$ be increasing continuous functions.*

1. *Compute sets I_p and I_q.*

Further, compute $\operatorname{char}(p)$ *and* $\operatorname{char}(q)$ *as above and find out whether* $\operatorname{char}(p)\varrho \operatorname{char}(q)$ *is satisfied. If not, then there is no solution and the construction ends. Thus, let* $\operatorname{char}(p)\varrho \operatorname{char}(q)$ *be satisfied now.*

2. *Determinate the decompositions $\mathbb{R}_p$ and $\mathbb{R}_q$ of $\mathbb{R}$ and compute the interval characteristic functions $\chi_p^{(i)}$ and $\chi_q^{(i)}$.*

3. *For p, compute the set G of the so called generators of the construction in the following way: for any $J \in \mathbb{R}_p$ with* $\operatorname{card} J > 1$, *choose $x_J \in J$ arbitrary and put*

$$G_J = \begin{cases} [x_J, p(x_J)) & \text{if } x_J < p(x_J), \\ (p(x_J), x_J] & \text{if } p(x_J) < x_J; \end{cases}$$

then set

$$G = (I_p \cap \mathbb{R}) \cup (-\infty, p(-\infty)] \cup [p(\infty), \infty) \cup \bigcup_{J \in \mathbb{R}_p, \operatorname{card} J > 1} G_J$$

(where, obviously, the sets $I_p \cap \mathbb{R}$, $(-\infty, p(-\infty)]$ and $[p(\infty), \infty)$ can be empty).

4. *Choose a function $\gamma\colon\ G \to \mathbb{R}$ with the following property:*
if $x \in G$ is arbitrary and $x \in J \in \mathbb{R}_p$ and $\gamma(x) \in K \in \mathbb{R}_q$ hold, then

$$\chi_p^{(i)}(J) \leqslant \chi_q^{(i)}(K)$$

is satisfied.

5. *Define a function $f_\gamma\colon\ \mathbb{R} \to \mathbb{R}$ in the following way: let $x \in \mathbb{R}$ be arbitrary;*
if $x \in I_p$, then put $f_\gamma(x) = \gamma(x)$;
if $x \notin I_p$, then there is $m \in \mathbb{Z}$ such that $p^{-m}(x) \in G$ (by Corollary 21) *and put*

$$f_\gamma(x) = q^m(\gamma(p^{-m}(x))).$$

The function f_γ is a solution of the equation $f(p(x)) = q(f(x))$.

Example 23. We want to find some solutions of the equation

$$f(2x+1) = 3f(x) + 1. \tag{2}$$

We see that the given functions $p(x) = 2x + 1$ and $q(x) = 3x + 1$ are increasing and continuous on $\mathbb{R}$. Therefore, we can use Construction 22. By means of this example, we want to demonstrate this construction; it will proceed up to the end of this section.

Of course, it would not be too time-consuming to specify Construction 22 in such a way that it could be formulated for any linear increasing functions p and q. However, we prefer a shorter demonstration of Construction 22 directly on the basis of a particular example.

For equation (2), step 1 of Construction 22 is the following.

First, since $p(-1) = -1$ and $q(-\frac{1}{2}) = -\frac{1}{2}$, we have $I_p = \{-\infty, -1, \infty\}$ and $I_q = \{-\infty, -\frac{1}{2}, \infty\}$.

Further, as a consequence, $\operatorname{char}(p) = \{\infty_1, \infty_2\}$ and $\operatorname{char}(q) = \{\infty_1, \infty_2\}$ hold by Lemma 15 a). Therefore, $\operatorname{char}(p)\ \varrho\ \operatorname{char}(q)$ is satisfied.

Now, step 2 of Construction 22 contains a consequence of step 1.

Namely, we have $\mathbb{R}_p = \{(\infty,-1),\{-1\},(-1,\infty)\}$ and $\mathbb{R}_q = \big\{(\infty,-\frac{1}{2}),\{-\frac{1}{2}\}, (-\frac{1}{2},\infty)\big\}$ by the definition of these decompositions of $\mathbb{R}$. Further, by the definition of interval characteristic functions $\chi_p^{(i)}$ and $\chi_q^{(i)}$, we obtain

$$\begin{aligned}
\chi_p^{(i)}(\{-1\}) &= \infty_2, & \chi_q^{(i)}\Big(\Big\{-\frac{1}{2}\Big\}\Big) &= \infty_2,\\
\chi_p^{(i)}((-\infty,-1)) &= \infty_1, & \chi_q^{(i)}\Big(\Big(-\infty,-\frac{1}{2}\Big)\Big) &= \infty_1,\\
\chi_p^{(i)}((-1,\infty)) &= \infty_1, & \chi_q^{(i)}\Big(\Big(-\frac{1}{2},\infty\Big)\Big) &= \infty_1.
\end{aligned}$$

Further, for step 3 of Construction 22 we proceed as follows. We choose $-2 \in (-\infty,-1)$ and $0 \in (-1,\infty)$. Then $p(-2) = -3$ and $p(0) = 1$ hold and since $I_p \cap \mathbb{R} = \{-1\}$ and $(-\infty, p(-\infty)] \cup [p(\infty),\infty) = \emptyset$ is satisfied the set of (such chosen) generators G is

$$G = (-3,-2] \cup \{-1\} \cup [0,1).$$

In our next computation, we want to be briefer and so we will confine the construction to the interval $(-1,\infty)$ only.

Then step 4 of Construction 22 is the following.

For the construction on the interval $(-1,\infty)$ the set of all the chosen generators is the interval $[0,1)$. Further, $q(0) = 1$ holds and thus, one of the simplest constructions of a partial solution of (2) on $(-1,\infty)$ is to choose a function $\gamma_1\colon [0,1) \to \mathbb{R}$ in step 4 such that $\gamma_1 = \mathrm{id}_{[0,1)}$ holds. Indeed, since $[0,1) \subset (-\frac{1}{2},\infty)$ and, by step 2, $\chi_p^{(i)}((-1,\infty)) = \chi_q^{(i)}\big((-\frac{1}{2},\infty)\big)$ hold, the condition required in step 4 (for functions $\chi_p^{(i)}$ and $\chi_q^{(i)}$ and relation $\leqslant$) is satisfied trivially.

For our next computations in step 5 of Construction 22, we can use the following assertion for a linear function $r(x) = kx + l$ with $k \neq 1$. In this assertion, point x_r plays the role that it is the only point of $r(x)$ with the property $r(x) = x$, i.e. the only cycle element of r (this is satisfied for $x_r = l/(1-k)$).

Proposition. *Let $k, l \in \mathbb{R}$ be arbitrary such that $k \neq 0$ and $k \neq 1$. Let $r(x) = kx + l$ be a linear function defined on $\mathbb{R}$. Denote $x_r = l/(1-k)$. Let $m \in \mathbb{Z}$ be arbitrary. Then $r^m(x) = k^m(x - x_r) + x_r$ holds for any $x \in \mathbb{R}$.*

Indeed, for $m = 0$ we have $r^0(x) = x - x_r + x_r = x$; if $m \in \mathbb{N}$ and the formula is satisfied for m, then $r^{m+1}(x) = r(r^m(x)) = k(k^m(x-x_r)+x_r)+l = k^{m+1}(x-x_r)+x_r$ because $kx_r + l = x_r$ holds.

Further, $r^{-1}(x) = x/k - l/k$ holds and at the same time, this is the above formula for the exponent -1, i.e. $r^{-1}(x) = k^{-1}(x - x_r) + x_r = x/k - x_r/k + x_r = x/k - l/k$ because $x_r - kx_r = l$ holds.

Finally, let the formula be satisfied for $-n$ where $n \in \mathbb{N}$, $n \geqslant 1$. Then $r^{-(n+1)}(x) = r^{-1}(r^{-n}(x)) = k^{-1}(k^{-n}(x-x_r)+x_r)-l/k = k^{-(n+1)}(x-x_r)+x_r$ because $x_r/k-l/k = x_r$ holds. This completes the proof.

Now, we come back to step 5 of Construction 22 in our case for the interval $(-1, \infty)$.

We need functions p^m and q^m for any $m \in \mathbb{Z}$ and so we will compute them first.

Using the above proposition, we have $x_p = -1$ and so

$$p^m(x) = 2^m x + 2^m - 1$$

for any $m \in \mathbb{Z}$. Furthermore, $x_q = -\frac{1}{2}$ which implies

$$q^m(x) = 3^m\Big(x + \frac{1}{2}\Big) - \frac{1}{2} = 3^m x + \frac{1}{2}(3^m - 1)$$

for any $m \in \mathbb{Z}$.

Therefore, $p^m(0) = 2^m - 1$ for any $m \in \mathbb{Z}$ and so we obtain

$$\begin{aligned} p(0) &= 1, & p^{-1}(0) &= -\frac{1}{2},\\ p^2(0) &= 3, & p^{-2}(0) &= -\frac{3}{4},\\ p^3(0) &= 7, & p^{-3}(0) &= -\frac{7}{8},\\ p^4(0) &= 15, & p^{-4}(0) &= -\frac{15}{16},\\ &\vdots & &\vdots \end{aligned}$$

Hence by Corollary 20, we have

$$\begin{aligned} p([0,1)) &= [1,3), & p^{-1}([0,1)) &= \Big[-\frac{1}{2}, 0\Big),\\ p^2([0,1)) &= [3,7), & p^{-2}([0,1)) &= \Big[-\frac{3}{4}, -\frac{1}{2}\Big),\\ p^3([0,1)) &= [7,15), & p^{-3}([0,1)) &= \Big[-\frac{7}{8}, -\frac{3}{4}\Big),\\ &\vdots & &\vdots \end{aligned}$$

Furthermore, the iterations of p and q are the functions

$$\begin{array}{llll}
p(x) = 2x+1, & p^{-1}(x) = \dfrac{x}{2} - \dfrac{1}{2}, & q(x) = 3x+1, & q^{-1}(x) = \dfrac{x}{3} - \dfrac{1}{3}, \\
p^2(x) = 4x+3, & p^{-2}(x) = \dfrac{x}{4} - \dfrac{3}{4}, & q^2(x) = 9x+4, & q^{-2}(x) = \dfrac{x}{9} - \dfrac{4}{9}, \\
p^3(x) = 8x+7, & p^{-3}(x) = \dfrac{x}{8} - \dfrac{7}{8}, & q^3(x) = 27x+13, & q^{-3}(x) = \dfrac{x}{27} - \dfrac{13}{27}, \\
\vdots & \vdots & \vdots & \vdots
\end{array}$$

By step 5, for the chosen function $\gamma_1 = \mathrm{id}_{[0,1)}$, we compute the function f_1 such that

$$f_1(x) = q^m(p^{-m}(x))$$

holds for any $m \in \mathbb{Z}$ and any $x \in [p^m(0), p^{m+1}(0))$.

Hence for instance, we obtain the following parts of f_1:

if $m = 1$, i.e. $x \in [1,3)$, then $f_1(x) = q(p^{-1}(x)) = 3\big(\frac{x}{2} - \frac{1}{2}\big) + 1 = 3\frac{x}{2} - \frac{1}{2}$;

if $m = 2$, i.e. $x \in [3,7)$, then $f_1(x) = q^2(p^{-2}(x)) = 9\big(\frac{x}{4} - \frac{3}{4}\big) + 4 = \frac{9}{4}x - \frac{11}{4}$;

if $m = 3$, i.e. $x \in [7,15)$, then $f_1(x) = q^3(p^{-3}(x)) = 27\big(\frac{x}{8} - \frac{7}{8}\big) + 13 = \frac{27}{8}x - \frac{85}{8}$;

if $m = -1$, i.e. $x \in \big[-\frac{1}{2}, 0\big)$, then $f_1(x) = q^{-1}(p(x)) = \frac{1}{3}(2x+1) - \frac{1}{3} = \frac{2}{3}x$;

if $m = -2$, i.e. $x \in \big[-\frac{3}{4}, -\frac{1}{2}\big)$, then $f_1(x) = q^{-2}(p^2(x)) = \frac{1}{9}(4x+3) - \frac{4}{9} = \frac{4}{9}x - \frac{1}{9}$;

if $m = -3$, i.e. $x \in \big[-\frac{7}{8}, -\frac{3}{4}\big)$, then $f_1(x) = q^{-3}(p^3(x)) = \frac{1}{27}(8x+7) - \frac{13}{27} = \frac{8}{27}x - \frac{6}{27}$.

Altogether, the partial solution f_1 of equation (2) on $(-1,\infty)$ is

$$f_1(x) = \begin{cases} \quad\vdots & \\ \frac{27}{8}x - \frac{85}{8} & \text{if } x \in [7,15), \\ \frac{9}{4}x - \frac{11}{4} & \text{if } x \in [3,7), \\ \frac{3}{2}x - \frac{1}{2} & \text{if } x \in [1,3), \\ x & \text{if } x \in [0,1), \\ \frac{2}{3}x & \text{if } x \in [-\frac{1}{2},0), \\ \frac{4}{9}x - \frac{1}{9} & \text{if } x \in [-\frac{3}{4},-\frac{1}{2}), \\ \frac{8}{27}x - \frac{13}{27} & \text{if } x \in [-\frac{7}{8},-\frac{3}{4}), \\ \quad\vdots & \end{cases}$$

Now, on the other hand, let $\mathbb{Q}$ denote the set of all rational numbers. We define $\gamma_2\colon [0,1) \to [0,1)$ as a "half Dirichlet" function in the following way. If $x \in [0,1)$ is arbitrary, then we put

$$\gamma_2(x) = \begin{cases} \frac{1}{2} & \text{if } x \in \mathbb{Q}, \\ 0 & \text{otherwise.} \end{cases}$$

By the above computations—in steps 1 up to 3 of the construction, we have the following result in this case.

Let $m \in \mathbb{Z}$ be arbitrary. If $x \in [p^m(0), p^{m+1}(0))$ is arbitrary, then

$$f_2(x) = q^m(\gamma_2(p^{-m}(x))) = 3^m \gamma_2(2^{-m}x + 2^{-m} - 1) + \frac{1}{2}(3^m - 1).$$

Thus, let $x \in [p^m(0), p^{m+1}(0))$ and $x \in \mathbb{Q}$. Then $2^{-m}x + 2^{-m} - 1 \in \mathbb{Q}$ too and $\gamma_2(2^{-m}x + 2^{-m} - 1) = \frac{1}{2}$ by the definition of γ_2. Then

$$f_2(x) = 3^m \frac{1}{2} + \frac{1}{2}(3^m - 1) = 3^m - \frac{1}{2}.$$

Let, on the other hand, $x \in [p^m(0), p^{m+1}(0))$ and $x \notin \mathbb{Q}$. Then $2^{-m}x+2^{-m}-1 \notin \mathbb{Q}$ too, $\gamma_2(2^{-m}x + 2^{-m} - 1) = 0$ by the definition of γ_2 and so

$$f_2(x) = \frac{1}{2}(3^m - 1).$$

Altogether, we obtain the following new partial solution f_2 of (2) on $(-1, \infty)$.

$$f_2(x) = \begin{cases} \vdots & \\ \frac{5}{2} & \text{if } \ x \in [1,3) \cap \mathbb{Q}, \\ 1 & \text{if } \ x \in [1,3) - \mathbb{Q}, \\ \frac{1}{2} & \text{if } \ x \in [0,1) \cap \mathbb{Q}, \\ 0 & \text{if } \ x \in [0,1) - \mathbb{Q}, \\ -\frac{5}{2} & \text{if } \ x \in \big[-\frac{1}{2}, 0\big) \cap \mathbb{Q}, \\ -\frac{1}{3} & \text{if } \ x \in \big[-\frac{1}{2}, 0\big) - \mathbb{Q}, \\ \vdots & \end{cases}$$

Partial solutions of equation (2) on the interval $(-1, \infty)$ can be used for simple construction of partial solutions on the interval $(-\infty, -1)$ by means of point symmetry with respect to the point $(-1, -\frac{1}{2})$. A point symmetry of subsets of the plane $\mathbb{R} \times \mathbb{R}$ with respect to the point $(x_0, y_0) \in \mathbb{R} \times \mathbb{R}$ uses the linear transformation

$$t = x_0 - (x - x_0) = 2x_0 - x,$$
$$u = y_0 - (y - y_0) = 2y_0 - y$$

for any $(x, y) \in \mathbb{R} \times \mathbb{R}$.

Therefore, the point symmetry of the graph of a function f defined on $\mathbb{R}$ with respect to a point (x_0, y_0) is defined by the identity $f(2x_0 - x) = 2y_0 - f(x)$ where $x \in \mathbb{R}$ is arbitrary.

Then, for instance, the graph of any linear function $q(x) = kx+l$ is point symmetric with respect to a point $(x_0, q(x_0))$ where $x_0 \in \mathbb{R}$ is arbitrary (because $k(2x_0-x)+l = 2kx_0 - kx + l - l + l = 2(kx_0 + l) - (kx + l)$ holds).

The point symmetry and its properties could be investigated generally but now, we can show by simple means that constructions by point symmetry are possible in our case.

Proposition. *Let $k, l, k', l' \in \mathbb{R}$ be arbitrary such that $k, k' \neq 1$. Let $r(x) = kx+l$ and $r'(x) = k'x + l'$ be linear functions defined on $\mathbb{R}$. Denote $x_r = l/(1-k)$ and $x_{r'} = l'/(1-k')$. Let $f\colon (x_r, \infty) \to \mathbb{R}$ be a partial solution of equation*

$$f(kx + l) = k'f(x) + l'. \tag{3}$$

Define a function $f^*\colon (-\infty, x_r) \to \mathbb{R}$ such that $f^*(x) = -f(2x_r - x) + 2x_{r'}$ holds for any $x \in (-\infty, x_r)$. Then f^* is a partial solution of (3) *on $(-\infty, x_r)$.*

Indeed, let f be a partial solution of equation (3) on (x_r, ∞). Then we prove the identity

$$f^*(kx + l) = k'f^*(x) + l' \tag{$*$}$$

for any $x \in (-\infty, x_r)$ if the function f^* is defined as above.

Let $x \in (-\infty, x_r)$ be arbitrary. By the definition of f^*, the left hand side of $(*)$ has the form

$$f^*(kx + l) = -f(2x_r - (kx + l)) + 2x_{r'}. \tag{$**$}$$

We will deal with the term $f(2x_r - (kx + l))$ now.

First, we observe the argument of f in this term. The points x_r and $x_{r'}$ are defined so that $kx_r + l = x_r$ and $k'x_{r'} + l' = x_{r'}$ hold. Hence $2x_r - l = 2(kx_r + l) - l = 2kx_r + l$ and, similarly, $2x_{r'} - l' = 2k'x_{r'} + l'$ hold. Therefore, the argument of f has a new form $2x_r - (kx + l) = -kx + \underbrace{2x_r - l} = k(-x + 2x_r) + l$. Thus,

$$f(2x_r - (kx + l)) = f(k(-x + 2x_r) + l). \tag{$***$}$$

Further, $-x + 2x_r \in (x_r, \infty)$ holds and we can use the condition that f is a partial solution of equation (3) on (x_r, ∞). Then we obtain

$$\begin{aligned} f(k(-x+2x_r)+l) &= k'f(-x+2x_r) + l' = k'f(-x+2x_r) - 2k'x_{r'} + \underbrace{2k'x_{r'} + l'} \\ &= k'(f(-x+2x_r) - 2x_{r'}) + 2x_{r'} - l' = k'(-f^*(x)) - l' + 2x_{r'}. \end{aligned}$$

Altogether, if we come back to $(**)$ and use $(***)$, we have

$$f^*(kx+l) = -(-k'f^*(x) - l' + 2x_{r'}) + 2x_{r'} = k'f^*(x) + l'$$

which is $(*)$ and which was to be proved.

Finally, we come back to equation (2). In particular, we have $x_p = -1$, $x_q = -\frac{1}{2}$ and, by the last proposition, any partial solution f_i of (2) on $(-1, \infty)$ for $i \in \{1, 2\}$ found above can be extended by

$$f_i^*(x) = -f_i(2x_p - x) + 2x_q = -(f_i(-(x+2)) + 1)$$

to the interval $(-\infty, -1)$.

In this way, we can obtain, for example, the set

$$\Big\{f_i^* \cup \Big\{\Big(-1, -\frac{1}{2}\Big)\Big\} \cup f_j\,;\ i, j \in \{1, 2\}\Big\}$$

of four *complete* solutions of equation (2).

In Example 23, we have construted some solutions of the given functional equation (2). But if we look at Construction 22, we see that—under the assumption of the existence of more than one solution—"functions γ's" in step 4 of Construction 22 play an important role in the construction. The set of such functions, which can be used there, is uncountable and this suggests that the set of all solutions of the given equation (1) is uncountable as well (cf. [13] or, for instance, [8]).

3. What should be done next

Our investigations should be extended to partially defined strictly increasing continuous functions p, q on $\mathbb{R}$—e.g. $\sqrt{x}$, $\ln x$ and so on. It is clear that such real functions will need more values of their corresponding characteristic functions than the six of them which were used for instance in Figure 2.

The other topics should be investigations of strictly decreasing continuous functions, and further, more general continuous functions, such as say e^{-x}, x^2, $1/x$, $\arctan(\ln x)$, ...

Some of these cases can be very easy. For other ones, the theory of mono-unary algebras must take effect more and be used wider than it was in the case of strictly increasing continuous functions in the present paper. There are already investigations of some of these problems—see e.g. [1]–[6].

Any of these considerations leads to a construction similar to the present one. Since, clearly, constructions are algorithms, any of such results of our considerations are programmable. These programs would be, above all, suitable for computations of local parts of solutions of given equations which, moreover, satisfy additional and for specific problems important conditions. In fact, this is one of the main aims of our investigations.

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