Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica

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Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica, Vol. 51 (2012), No. 2, 87--106

Persistent URL: http://dml.cz/dmlcz/143070

Acta Univ. Palacki. Olomuc., Fac. rer. nat., Mathematica **51**, 2 (2012) 87–106

Additional Experiment and Linear Statistical Models*

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(Received November 23, 2011)

Abstract

An accuracy of parameter estimates need not be sufficient for their unforeseen utilization. Therefore some additional measurement is necessary in order to attain the required precision. The problem is to express the correction to the original estimates in an explicit form.

Key words: additional experiment, linear statistical model, constraints

2000 Mathematics Subject Classification: 62J05

1 Introduction

A linear statistical model is considered in the form

$$\mathbf{Y}_1 \sim_n (\mathbf{X}_1\boldsymbol{\beta}, \boldsymbol{\Sigma}_1),$$

where $\mathbf{Y}_1$ is an n-dimensional random vector (observation vector) with the mean value $E(\mathbf{Y}_1) = \mathbf{X}_1\boldsymbol{\beta}$ and the covariance matrix $\mathrm{Var}(\mathbf{Y}_1) = \boldsymbol{\Sigma}_1$. The $n \times k$ matrix $\mathbf{X}_1$ is given, the k-dimensional vector $\boldsymbol{\beta}$ is unknown and the matrix $\boldsymbol{\Sigma}_1$ is known.

In the following text it is assumed that the rank $r(\mathbf{X}_1) = k \leq n$ and the matrix $\boldsymbol{\Sigma}_1$ is positive definite.

The accuracy of the estimator $\widehat{\boldsymbol{\beta}}$ is characterized by its covariance matrix $\mathrm{Var}(\widehat{\boldsymbol{\beta}})$. If it is not satisfactory, then it is necessary to realize an additional experiment, e.g.

$$\mathbf{Y}_2 \sim_m (\mathbf{X}_2\boldsymbol{\beta}, \boldsymbol{\Sigma}_2).$$

*Supported by the Council of the Czech Government MSM 6 198 959 214

(Another forms of additional experiments are described in the following sections.)

Thus the joint model (original and additional) can be written as

$$\begin{pmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{pmatrix} \sim_{n+m} \left[\begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{pmatrix} \boldsymbol{\beta}, \begin{pmatrix} \boldsymbol{\Sigma}_1, & \mathbf{0} \\ \mathbf{0}, & \boldsymbol{\Sigma}_2 \end{pmatrix} \right].$$

The covariance matrix of the estimator $\widehat{\boldsymbol{\beta}}$ in this model is obviously more satisfactory than the original covariance matrix.

In the following text the following notation will be used.

$\widehat{\boldsymbol{\beta}}$ ... the best linear unbiased estimator (BLUE) in a model without constraints;

$\widehat{\widehat{\boldsymbol{\beta}}}$ ... the BLUE in the model with constraints;

$\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1)$, $\widehat{\widehat{\boldsymbol{\beta}}}(\mathbf{Y}_1, \mathbf{Y}_2)$ ... the BLUEs based on the observation vector $\mathbf{Y}_1$ and $\mathbf{Y}_1, \mathbf{Y}_2$, respectively;

$\mathbf{A}^+$ ... the Moore–Penrose generalized inverse of the matrix $\mathbf{A}$ (i.e. $\mathbf{A}\mathbf{A}^+\mathbf{A} = \mathbf{A}$, $\mathbf{A}^+\mathbf{A}\mathbf{A}^+ = \mathbf{A}^+$, $\mathbf{A}\mathbf{A}^+ = (\mathbf{A}\mathbf{A}^+)'$, $\mathbf{A}^+\mathbf{A} = (\mathbf{A}^+\mathbf{A})'$);

$\mathbf{A}^-_{m(N)}$ ... the minimum $\mathbf{N}$-seminorm generalized inverse of the matrix $\mathbf{A}$ (i.e. $\mathbf{A}\mathbf{A}^-_{m(N)}\mathbf{A} = \mathbf{A}$, $\mathbf{N}\mathbf{A}^-_{m(N)}\mathbf{A} = (\mathbf{N}\mathbf{A}^-_{m(N)}\mathbf{A})'$, $\mathbf{N}$ is a positive semidefinite matrix; in more deatil see in [3]);

$\mathbf{b}_{q,1} + \mathbf{B}_{q,k}\boldsymbol{\beta} = \mathbf{0}$ ... constraints in the original model;

$\mathbf{g}_{r,1} + \mathbf{G}_{r,k}\boldsymbol{\beta} = \mathbf{0}$ ... constraints in the additional model;

$\mathcal{M}(\mathbf{A})$ denotes the column space of the matrix $\mathbf{A}$, i.e.

$$\mathcal{M}(\mathbf{A}) = \{\mathbf{A}\mathbf{u} \colon \mathbf{u} \in R^n\}.$$

The original model can be either of the form (model without constraints)

$$\mathbf{Y}_1 \sim_n (\mathbf{X}_1\boldsymbol{\beta}, \boldsymbol{\Sigma}_1),$$

or of the form (model with constraints)

$$\mathbf{Y}_1 \sim_n (\mathbf{X}_1\boldsymbol{\beta}, \boldsymbol{\Sigma}_1), \quad \mathbf{b} + \mathbf{B}\boldsymbol{\beta} = \mathbf{0},$$

where the rank of the matrix $\mathbf{B}$ is $r(\mathbf{B}) = q < k$.

The additional model can be either of the form (model without constraints)

$$\mathbf{Y}_2 \sim_m (\mathbf{X}_2\boldsymbol{\beta}, \boldsymbol{\Sigma}_2), \quad r(\mathbf{X}_2) = k < m, \quad \boldsymbol{\Sigma}_2 \text{ is positive definite},$$

or of the forms

model with additional parameters $\boldsymbol{\gamma} \in R^l$

$$\mathbf{Y}_2 \sim_m \left[(\mathbf{D}_{m,k}, \mathbf{X}_{2,(m,l)}) \begin{pmatrix} \boldsymbol{\beta} \\ \boldsymbol{\gamma} \end{pmatrix}, \boldsymbol{\Sigma}_2 \right], \ r(\mathbf{D}, \mathbf{X}_2) = k + l < m,$$
$$\text{the matrix } \boldsymbol{\Sigma}_2 \text{ is positive definite},$$

model with additional constraints

$$\mathbf{Y}_2 \sim_m (\mathbf{X}_2\boldsymbol{\beta}, \boldsymbol{\Sigma}_2), \ \mathbf{g}_{r,1} + \mathbf{G}_{r,k}\boldsymbol{\beta} = \mathbf{0}, \ r(\mathbf{X}_2) = l < m, \ r(\mathbf{G}) = r < k,$$

models with additional parameters and additional constraints

$$\mathbf{Y}_2 \sim_m \left[(\mathbf{D}_{m,k}, \mathbf{X}_{2,(m,l)}) \begin{pmatrix} \boldsymbol{\beta} \\ \gamma \end{pmatrix}, \boldsymbol{\Sigma}_2 \right], \ \mathbf{g}_{r,1} + \mathbf{G}_{r,k}\boldsymbol{\beta} = \mathbf{0},$$
$$r(\mathbf{D}, \mathbf{X}_2) = k + l < m, \ \boldsymbol{\Sigma}_2 \text{ is positive definite,}$$
$$\mathbf{Y}_2 \sim_m \left[(\mathbf{D}_{m,k}, \mathbf{X}_{2,(m,l)}) \begin{pmatrix} \boldsymbol{\beta} \\ \gamma \end{pmatrix}, \boldsymbol{\Sigma}_2 \right], \ \mathbf{g}_{r,1} + \mathbf{G}_{1,(r,k)}\boldsymbol{\beta} + \mathbf{G}_{2,(r,l)}\gamma = \mathbf{0},$$
$$r(\mathbf{D}, \mathbf{X}_2) = k + l < m, \ \boldsymbol{\Sigma}_2 \text{ is positive definite,}$$
$$r(\mathbf{G}_1, \mathbf{G}_2) = r, \ r(\mathbf{G}_2) = l < r.$$

The problem is to find the vector $\mathbf{k}$ either in the equation

$$\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1, \mathbf{Y}_2) = \widehat{\boldsymbol{\beta}}(\mathbf{Y}_1) + \mathbf{k}, \quad \text{or} \quad \widehat{\widehat{\boldsymbol{\beta}}}(\mathbf{Y}_1, \mathbf{Y}_2) = \widehat{\widehat{\boldsymbol{\beta}}}(\mathbf{Y}_1) + \mathbf{k},$$

or

$$\widehat{\widehat{\boldsymbol{\beta}}}(\mathbf{Y}_1, \mathbf{Y}_2) = \widehat{\boldsymbol{\beta}}(\mathbf{Y}_1) + \mathbf{k}, \quad \text{or} \quad \widehat{\widehat{\widehat{\boldsymbol{\beta}}}}(\mathbf{Y}_1, \mathbf{Y}_2) = \widehat{\widehat{\boldsymbol{\beta}}}(\mathbf{Y}_1) + \mathbf{k}$$

($\widehat{\widehat{\widehat{\ }}}$ denotes an estimator respecting two constraints) in dependence on the form of the joint model.

All models considered are assumed to be regular.

2 Original models without constraints

Lemma 1 *Let* $\mathbf{A}$ *be an* $n \times n$ *positive semidefinite matrix,* $\mathbf{C}$ *be an* $m \times m$ *positive semidefinite matrix and* $\mathbf{B}$ *be an* $n \times m$ *matrix with the properties* $\mathcal{M}(\mathbf{B}) \subset \mathcal{M}(\mathbf{A})$ *and* $\mathcal{M}(\mathbf{B}') \subset \mathcal{M}(\mathbf{C})$*. Then*

$$(\mathbf{A} - \mathbf{B}\mathbf{C}^+\mathbf{B}')^+ = \mathbf{A}^+ + \mathbf{A}^+\mathbf{B}(\mathbf{C} - \mathbf{B}'\mathbf{A}^+\mathbf{B})^+\mathbf{B}'\mathbf{A}^+,$$
$$(\mathbf{A} - \mathbf{B}\mathbf{C}^+\mathbf{B}')^+\mathbf{B}\mathbf{C}^+ = \mathbf{A}^+\mathbf{B}(\mathbf{C} - \mathbf{B}'\mathbf{A}^+\mathbf{B})^+.$$

Proof It is sufficient to verify the properties of the Moore–Penrose generalized inverse of a matrix (in more detail see [3]). □

Lemma 2 *Let* $\mathbf{A}$ *be any* $n \times k$ *matrix,* $\mathbf{B}$ *be any* $q \times k$ *matrix and* $\boldsymbol{\Sigma}$ *be any* $n \times n$ *positive semidefinite matrix. Then*

$$\left[(\mathbf{A}', \mathbf{B}')^-_{m\begin{pmatrix} \Sigma, & 0 \\ 0, & 0 \end{pmatrix}} \right]' = \left(\left[(\mathbf{M}_{B'}\mathbf{A}')^-_{m(\Sigma)} \right]', \left\{ \mathbf{I} - \left[(\mathbf{M}_{B'}\mathbf{A}')^-_{m(\Sigma)} \right]' \mathbf{A} \right\} \mathbf{B}^+ \right).$$

Here $\mathbf{M}_{B'} = \mathbf{I} - \mathbf{P}_{B'}, \mathbf{P}_{B'} = \mathbf{B}'(\mathbf{B}')^+$.

Proof It is valid that

$$\begin{pmatrix} \mathbf{A} \\ \mathbf{B} \end{pmatrix} \left(\left[(\mathbf{M}_{B'}\mathbf{A}')^-_{m(\Sigma)} \right]', \left\{ \mathbf{I} - \left[(\mathbf{M}_{B'}\mathbf{A}')^-_{m(\Sigma)} \right]' \mathbf{A} \right\} \mathbf{B}^+ \right) \begin{pmatrix} \mathbf{A} \\ \mathbf{B} \end{pmatrix}$$
$$= \begin{pmatrix} \mathbf{A}\mathbf{M}_{B'} \left[(\mathbf{M}_{B'}\mathbf{A}')^-_{m(\Sigma)} \right]' \mathbf{A}\mathbf{M}_{B'} + \mathbf{A}\mathbf{P}_{B'} \\ \mathbf{B} \end{pmatrix} = \begin{pmatrix} \mathbf{A} \\ \mathbf{B} \end{pmatrix}$$

and

$$\begin{pmatrix}\mathbf{A}\\ \mathbf{B}\end{pmatrix}\left(\left[(\mathbf{M}_{B'}\mathbf{A}')^-_{m(\Sigma)}\right]', \left\{\mathbf{I}-\left[(\mathbf{M}_{B'}\mathbf{A}')^-_{m(\Sigma)}\right]'\mathbf{A}\right\}\mathbf{B}^+\right)\begin{pmatrix}\mathbf{\Sigma}, & \mathbf{0}\\ \mathbf{0}, & \mathbf{0}\end{pmatrix}$$
$$= \begin{pmatrix}\mathbf{A}\mathbf{M}_{B'}\left[(\mathbf{M}_{B'}\mathbf{A}')^-_{m(\Sigma)}\right]'\mathbf{\Sigma}, & \mathbf{0}\\ \mathbf{0}, & \mathbf{0}\end{pmatrix}$$

is a symmetric matrix.

The relationship

$$\left[(\mathbf{M}_{B'}\mathbf{A}')^-_{m(\Sigma)}\right]' = \mathbf{M}_{B'}\left[(\mathbf{M}_{B'}\mathbf{A}')^-_{m(\Sigma)}\right]'$$

was utilized. □

Lemma 3 *Let the model*

$$\begin{pmatrix}\tilde{\boldsymbol{\beta}}\\ -\mathbf{g}\end{pmatrix} \sim_{k+q}\left[\begin{pmatrix}\mathbf{I}\\ \mathbf{G}\end{pmatrix}\boldsymbol{\beta}, \begin{pmatrix}\mathbf{W}, & \mathbf{0}\\ \mathbf{0}, & \mathbf{0}\end{pmatrix}\right]$$

be considered ($\mathbf{W}$ *need not be regular and the rank* $r(\mathbf{G}_{q,k})$ *need not be* $q<k$*). The BLUE of* $\boldsymbol{\beta}$ *is*

$$\hat{\hat{\boldsymbol{\beta}}} = \tilde{\boldsymbol{\beta}} - \mathbf{W}\mathbf{G}'(\mathbf{G}\mathbf{W}\mathbf{G}')^+(\mathbf{G}\tilde{\boldsymbol{\beta}}+\mathbf{g})$$

if $\mathbf{W}$ *is p.d. and*

$$\mathrm{Var}(\hat{\hat{\boldsymbol{\beta}}}) = \mathbf{W} - \mathbf{W}\mathbf{G}'(\mathbf{G}\mathbf{W}\mathbf{G}')^+\mathbf{G}\mathbf{W}$$

if $\mathbf{W}$ *is p.d.*

In general

$$\hat{\hat{\boldsymbol{\beta}}} = \tilde{\boldsymbol{\beta}} - \left[(\mathbf{W}+\mathbf{M}_{G'})^+ + \mathbf{G}'\mathbf{G}\right]^{-1}\mathbf{G}'\left\{\mathbf{G}\left[(\mathbf{W}+\mathbf{M}_{G'})^+ + \mathbf{G}'\mathbf{G}\right]^{-1}\mathbf{G}'\right\}^+(\mathbf{G}\tilde{\boldsymbol{\beta}}+\mathbf{g})$$

and

$$\mathrm{Var}(\hat{\hat{\boldsymbol{\beta}}}) = \left[(\mathbf{W}+\mathbf{M}_{G'})^+ + \mathbf{G}'\mathbf{G}\right]^{-1} - \left[(\mathbf{W}+\mathbf{M}_{G'})^+ + \mathbf{G}'\mathbf{G}\right]^{-1}\mathbf{G}'$$
$$\times\left\{\mathbf{G}\left[(\mathbf{W}+\mathbf{M}_{G'})^+ + \mathbf{G}'\mathbf{G}\right]^{-1}\mathbf{G}'\right\}^+\mathbf{G}\left[(\mathbf{W}+\mathbf{M}_{G'})^+ + \mathbf{G}'\mathbf{G}\right]^{-1} - \mathbf{M}_{G'}.$$

Proof Since $\boldsymbol{\beta}$ is unbiasedly estimable (in more detail see in [1], p. 337 and 346), the BLUE in the model with constraints is (see the preceding lemma)

$$\hat{\hat{\boldsymbol{\beta}}} = \left[(\mathbf{I}, \mathbf{G}')^-_{m\begin{pmatrix} W, & 0 \\ 0, & 0 \end{pmatrix}}\right]' \begin{pmatrix} \tilde{\boldsymbol{\beta}} \\ -\mathbf{g} \end{pmatrix}$$

$$= \left(\left[(\mathbf{M}_{G'})^-_{m(W)}\right]', \left\{\mathbf{I} - \left[(\mathbf{M}_{G'})^-_{m(W)}\right]'\right\}\mathbf{G}^+\right) \begin{pmatrix} \tilde{\boldsymbol{\beta}} \\ -\mathbf{g} \end{pmatrix} = \left[\mathbf{M}_{G'}(\mathbf{W}+\mathbf{M}_{G'})^+\mathbf{M}_{G'}\right]^+$$

$$\times \mathbf{M}_{G'}(\mathbf{W}+\mathbf{M}_{G'})^+\tilde{\boldsymbol{\beta}} + \left\{\mathbf{I} - \left[\mathbf{M}_{G'}(\mathbf{W}+\mathbf{M}_{G'})^+\mathbf{M}_{G'}\right]^+(\mathbf{W}+\mathbf{M}_{G'})^+\right\}\mathbf{G}^+(-\mathbf{g})$$

$$= \Big\{\left[(\mathbf{W}+\mathbf{M}_{G'})^+ + \mathbf{G}'\mathbf{G}\right]^{-1} - \left[(\mathbf{W}+\mathbf{M}_{G'})^+ + \mathbf{G}'\mathbf{G}\right]^{-1}\mathbf{G}'$$

$$\times \left\{\mathbf{G}\left[(\mathbf{W}+\mathbf{M}_{G'})^+ + \mathbf{G}'\mathbf{G}\right]^{-1}\mathbf{G}'\right\}^+\mathbf{G}\left[(\mathbf{W}+\mathbf{M}_{G'})^+ + \mathbf{G}'\mathbf{G}\right]^{-1}\Big\}$$

$$\times\left[(\mathbf{W}+\mathbf{M}_{G'})^+ + \mathbf{G}'\mathbf{G}\right]\tilde{\boldsymbol{\beta}} + \Big(\mathbf{I} - \Big\{\left[(\mathbf{W}+\mathbf{M}_{G'})^+ + \mathbf{G}'\mathbf{G}\right]^{-1} - \left[(\mathbf{W}+\mathbf{M}_{G'})^+ + \mathbf{G}'\mathbf{G}\right]^{-1}$$

$$\times \mathbf{G}'\left\{\mathbf{G}\left[(\mathbf{W}+\mathbf{M}_{G'})^+ + \mathbf{G}'\mathbf{G}\right]^{-1}\mathbf{G}'\right\}^+\mathbf{G}\left[(\mathbf{W}+\mathbf{M}_{G'})^+ + \mathbf{G}'\mathbf{G}\right]^{-1}\Big\}$$

$$\times \left[(\mathbf{W}+\mathbf{M}_{G'})^+ + \mathbf{G}'\mathbf{G}\right]\Big)\mathbf{G}^+(-\mathbf{g})$$

$$= \tilde{\boldsymbol{\beta}} - \left[(\mathbf{W}+\mathbf{M}_{G'})^+ + \mathbf{G}'\mathbf{G}\right]^{-1}\mathbf{G}'\left\{\mathbf{G}\left[(\mathbf{W}+\mathbf{M}_{G'})^+ + \mathbf{G}'\mathbf{G}\right]^{-1}\mathbf{G}'\right\}^+(\mathbf{G}\tilde{\boldsymbol{\beta}} + \mathbf{g}).$$

The expression for $\mathrm{Var}(\hat{\hat{\boldsymbol{\beta}}})$ can be obtained easily.

If $\mathbf{W}$ is p.d., then (see [1], p. 337)

$$\left[(\mathbf{M}_{G'})^-_{m(W)}\right]' = (\mathbf{M}_{G'}\mathbf{W}^{-1}\mathbf{M}_{G'})^+\mathbf{M}_{G'}\mathbf{W}^{-1} = \mathbf{I} - \mathbf{W}\mathbf{G}'(\mathbf{G}\mathbf{W}\mathbf{G}')^+\mathbf{G}$$

and the proof can be proceeded analogously. □

In the following text $\mathbf{C}_1 = \mathbf{X}_1'\boldsymbol{\Sigma}_1^{-1}\mathbf{X}_1,\ \mathbf{C}_2 = \mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{X}_2$.

Theorem 1 *If*

$$\begin{pmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{pmatrix} \sim_{n+m} \left[\begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{pmatrix}\boldsymbol{\beta}, \begin{pmatrix} \boldsymbol{\Sigma}_1, & \mathbf{0} \\ \mathbf{0}, & \boldsymbol{\Sigma}_2 \end{pmatrix}\right],$$

then

$$\hat{\boldsymbol{\beta}}(\mathbf{Y}_1, \mathbf{Y}_2) = \hat{\boldsymbol{\beta}}(\mathbf{Y}_1) + (\mathbf{C}_1 + \mathbf{C}_2)^{-1}\mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\left[\mathbf{Y}_2 - \mathbf{X}_2\hat{\boldsymbol{\beta}}(\mathbf{Y}_1)\right]$$

and

$$\mathrm{Var}\left[\hat{\boldsymbol{\beta}}(\mathbf{Y}_1, \mathbf{Y}_2)\right] = \mathrm{Var}\left[\hat{\boldsymbol{\beta}}(\mathbf{Y}_1)\right] - \mathbf{C}_1^{-1}\mathbf{X}_2'(\boldsymbol{\Sigma}_2 + \mathbf{X}_2\mathbf{C}_1^{-1}\mathbf{X}_2')^{-1}\mathbf{X}_2\mathbf{C}_1^{-1}.$$

Proof

$$\begin{aligned}\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1,\mathbf{Y}_2) &= (\mathbf{C}_1+\mathbf{C}_2)^{-1}(\mathbf{X}_1'\boldsymbol{\Sigma}_1^{-1}\mathbf{Y}_1+\mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{Y}_2)\\ &= \big[\mathbf{C}_1^{-1}-\mathbf{C}_1^{-1}\mathbf{X}_2'(\boldsymbol{\Sigma}_2+\mathbf{X}_2\mathbf{C}_1^{-1}\mathbf{X}_2')^{-1}\mathbf{X}_2\mathbf{C}_1^{-1}\big](\mathbf{X}_1'\boldsymbol{\Sigma}_1^{-1}\mathbf{Y}_1+\mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{Y}_2)\\ &= \widehat{\boldsymbol{\beta}}(\mathbf{Y}_1)-\mathbf{C}_1^{-1}\mathbf{X}_2'(\boldsymbol{\Sigma}_2+\mathbf{X}_2\mathbf{C}_1^{-1}\mathbf{X}_2')^{-1}\mathbf{X}_2\mathbf{C}_1^{-1}\mathbf{X}_1'\boldsymbol{\Sigma}_1^{-1}\mathbf{Y}_1+(\mathbf{C}_1+\mathbf{C}_2)^{-1}\mathbf{X}_2'\boldsymbol{\Sigma}_2^{-}\mathbf{Y}_2\\ &= \widehat{\boldsymbol{\beta}}(\mathbf{Y}_1)-(\mathbf{C}_1+\mathbf{C}_2)^{-1}\mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{X}_2\mathbf{C}_1^{-1}\mathbf{X}_1'\boldsymbol{\Sigma}_1^{-1}\mathbf{Y}_1+(\mathbf{C}_1+\mathbf{C}_2)^{-1}\mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{Y}_2\\ &= \widehat{\boldsymbol{\beta}}(\mathbf{Y}_1)+(\mathbf{C}_1+\mathbf{C}_2)^{-1}\mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\big[\mathbf{Y}_2-\mathbf{X}_2\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1)\big].\end{aligned}$$

Since

$$\mathrm{Var}\big[\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1,\mathbf{Y}_2)\big]=(\mathbf{C}_1+\mathbf{C}_2)^{-1}$$

and

$$(\mathbf{C}_1+\mathbf{C}_2)^{-1}=\mathbf{C}_1^{-1}-\mathbf{C}_1^{-1}\mathbf{X}_2'(\boldsymbol{\Sigma}_2+\mathbf{X}_2\mathbf{C}_1^{-1}\mathbf{X}_2')^{-1}\mathbf{X}_2\mathbf{C}_1^{-1},$$

we have

$$\mathrm{Var}\big[\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1,\mathbf{Y}_2)\big]=\mathrm{Var}\big[\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1)\big]-\mathbf{C}_1^{-1}\mathbf{X}_2'(\boldsymbol{\Sigma}_2+\mathbf{X}_2\mathbf{C}_1^{-1}\mathbf{X}_2')^{-1}\mathbf{X}_2\mathbf{C}_1^{-1}.$$

□

Theorem 2 *If*

$$\begin{pmatrix}\mathbf{Y}_1\\ \mathbf{Y}_2\end{pmatrix}\sim_{n+m}\left[\begin{pmatrix}\mathbf{X}_1, & \mathbf{0}\\ \mathbf{D}, & \mathbf{X}_2\end{pmatrix}\begin{pmatrix}\boldsymbol{\beta}\\ \gamma\end{pmatrix},\begin{pmatrix}\boldsymbol{\Sigma}_1, & \mathbf{0}\\ \mathbf{0}, & \boldsymbol{\Sigma}_2\end{pmatrix}\right],$$

then

$$\begin{aligned}\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1,\mathbf{Y}_2) &= \widehat{\boldsymbol{\beta}}(\mathbf{Y}_1)-\mathbf{C}_1^{-1}\mathbf{D}'\big[\mathbf{M}_{X_2}(\boldsymbol{\Sigma}_2+\mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}')\mathbf{M}_{X_2}\big]^{+}\mathbf{D}\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1)\\ &\quad+\big[\mathbf{C}_1+\mathbf{D}'(\mathbf{M}_{X_2}\boldsymbol{\Sigma}_2\mathbf{M}_{X_2})^{+}\mathbf{D}\big]^{-1}\mathbf{D}'(\mathbf{M}_{X_2}\boldsymbol{\Sigma}_2\mathbf{M}_{X_2})^{+}\mathbf{Y}_2\end{aligned}$$

and

$$\mathrm{Var}\big[\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1,\mathbf{Y}_2)\big]=\mathrm{Var}\big[\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1)\big]-\mathbf{C}_1^{-1}\mathbf{D}'\big[\mathbf{M}_{X_2}(\boldsymbol{\Sigma}_2+\mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}')\mathbf{M}_{X_2}\big]^{+}\mathbf{D}\mathbf{C}_1^{-1}.$$

Proof

$$\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1,\mathbf{Y}_2)=(\mathbf{I},\mathbf{0})\begin{pmatrix}\boxed{11}, & \boxed{12}\\ \boxed{21}, & \boxed{22}\end{pmatrix}\begin{pmatrix}\mathbf{X}_1'\boldsymbol{\Sigma}_1^{-1}\mathbf{Y}_1+\mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{Y}_2\\ \mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{Y}_2\end{pmatrix},$$

where

$$\begin{aligned}\begin{pmatrix}\boxed{11}, & \boxed{12}\\ \boxed{21}, & \boxed{22}\end{pmatrix} &= \left[\begin{pmatrix}\mathbf{X}_1', & \mathbf{D}'\\ \mathbf{0}, & \mathbf{X}_2'\end{pmatrix}\begin{pmatrix}\boldsymbol{\Sigma}_1^{-1}, & \mathbf{0}\\ \mathbf{0}, & \boldsymbol{\Sigma}_2^{-1}\end{pmatrix}\begin{pmatrix}\mathbf{X}_1, & \mathbf{0}\\ \mathbf{D}, & \mathbf{X}_2\end{pmatrix}\right]^{-1}\\ &= \begin{pmatrix}\mathbf{C}_1+\mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{D}, & \mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{X}_2\\ \mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{D}, & \mathbf{C}_2\end{pmatrix}^{-1},\end{aligned}$$

$$\boxed{11} = \left[\mathbf{C}_1 + \mathbf{D}'(\mathbf{M}_{X_2}\boldsymbol{\Sigma}_2\mathbf{M}_{X_2})^+\mathbf{D}\right]^{-1},$$
$$\boxed{12} = -\left[\mathbf{C}_1 + \mathbf{D}'(\mathbf{M}_{X_2}\boldsymbol{\Sigma}_2\mathbf{M}_{X_2})^+\mathbf{D}\right]^{-1}\mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{X}_2\mathbf{C}_2^{-1},$$
$$\boxed{21} = -\mathbf{C}_2^{-1}\mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{D}\left[\mathbf{C}_1 + \mathbf{D}'(\mathbf{M}_{X_2}\boldsymbol{\Sigma}_2\mathbf{M}_{X_2})^+\mathbf{D}\right]^{-1},$$
$$\boxed{22} = \left[\mathbf{C}_2 - \mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{D}(\mathbf{C}_1 + \mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{D})^{-1}\mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{X}_2\right]^{-1}$$
$$= \left[\mathbf{X}_2'(\boldsymbol{\Sigma}_2 + \mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}')^{-1}\mathbf{X}_2\right]^{-1}.$$

Thus

$$\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1,\mathbf{Y}_2) = \left[\mathbf{C}_1 + \mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{D} - \mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{X}_2\mathbf{C}_2^{-1}\mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{D}\right]^{-1}(\mathbf{X}_1'\boldsymbol{\Sigma}_1^{-1}\mathbf{Y}_1$$
$$+\mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{Y}_2 - \mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{X}_2\mathbf{C}_2^{-1}\mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{Y}_2)$$
$$= \Big\{(\mathbf{C}_1 + \mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{D})^{-1} + (\mathbf{C}_1 + \mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{D})^{-1}\mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{X}_2\big[\mathbf{C}_2 - \mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{D}$$
$$\times (\mathbf{C}_1 + \mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{D})^{-1}\mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{X}_2\big]^{-1}\mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{D}(\mathbf{C}_1 + \mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{D})^{-1}\Big\}\mathbf{X}_1'\boldsymbol{\Sigma}_1^{-1}\mathbf{Y}_1$$
$$+ \left[\mathbf{C}_1 + \mathbf{D}'(\mathbf{M}_{X_2}\boldsymbol{\Sigma}_2\mathbf{M}_{X_2})^+\mathbf{D}\right]^{-1}\mathbf{D}'(\mathbf{M}_{X_2}\boldsymbol{\Sigma}_2\mathbf{M}_{X_2})^+\mathbf{Y}_2$$
$$= \widehat{\boldsymbol{\beta}}(\mathbf{Y}_1) - \mathbf{C}_1^{-1}\mathbf{D}'(\boldsymbol{\Sigma}_2 + \mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}')^{-1}\mathbf{D}\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1) + \mathbf{C}_1^{-1}\mathbf{D}'(\boldsymbol{\Sigma}_2 + \mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}')^{-1}\mathbf{X}_2$$
$$\times \left[\mathbf{X}_2'(\boldsymbol{\Sigma}_2 + \mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}')^{-1}\mathbf{X}_2\right]^{-1}\mathbf{X}_2'(\boldsymbol{\Sigma}_2 + \mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}')^{-1}\mathbf{D}\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1)$$
$$+ \left[\mathbf{C}_1 + \mathbf{D}'(\mathbf{M}_{X_2}\boldsymbol{\Sigma}_2\mathbf{M}_{X_2})^+\mathbf{D}\right]^{-1}\mathbf{D}'(\mathbf{M}_{X_2}\boldsymbol{\Sigma}_2\mathbf{M}_{X_2})^+\mathbf{Y}_2$$
$$= \widehat{\boldsymbol{\beta}}(\mathbf{Y}_1) - \mathbf{C}_1^{-1}\mathbf{D}'\left[\mathbf{M}_{X_2}(\boldsymbol{\Sigma}_2 + \mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}')\mathbf{M}_{X_2}\right]^+\mathbf{D}\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1)$$
$$+ \left[\mathbf{C}_1 + \mathbf{D}'(\mathbf{M}_{X_2}\boldsymbol{\Sigma}_2\mathbf{M}_{X_2})^+\mathbf{D}\right]^{-1}\mathbf{D}'(\mathbf{M}_{X_2}\boldsymbol{\Sigma}_2\mathbf{M}_{X_2})^+\mathbf{Y}_2.$$

$$\mathrm{Var}\left[\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1,\mathbf{Y}_2)\right] = (\mathbf{I},\mathbf{0})\begin{pmatrix}\mathbf{C}_1 + \mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{D}, & \mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{X}_2\\ \mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{D}, & \mathbf{C}_2\end{pmatrix}^{-1}\begin{pmatrix}\mathbf{I}\\ \mathbf{0}\end{pmatrix}$$
$$= (\mathbf{C}_1 + \mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{D})^{-1} + (\mathbf{C}_1 + \mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{D})^{-1}\mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{X}_2\big[\mathbf{C}_2 - \mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{D}$$
$$\times (\mathbf{C}_1 + \mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{D})^{-1}\mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{X}_2\big]^{-1}\mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{D}(\mathbf{C}_1 + \mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{D})^{-1}$$
$$= \mathbf{C}_1^{-1} - \mathbf{C}_1^{-1}\mathbf{D}'(\boldsymbol{\Sigma}_2 + \mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}')^{-1}\mathbf{D}\mathbf{C}_1^{-1} + \mathbf{C}_1^{-1}\mathbf{D}'(\boldsymbol{\Sigma}_2 + \mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}')^{-1}\mathbf{X}_2$$
$$\times \left[\mathbf{X}_2'(\boldsymbol{\Sigma}_2 + \mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}')^{-1}\mathbf{X}_2\right]^{-1}\mathbf{X}_2'(\boldsymbol{\Sigma}_2 + \mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}')^{-1}\mathbf{D}\mathbf{C}_1^{-1}$$
$$= \mathbf{C}_1^{-1} - \mathbf{C}_1^{-1}\mathbf{D}'\left[\mathbf{M}_{X_2}(\boldsymbol{\Sigma}_2 + \mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}')\mathbf{M}_{X_2}\right]^+\mathbf{D}\mathbf{C}_1^{-1}.$$

□

Theorem 3 *If*

$$\begin{pmatrix}\mathbf{Y}_1\\ \mathbf{Y}_2\end{pmatrix} \sim_{n+m} \left[\begin{pmatrix}\mathbf{X}_1\\ \mathbf{X}_2\end{pmatrix}\boldsymbol{\beta}, \begin{pmatrix}\boldsymbol{\Sigma}_1, & \mathbf{0}\\ \mathbf{0}, & \boldsymbol{\Sigma}_2\end{pmatrix}\right], \quad \mathbf{g} + \mathbf{G}\boldsymbol{\beta} = \mathbf{0},$$

then

$$\widehat{\widehat{\boldsymbol{\beta}}}(\mathbf{Y}_1,\mathbf{Y}_2) = \widehat{\boldsymbol{\beta}}(\mathbf{Y}_1) - (\mathbf{C}_1 + \mathbf{C}_2)^{-1}\mathbf{G}'\left[\mathbf{G}(\mathbf{C}_1 + \mathbf{C}_2)^{-1}\mathbf{G}'\right]^{-1}\left[\mathbf{G}\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1) + \mathbf{g}\right]$$
$$+ \left[\mathbf{M}_{G'}(\mathbf{C}_1 + \mathbf{C}_2)\mathbf{M}_{G'}\right]^+\mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\left[\mathbf{Y}_2 - \mathbf{X}_2\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1)\right]$$

and

$$\operatorname{Var}\left[\widehat{\widehat{\boldsymbol\beta}}(\mathbf{Y}_1,\mathbf{Y}_2)\right] = \operatorname{Var}\left[\widehat{\boldsymbol\beta}(\mathbf{Y}_1)\right] - \mathbf{C}_1^{-1}\mathbf{X}_2'(\boldsymbol\Sigma_2 + \mathbf{X}_2\mathbf{C}_1^{-1}\mathbf{X}_2')^{-1}\mathbf{X}_2\mathbf{C}_1^{-1}$$
$$-(\mathbf{C}_1+\mathbf{C}_2)^{-1}\mathbf{G}'\left[\mathbf{G}(\mathbf{C}_1+\mathbf{C}_2)^{-1}\mathbf{G}'\right]^{-1}\mathbf{G}(\mathbf{C}_1+\mathbf{C}_2)^{-1}.$$

Proof With respect to Lemma 3

$$\widehat{\widehat{\boldsymbol\beta}}(\mathbf{Y}_1,\mathbf{Y}_2) = \widehat{\boldsymbol\beta}(\mathbf{Y}_1,\mathbf{Y}_2) - \mathbf{V}\mathbf{G}'(\mathbf{G}\mathbf{V}\mathbf{G}')^{+}\left[\mathbf{G}\widehat{\boldsymbol\beta}(\mathbf{Y}_1,\mathbf{Y}_2)+\mathbf{g}\right],$$

where $\mathbf{V} = (\mathbf{C}_1+\mathbf{C}_2)^{-1}$ and (see Theorem 1)

$$\widehat{\boldsymbol\beta}(\mathbf{Y}_1,\mathbf{Y}_2) = \widehat{\boldsymbol\beta}(\mathbf{Y}_1) + (\mathbf{C}_1+\mathbf{C}_2)^{-1}\mathbf{X}_2'\boldsymbol\Sigma_2^{-1}\left[\mathbf{Y}_2 - \mathbf{X}_2\widehat{\boldsymbol\beta}(\mathbf{Y}_1)\right]$$

Thus

$$\widehat{\widehat{\boldsymbol\beta}}(\mathbf{Y}_1,\mathbf{Y}_2) = \widehat{\boldsymbol\beta}(\mathbf{Y}_1) + (\mathbf{C}_1+\mathbf{C}_2)^{-1}\mathbf{X}_2'\boldsymbol\Sigma_2^{-1}\left[\mathbf{Y}_2 - \mathbf{X}_2\widehat{\boldsymbol\beta}(\mathbf{Y}_1)\right]$$
$$-(\mathbf{C}_1+\mathbf{C}_2)^{-1}\mathbf{G}'\left[\mathbf{G}(\mathbf{C}_1+\mathbf{C}_2)^{-1}\mathbf{G}'\right]^{-1}\left[\mathbf{G}\widehat{\boldsymbol\beta}(\mathbf{Y}_1,\mathbf{Y}_2)+\mathbf{g}\right]$$
$$= \widehat{\boldsymbol\beta}(\mathbf{Y}_1) + (\mathbf{C}_1+\mathbf{C}_2)^{-1}\mathbf{X}_2'\boldsymbol\Sigma_2^{-1}\left[\mathbf{Y}_2 - \mathbf{X}_2\widehat{\boldsymbol\beta}(\mathbf{Y}_1)\right]$$
$$-(\mathbf{C}_1+\mathbf{C}_2)^{-1}\mathbf{G}'\left[\mathbf{G}(\mathbf{C}_1+\mathbf{C}_2)^{-1}\mathbf{G}'\right]^{-1}$$
$$\times\left(\mathbf{G}\left\{\widehat{\boldsymbol\beta}(\mathbf{Y}_1) + (\mathbf{C}_1+\mathbf{C}_2)^{-1}\mathbf{X}_2'\boldsymbol\Sigma_2^{-1}\left[\mathbf{Y}_2 - \mathbf{X}_2\widehat{\boldsymbol\beta}(\mathbf{Y}_1)\right]\right\}+\mathbf{g}\right)$$
$$= \widehat{\boldsymbol\beta}(\mathbf{Y}_1) + \left[\mathbf{M}_{G'}(\mathbf{C}_1+\mathbf{C}_2)\mathbf{M}_{G'}\right]^{+}\mathbf{X}_2'\boldsymbol\Sigma_2^{-1}\left[\mathbf{Y}_2 - \mathbf{X}_2\widehat{\boldsymbol\beta}(\mathbf{Y}_1)\right]$$
$$-(\mathbf{C}_1+\mathbf{C}_2)^{-1}\mathbf{G}'\left[\mathbf{G}(\mathbf{C}_1+\mathbf{C}_2)^{-1}\mathbf{G}'\right]^{-1}\left[\mathbf{G}\widehat{\boldsymbol\beta}(\mathbf{Y}_1)+\mathbf{g}\right].$$

With respect to Lemma 3

$$\operatorname{Var}\left[\widehat{\widehat{\boldsymbol\beta}}(\mathbf{Y}_1,\mathbf{Y}_2)\right] = \operatorname{Var}\left[\widehat{\boldsymbol\beta}(\mathbf{Y}_1,\mathbf{Y}_2)\right]$$
$$-\operatorname{Var}\left[\widehat{\boldsymbol\beta}(\mathbf{Y}_1,\mathbf{Y}_2)\right]\mathbf{G}'\left\{\mathbf{G}\operatorname{Var}\left[\widehat{\boldsymbol\beta}(\mathbf{Y}_1,\mathbf{Y}_2)\right]\mathbf{G}'\right\}^{-1}\mathbf{G}\operatorname{Var}\left[\widehat{\boldsymbol\beta}(\mathbf{Y}_1,\mathbf{Y}_2)\right]$$
$$= \operatorname{Var}\left[\widehat{\boldsymbol\beta}(\mathbf{Y}_1)\right] - \mathbf{C}_1^{-1}\mathbf{X}_2'(\boldsymbol\Sigma_2 + \mathbf{X}_2\mathbf{C}_1^{-1}\mathbf{X}_2')^{-1}\mathbf{X}_2\mathbf{C}_1^{-1}$$
$$-(\mathbf{C}_1+\mathbf{C}_2)^{-1}\mathbf{G}'\left[\mathbf{G}(\mathbf{C}_1+\mathbf{C}_2)^{-1}\mathbf{G}'\right]^{-1}\mathbf{G}(\mathbf{C}_1+\mathbf{C}_2)^{-1}.$$

□

Theorem 4 *If*

$$\begin{pmatrix}\mathbf{Y}_1\\ \mathbf{Y}_2\end{pmatrix} \sim_{n+m} \left[\begin{pmatrix}\mathbf{X}_1, & \mathbf{0}\\ \mathbf{D}, & \mathbf{X}_2\end{pmatrix}\begin{pmatrix}\boldsymbol\beta\\ \gamma\end{pmatrix}, \begin{pmatrix}\boldsymbol\Sigma_1, & \mathbf{0}\\ \mathbf{0}, & \boldsymbol\Sigma_2\end{pmatrix}\right], \quad \mathbf{g} + \mathbf{G}\boldsymbol\beta = \mathbf{0},$$

then

$$\widehat{\widehat{\boldsymbol\beta}}(\mathbf{Y}_1,\mathbf{Y}_2) = \widehat{\boldsymbol\beta}(\mathbf{Y}_1) - \mathbf{C}_1^{-1}\mathbf{D}'\left[\mathbf{M}_{X_2}(\boldsymbol\Sigma_2 + \mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}')\mathbf{M}_{X_2}\right]^{+}\mathbf{D}\widehat{\boldsymbol\beta}(\mathbf{Y}_1)$$
$$+\left[\mathbf{C}_1 + \mathbf{D}'(\mathbf{M}_{X_2}\boldsymbol\Sigma_2\mathbf{M}_{X_2})^{+}\mathbf{D}\right]^{-1}\mathbf{D}'(\mathbf{M}_{X_2}\boldsymbol\Sigma_2\mathbf{M}_{X_2})^{+}\mathbf{Y}_2$$
$$-\left[\mathbf{C}_1 + \mathbf{D}'(\mathbf{M}_{X_2}\boldsymbol\Sigma_2\mathbf{M}_{X_2})^{+}\mathbf{D}\right]^{-1}\mathbf{G}'\left\{\mathbf{G}\left[\mathbf{C}_1 + \mathbf{D}'(\mathbf{M}_{X_2}\boldsymbol\Sigma_2\mathbf{M}_{X_2})^{+}\mathbf{D}\right]^{-1}\mathbf{G}'\right\}^{-1}$$
$$\times\left[\mathbf{G}\widehat{\boldsymbol\beta}(\mathbf{Y}_1,\mathbf{Y}_2)+\mathbf{g}\right],$$

where

$$\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1,\mathbf{Y}_2) = \widehat{\boldsymbol{\beta}}(\mathbf{Y}_1) - \mathbf{C}_1^{-1}\mathbf{D}'\big[\mathbf{M}_{X_2}(\boldsymbol{\Sigma}_2 + \mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}')\mathbf{M}_{X_2}\big]^{+}\mathbf{D}\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1)$$
$$+ \big[\mathbf{C}_1 + \mathbf{D}'(\mathbf{M}_{X_2}\boldsymbol{\Sigma}_2\mathbf{M}_{X_2})^{+}\mathbf{D}\big]^{-1}\mathbf{D}'(\mathbf{M}_{X_2}\boldsymbol{\Sigma}_2\mathbf{M}_{X_2})^{+}\mathbf{Y}_2.$$

$$\mathrm{Var}\,\big[\widehat{\widehat{\boldsymbol{\beta}}}(\mathbf{Y}_1,\mathbf{Y}_2)\big] = \mathrm{Var}\,\big[\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1)\big] - \mathbf{C}_1^{-1}\mathbf{D}'\Big[\mathbf{M}_{X_2}(\boldsymbol{\Sigma}_2 + \mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}')\mathbf{M}_{X_2}\Big]^{+}\mathbf{D}\mathbf{C}_1^{-1}$$
$$- \mathrm{Var}\,\big[\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1,\mathbf{Y}_2)\big]\mathbf{G}'\Big\{\mathbf{G}\,\mathrm{Var}\,\big[\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1,\mathbf{Y}_2)\big]\mathbf{G}'\Big\}^{-1}\mathbf{G}\,\mathrm{Var}\,\big[\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1,\mathbf{Y}_2)\big],$$

and

$$\mathrm{Var}\,\big[\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1,\mathbf{Y}_2)\big] = \mathbf{C}_1^{-1} - \mathbf{C}_1^{-1}\mathbf{D}'\big[\mathbf{M}_{X_2}(\boldsymbol{\Sigma}_2 + \mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}')\mathbf{M}_{X_2}\big]^{+}\mathbf{D}\mathbf{C}_1^{-1}.$$

Proof

$$\widehat{\widehat{\boldsymbol{\beta}}}(\mathbf{Y}_1,\mathbf{Y}_2)\big] = \widehat{\boldsymbol{\beta}}(\mathbf{Y}_1,\mathbf{Y}_2) - \mathbf{V}\mathbf{G}'(\mathbf{G}\mathbf{V}\mathbf{G}')^{-1}\big[\mathbf{G}\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1,\mathbf{Y}_2) + \mathbf{g}\big],$$

where

$$\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1,\mathbf{Y}_2) = \widehat{\boldsymbol{\beta}}(\mathbf{Y}_1) - \mathbf{C}_1^{-1}\mathbf{D}'\big[\mathbf{M}_{X_2}(\boldsymbol{\Sigma}_2 + \mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}')\mathbf{M}_{X_2}\big]^{+}\mathbf{D}\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1)$$
$$+ \big[\mathbf{C}_1 + \mathbf{D}'(\mathbf{M}_{X_2}\boldsymbol{\Sigma}_2\mathbf{M}_{X_2})^{+}\mathbf{D}\big]^{-1}\mathbf{D}'(\mathbf{M}_{X_2}\boldsymbol{\Sigma}_2\mathbf{M}_{X_2})^{+}\mathbf{Y}_2,$$

(see Theorem 2)

$$\mathbf{V} = \mathrm{Var}\,\big[\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1,\mathbf{Y}_2)\big] = \big[\mathbf{C}_1 + \mathbf{D}'(\mathbf{M}_{X_2}\boldsymbol{\Sigma}_2\mathbf{M}_{X_2})^{+}\mathbf{D}\big]^{-1}.$$

Now the proof can be easily finished. □

Theorem 5 *If*

$$\begin{pmatrix}\mathbf{Y}_1\\ \mathbf{Y}_2\end{pmatrix} \sim_{n+m} \left[\begin{pmatrix}\mathbf{X}_1, & \mathbf{0}\\ \mathbf{D}, & \mathbf{X}_2\end{pmatrix}\begin{pmatrix}\boldsymbol{\beta}\\ \boldsymbol{\gamma}\end{pmatrix}, \begin{pmatrix}\boldsymbol{\Sigma}_1, & \mathbf{0}\\ \mathbf{0}, & \boldsymbol{\Sigma}_2\end{pmatrix}\right], \quad \mathbf{g} + \mathbf{G}_1\boldsymbol{\beta} + \mathbf{G}_2\boldsymbol{\gamma} = \mathbf{0},$$

then

$$\widehat{\widehat{\boldsymbol{\beta}}}(\mathbf{Y}_1,\mathbf{Y}_2) = \widehat{\boldsymbol{\beta}}(\mathbf{Y}_1) - \mathbf{C}_1'\mathbf{D}\big[\mathbf{M}_{X_2}(\boldsymbol{\Sigma}_2 + \mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}')\mathbf{M}_{X_2}\big]^{+}\mathbf{D}\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1)$$
$$+ \big[\mathbf{C}_1 + \mathbf{D}'(\mathbf{M}_{X_2}\boldsymbol{\Sigma}_2\mathbf{M}_{X_2})^{+}\mathbf{D}\big]^{-1}\mathbf{D}'(\mathbf{M}_{X_2}\boldsymbol{\Sigma}_2\mathbf{M}_{X_2})^{+}\mathbf{Y}_2 - (\mathbf{V}_{1,1},\mathbf{V}_{1,2})\begin{pmatrix}\mathbf{G}_1'\\ \mathbf{G}_2'\end{pmatrix}$$
$$\times\left[(\mathbf{G}_1,\mathbf{G}_2)\begin{pmatrix}\mathbf{V}_{1,1}, & \mathbf{V}_{1,2}\\ \mathbf{V}_{2,1}, & \mathbf{V}_{2,2}\end{pmatrix}\begin{pmatrix}\mathbf{G}_1'\\ \mathbf{G}_2'\end{pmatrix}\right]^{-1}\left[(\mathbf{G}_1,\mathbf{G}_2)\begin{pmatrix}\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1,\mathbf{Y}_2)\\ \widehat{\boldsymbol{\gamma}}(\mathbf{Y}_1,\mathbf{Y}_2)\end{pmatrix} + \mathbf{g}\right],$$

where

$$\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1,\mathbf{Y}_2) = \widehat{\boldsymbol{\beta}}(\mathbf{Y}_1) - \mathbf{C}_1\mathbf{D}\big[\mathbf{M}_{X_2}(\boldsymbol{\Sigma}_2 + \mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}')\mathbf{M}_{X_2}\big]^{+}\mathbf{D}\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1)$$
$$+ \big[\mathbf{C}_1 + \mathbf{D}'(\mathbf{M}_{X_2}\boldsymbol{\Sigma}_2\mathbf{M}_{X_2})^{+}\mathbf{D}\big]^{-1}\mathbf{D}'(\mathbf{M}_{X_2}\boldsymbol{\Sigma}_2\mathbf{M}_{X_2})^{+}\mathbf{Y}_2,$$

$$\widehat{\boldsymbol{\gamma}}(\mathbf{Y}_1,\mathbf{Y}_2) = \mathbf{C}_2^{-1}\mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{Y}_2 - \mathbf{C}_2^{-1}\mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{D}\big[\mathbf{C}_1 + \mathbf{D}'(\mathbf{M}_{X_2}\boldsymbol{\Sigma}_2\mathbf{M}_{X_2})^{+}\mathbf{D}\big]^{-1}$$
$$\times\big[\mathbf{X}_1'\boldsymbol{\Sigma}_1^{-1}\mathbf{Y}_1 + \mathbf{D}'(\mathbf{M}_{X_2}\boldsymbol{\Sigma}_2\mathbf{M}_{X_2})^{+}\mathbf{Y}_2\big],$$

$$\begin{pmatrix}\mathbf{V}_{1,1}, & \mathbf{V}_{1,2}\\ \mathbf{V}_{2,1}, & \mathbf{V}_{2,2}\end{pmatrix} = \mathrm{Var}\begin{pmatrix}\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1,\mathbf{Y}_2)\\ \widehat{\boldsymbol{\gamma}}(\mathbf{Y}_1,\mathbf{Y}_2)\end{pmatrix},$$

$$\begin{aligned}\mathbf{V}_{1,1} &= \left[\mathbf{C}_1 + \mathbf{D}'(\mathbf{M}_{X_2}\boldsymbol{\Sigma}_2\mathbf{M}_{X_2})^+\mathbf{D}\right]^{-1},\\ \mathbf{V}_{1,2} &= -\left[\mathbf{C}_1 + \mathbf{D}'(\mathbf{M}_{X_2}\boldsymbol{\Sigma}_2\mathbf{M}_{X_2})^+\mathbf{D}\right]^{-1}\mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{X}_2\mathbf{C}_2^{-1},\\ \mathbf{V}_{2,2} &= \left[\mathbf{X}_2'(\boldsymbol{\Sigma}_2 + \mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}')^{-1}\mathbf{X}_2\right]^{-1}\end{aligned}$$

and

$$-\mathbf{C}_2^{-1}\mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{D}\left[\mathbf{C}_1 + \mathbf{D}'(\mathbf{M}_{X_2}\boldsymbol{\Sigma}_2\mathbf{M}_{X_2})^+\mathbf{D}\right]^{-1} =$$
$$-\left[\mathbf{X}_2'(\boldsymbol{\Sigma}_2 + \mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}')^{-1}\mathbf{X}_2\right]^{-1}\mathbf{X}_2'(\boldsymbol{\Sigma}_2 + \mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}')^{-1}\mathbf{D}\mathbf{C}_1^{-1}.$$

Proof It is valid that

$$\begin{pmatrix}\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1,\mathbf{Y}_2)\\ \widehat{\gamma}(\mathbf{Y}_1,\mathbf{Y}_2)\end{pmatrix} = \left[\begin{pmatrix}\mathbf{X}_1', & \mathbf{D}'\\ \mathbf{0}, & \mathbf{X}_2'\end{pmatrix}\begin{pmatrix}\boldsymbol{\Sigma}_1^{-1}, & \mathbf{0}\\ \mathbf{0}, & \boldsymbol{\Sigma}_2^{-1}\end{pmatrix}\begin{pmatrix}\mathbf{X}_1, & \mathbf{0}\\ \mathbf{D}, & \mathbf{X}_2\end{pmatrix}\right]^{-1}$$
$$\times\begin{pmatrix}\mathbf{X}_1'\boldsymbol{\Sigma}_1^{-1}\mathbf{Y}_1 + \mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{Y}_2\\ \mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{Y}_2\end{pmatrix}.$$

Now the expressions for $\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1,\mathbf{Y}_2)$ and $\widehat{\gamma}(\mathbf{Y}_1,\mathbf{Y}_2)$ and also for

$$\begin{pmatrix}\mathbf{V}_{1,1}, & \mathbf{V}_{1,2}\\ \mathbf{V}_{2,1}, & \mathbf{V}_{2,2}\end{pmatrix}$$

can be obtained easily.

With respect to Lemma 3

$$\begin{pmatrix}\widehat{\widehat{\boldsymbol{\beta}}}(\mathbf{Y}_1,\mathbf{Y}_2)\\ \widehat{\widehat{\gamma}}(\mathbf{Y}_1,\mathbf{Y}_2)\end{pmatrix} = \begin{pmatrix}\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1,\mathbf{Y}_2)\\ \widehat{\gamma}(\mathbf{Y}_1,\mathbf{Y}_2)\end{pmatrix} - \begin{pmatrix}\mathbf{V}_{1,1}, & \mathbf{V}_{1,2}\\ \mathbf{V}_{2,1}, & \mathbf{V}_{2,2}\end{pmatrix}\begin{pmatrix}\mathbf{G}_1'\\ \mathbf{G}_2'\end{pmatrix}$$
$$\times\left[(\mathbf{G}_1,\mathbf{G}_2)\begin{pmatrix}\mathbf{V}_{1,1}, & \mathbf{V}_{1,2}\\ \mathbf{V}_{2,1}, & \mathbf{V}_{2,2}\end{pmatrix}\begin{pmatrix}\mathbf{G}_1'\\ \mathbf{G}_2'\end{pmatrix}\right]^{-1}\left[(\mathbf{G}_1,\mathbf{G}_2)\begin{pmatrix}\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1,\mathbf{Y}_2)\\ \widehat{\gamma}(\mathbf{Y}_1,\mathbf{Y}_2)\end{pmatrix} + \mathbf{g}\right].$$

□

3 Original models with constraints

Theorem 6 *If*

$$\begin{pmatrix}\mathbf{Y}_1\\ \mathbf{Y}_2\end{pmatrix} \sim_{n+m}\left[\begin{pmatrix}\mathbf{X}_1\\ \mathbf{X}_2\end{pmatrix}\boldsymbol{\beta}, \begin{pmatrix}\boldsymbol{\Sigma}_1, & \mathbf{0}\\ \mathbf{0}, & \boldsymbol{\Sigma}_2\end{pmatrix}\right], \quad \mathbf{b} + \mathbf{B}\boldsymbol{\beta} = \mathbf{0},$$

then

$$\widehat{\widehat{\boldsymbol{\beta}}}(\mathbf{Y}_1,\mathbf{Y}_2) = \widehat{\widehat{\boldsymbol{\beta}}}(\mathbf{Y}_1) + \left[\mathbf{M}_{B'}(\mathbf{C}_1+\mathbf{C}_2)\mathbf{M}_{B'}\right]^+\mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\left[\mathbf{Y}_2 - \mathbf{X}_2\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1)\right]$$
$$+\left(\mathbf{B}^-_{m(C_1)} - \mathbf{B}^-_{m(C_1+C_2)}\right)\left[\mathbf{B}\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1) + \mathbf{b}\right]$$

and

$$\mathrm{Var}\left[\widehat{\widehat{\boldsymbol{\beta}}}(\mathbf{Y}_1,\mathbf{Y}_2)\right] = \mathrm{Var}\left[\widehat{\widehat{\boldsymbol{\beta}}}(\mathbf{Y}_1)\right] - \left\{(\mathbf{M}_{B'}\mathbf{C}_1\mathbf{M}_{B'})^+ - \left[\mathbf{M}_{B'}(\mathbf{C}_1+\mathbf{C}_2)\mathbf{M}_{B'}\right]^+\right\}.$$

Proof Regarding Theorem 1 and Lemma 3 we have

$$\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1, \mathbf{Y}_2) = (\mathbf{C}_1 + \mathbf{C}_2)^{-1}(\mathbf{X}_1'\boldsymbol{\Sigma}_1^{-1}\mathbf{Y}_1 + \mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{Y}_2),$$

$$\widehat{\widehat{\boldsymbol{\beta}}}(\mathbf{Y}_1, \mathbf{Y}_2) = \widehat{\boldsymbol{\beta}}(\mathbf{Y}_1, \mathbf{Y}_2) - (\mathbf{C}_1+\mathbf{C}_2)^{-1}\mathbf{B}'\big[\mathbf{B}(\mathbf{C}_1+\mathbf{C}_2)^{-1}\mathbf{B}'\big]^{-1}\big[\mathbf{B}\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1, \mathbf{Y}_2)+\mathbf{b}\big]$$

(see also in [2], p. 152).

Thus

$$\begin{aligned}
\widehat{\widehat{\boldsymbol{\beta}}}(\mathbf{Y}_1, \mathbf{Y}_2) &= \widehat{\boldsymbol{\beta}}(\mathbf{Y}_1) + (\mathbf{C}_1 + \mathbf{C}_2)^{-1}\mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\big[\mathbf{Y}_2 - \mathbf{X}_2\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1)\big] \\
&\quad - (\mathbf{C}_1 + \mathbf{C}_2)^{-1}\mathbf{B}'\big[\mathbf{B}(\mathbf{C}_1 + \mathbf{C}_2)^{-1}\mathbf{B}'\big]^{-1} \\
&\quad \times \Big\{\mathbf{B}\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1) + \mathbf{b} + \mathbf{B}(\mathbf{C}_1 + \mathbf{C}_2)^{-1}\mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\big[\mathbf{Y}_2 - \mathbf{X}_2\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1)\big]\Big\}
\end{aligned}$$

$$= \widehat{\boldsymbol{\beta}}(\mathbf{Y}_1) + \Big\{(\mathbf{C}_1+\mathbf{C}_2)^{-1} - (\mathbf{C}_1+\mathbf{C}_2)^{-1}\mathbf{B}'\big[\mathbf{B}(\mathbf{C}_1+\mathbf{C}_2)^{-1}\mathbf{B}'\big]^{-1}\mathbf{B}(\mathbf{C}_1+\mathbf{C}_2)^{-1}\Big\}$$

$$\times \mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\big[\mathbf{Y}_2 - \mathbf{X}_2\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1)\big] - (\mathbf{C}_1+\mathbf{C}_2)^{-1}\mathbf{B}'\big[\mathbf{B}(\mathbf{C}_1+\mathbf{C}_2)^{-1}\mathbf{B}'\big]^{-1}\big[\mathbf{B}\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1)+\mathbf{b}\big]$$

$$\begin{aligned}
&= \widehat{\boldsymbol{\beta}}(\mathbf{Y}_1) + \big[\mathbf{M}_{B'}(\mathbf{C}_1 + \mathbf{C}_2)\mathbf{M}_{B'}\big]^{+}\mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\big[\mathbf{Y}_2 - \mathbf{X}_2\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1)\big] \\
&\quad - (\mathbf{C}_1 + \mathbf{C}_2)^{-1}\mathbf{B}'\big[\mathbf{B}(\mathbf{C}_1 + \mathbf{C}_2)^{-1}\mathbf{B}'\big]^{-1}\big[\mathbf{B}\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1) + \mathbf{b}\big].
\end{aligned}$$

Now it is necessary to re-establish the expression

$$\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1) - (\mathbf{C}_1 + \mathbf{C}_2)^{-1}\mathbf{B}'\big[\mathbf{B}(\mathbf{C}_1 + \mathbf{C}_2)^{-1}\mathbf{B}'\big]^{-1}\big[\mathbf{B}\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1) + \mathbf{b}\big];$$

it is valid that

$$\begin{aligned}
&\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1) - (\mathbf{C}_1 + \mathbf{C}_2)^{-1}\mathbf{B}'\big[\mathbf{B}(\mathbf{C}_1 + \mathbf{C}_2)^{-1}\mathbf{B}'\big]^{-1}\big[\mathbf{B}\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1) + \mathbf{b}\big] \\
&= \widehat{\widehat{\boldsymbol{\beta}}}(\mathbf{Y}_1) + \Big\{\mathbf{C}_1^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}_1^{-1}\mathbf{B}')^{-1} - (\mathbf{C}_1 + \mathbf{C}_2)^{-1}\mathbf{B}'\big[\mathbf{B}(\mathbf{C}_1 + \mathbf{C}_2)^{-1}\mathbf{B}'\big]^{-1}\Big\} \\
&\quad \times \big[\mathbf{B}\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1) + \mathbf{b}\big] = \widehat{\widehat{\boldsymbol{\beta}}}(\mathbf{Y}_1) + \Big\{\mathbf{B}^-_{m(C_1)} - \mathbf{B}^-_{m(C_1+C_2)}\Big\}\big[\mathbf{B}\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1) + \mathbf{b}\big].
\end{aligned}$$

As far as $\operatorname{Var}\big[\widehat{\widehat{\boldsymbol{\beta}}}(\mathbf{Y}_1, \mathbf{Y}_2)\big]$ is concerned, it is valid that

$$\begin{aligned}
\operatorname{Var}\big[\widehat{\widehat{\boldsymbol{\beta}}}(\mathbf{Y}_1, \mathbf{Y}_2)\big] &= (\mathbf{C}_1+\mathbf{C}_2)^{-1} - (\mathbf{C}_1+\mathbf{C}_2)^{-1}\mathbf{B}'\big[\mathbf{B}(\mathbf{C}_1+\mathbf{C}_2)^{-1}\mathbf{B}'\big]^{-1}\mathbf{B}(\mathbf{C}_1+\mathbf{C}_2)^{-1} \\
&= \big[\mathbf{M}_{B'}(\mathbf{C}_1 + \mathbf{C}_2)\mathbf{M}_{B'}\big]^{+} = (\mathbf{M}_{B'}\mathbf{C}_1\mathbf{M}_{B'})^{+} \\
&\quad - \Big\{(\mathbf{M}_{B'}\mathbf{C}_1\mathbf{M}_{B'})^{+} - \big[\mathbf{M}_{B'}(\mathbf{C}_1 + \mathbf{C}_2)\mathbf{M}_{B'}\big]^{+}\Big\} \\
&= \operatorname{Var}\big[\widehat{\widehat{\boldsymbol{\beta}}}(\mathbf{Y}_1)\big] - \Big\{(\mathbf{M}_{B'}\mathbf{C}_1\mathbf{M}_{B'})^{+} - \big[\mathbf{M}_{B'}(\mathbf{C}_1 + \mathbf{C}_2)\mathbf{M}_{B'}\big]^{+}\Big\}.
\end{aligned}$$

□

Theorem 7 *If*

$$\begin{pmatrix}\mathbf{Y}_1\\ \mathbf{Y}_2\end{pmatrix} \sim_{n+m} \left[\begin{pmatrix}\mathbf{X}_1\\ \mathbf{X}_2\end{pmatrix}\boldsymbol{\beta}, \begin{pmatrix}\boldsymbol{\Sigma}_1, & \mathbf{0}\\ \mathbf{0}, & \boldsymbol{\Sigma}_2\end{pmatrix},\right], \quad \mathbf{b} + \mathbf{B}\boldsymbol{\beta} = \mathbf{0},\ \mathbf{g} + \mathbf{G}\boldsymbol{\beta} = \mathbf{0},$$

then ($\widehat{\widehat{\widehat{}}}$ *denotes the estimator satisfying both constraints)*

$$\widehat{\widehat{\widehat{\boldsymbol{\beta}}}}(\mathbf{Y}_1,\mathbf{Y}_2)=\widehat{\widehat{\boldsymbol{\beta}}}(\mathbf{Y}_1)+\mathbf{V}\mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\big[\mathbf{Y}_2-\mathbf{X}_2\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1)\big]$$
$$+\big[\mathbf{B}^-_{m(C_1)}-\mathbf{B}^-_{m(C_1+C_2)}\big]\big[\mathbf{B}\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1)+\mathbf{b}\big]$$
$$-\big[(\mathbf{V}+\mathbf{M}_{G'})^++\mathbf{G}'\mathbf{G}\big]^{-1}\mathbf{G}'\Big\{\mathbf{G}\big[(\mathbf{V}+\mathbf{M}_{G'})^++\mathbf{G}'\mathbf{G}\big]^{-1}\mathbf{G}'\Big\}^+\big[\mathbf{G}\widehat{\widehat{\boldsymbol{\beta}}}(\mathbf{Y}_1,\mathbf{Y}_2)+\mathbf{g}\big],$$

where

$$\mathbf{V}=\operatorname{Var}\big[\widehat{\widehat{\boldsymbol{\beta}}}(\mathbf{Y}_1,\mathbf{Y}_2)\big]$$
$$=(\mathbf{C}_1+\mathbf{C}_2)^{-1}-(\mathbf{C}_1+\mathbf{C}_2)^{-1}\mathbf{B}'\big[\mathbf{B}(\mathbf{C}_1+\mathbf{C}_2)^{-1}\mathbf{B}'\big]^{-1}\mathbf{B}(\mathbf{C}_1+\mathbf{C}_2)^{-1}$$
$$=\big[\mathbf{M}_{B'}(\mathbf{C}_1+\mathbf{C}_2)\mathbf{M}_{B'}\big]^+$$

and

$$\operatorname{Var}\big[\widehat{\widehat{\widehat{\boldsymbol{\beta}}}}(\mathbf{Y}_1,\mathbf{Y}_2)\big]=\big[(\mathbf{V}+\mathbf{M}_{G'})^++\mathbf{G}'\mathbf{G}\big]^{-1}-\big[(\mathbf{V}+\mathbf{M}_{G'})^++\mathbf{G}'\mathbf{G}\big]^{-1}\mathbf{G}'$$
$$\times\Big\{\mathbf{G}\big[(\mathbf{V}+\mathbf{M}_{G'})^++\mathbf{G}'\mathbf{G}\big]^{-1}\mathbf{G}'\Big\}^+\big[(\mathbf{V}+\mathbf{M}_{G'})^++\mathbf{G}'\mathbf{G}\big]^{-1}-\mathbf{M}_{G'}.$$

Proof If

$$\begin{pmatrix}\mathbf{Y}_1\\\mathbf{Y}_2\end{pmatrix}\sim_{n+m}\left[\begin{pmatrix}\mathbf{X}_1\\\mathbf{X}_2\end{pmatrix}\boldsymbol{\beta},\begin{pmatrix}\boldsymbol{\Sigma}_1, & \mathbf{0}\\ \mathbf{0}, & \boldsymbol{\Sigma}_2\end{pmatrix}\right],$$
$$\mathbf{b}+\mathbf{B}\boldsymbol{\beta}=\mathbf{0},\quad \mathbf{g}+\mathbf{G}\boldsymbol{\beta}=\mathbf{0},$$

then with respect to Theorem 6 and Lemma 3

$$\widehat{\widehat{\widehat{\boldsymbol{\beta}}}}(\mathbf{Y}_1,\mathbf{Y}_2)=\widehat{\widehat{\boldsymbol{\beta}}}(\mathbf{Y}_1)+\big[\mathbf{M}_{B'}(\mathbf{C}_1+\mathbf{C}_2)\mathbf{M}_{B'}\big]^+\mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\big[\mathbf{Y}_2-\mathbf{X}_2\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1)\big]$$
$$+\big(\mathbf{B}^-_{m(C_1)}-\mathbf{B}^-_{m(C_1+C_2)}\big)\big[\mathbf{B}\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1)+\mathbf{b}\big]$$
$$-\big[(\mathbf{V}+\mathbf{M}_{G'})^++\mathbf{G}'\mathbf{G}\big]^{-1}\mathbf{G}'\Big\{\mathbf{G}\big[(\mathbf{V}+\mathbf{M}_{G'})^++\mathbf{G}'\mathbf{G}\big]^{-1}\mathbf{G}'\Big\}^+\big[\mathbf{G}\widehat{\widehat{\boldsymbol{\beta}}}(\mathbf{Y}_1,\mathbf{Y}_2)+\mathbf{g}\big],$$

where

$$\mathbf{V}=\operatorname{Var}\big[\widehat{\widehat{\boldsymbol{\beta}}}(\mathbf{Y}_1,\mathbf{Y}_2)\big]$$
$$=(\mathbf{C}_1+\mathbf{C}_2)^{-1}-(\mathbf{C}_1+\mathbf{C}_2)^{-1}\mathbf{B}'\big[\mathbf{B}(\mathbf{C}_1+\mathbf{C}_2)^{-1}\mathbf{B}'\big]^{-1}\mathbf{B}(\mathbf{C}_1+\mathbf{C}_2)^{-1}$$

and

$$\widehat{\widehat{\boldsymbol{\beta}}}(\mathbf{Y}_1,\mathbf{Y}_2)=\widehat{\widehat{\boldsymbol{\beta}}}(\mathbf{Y}_1)+\big[\mathbf{M}_{B'}(\mathbf{C}_1+\mathbf{C}_2)\mathbf{M}_{B'}\big]^+\mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\big[\mathbf{Y}_2-\mathbf{X}_2\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1)\big]$$
$$+\Big\{\mathbf{B}^-_{m(C_1)}-\mathbf{B}^-_{m(C_1+C_2)}\Big\}\big[\mathbf{B}\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1)+\mathbf{b}\big].$$

The expression for $\operatorname{Var}\big[\widehat{\widehat{\widehat{\boldsymbol{\beta}}}}(\mathbf{Y}_1,\mathbf{Y}_2)\big]$ can be easily obtained. □

Theorem 8 *If*

$$\begin{pmatrix}\mathbf{Y}_1\\ \mathbf{Y}_2\end{pmatrix} \sim_{n+m} \left[\begin{pmatrix}\mathbf{X}_1, & \mathbf{0}\\ \mathbf{D}, & \mathbf{X}_2\end{pmatrix}\begin{pmatrix}\boldsymbol{\beta}\\ \gamma\end{pmatrix}, \begin{pmatrix}\boldsymbol{\Sigma}_1, & \mathbf{0}\\ \mathbf{0}, & \boldsymbol{\Sigma}_2\end{pmatrix}\right], \quad \mathbf{b}+\mathbf{B}\boldsymbol{\beta}=\mathbf{0},$$

then

$$\begin{aligned}\hat{\hat{\boldsymbol{\beta}}}(\mathbf{Y}_1,\mathbf{Y}_2) = \hat{\hat{\boldsymbol{\beta}}}(\mathbf{Y}_1) - \left[\mathbf{V}_1+\mathbf{I}-\mathbf{V}_1(\mathbf{V}_1+\mathbf{V}_2)^+\mathbf{V}_1\right]\left[\mathbf{I}-(\mathbf{V}_1+\mathbf{I})^{-1}\right]\\ \times\left[\mathbf{V}_2+\mathbf{I}-(\mathbf{V}_1+\mathbf{I})^{-1}\right]^+\left[\hat{\hat{\boldsymbol{\beta}}}(\mathbf{Y}_1)-\hat{\hat{\boldsymbol{\beta}}}(\mathbf{Y}_2)\right],\end{aligned}$$

where

$$\begin{aligned}\hat{\hat{\boldsymbol{\beta}}}(\mathbf{Y}_1) &= \hat{\boldsymbol{\beta}}(\mathbf{Y}_1) - \mathbf{C}_1^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}_1^{-1}\mathbf{B}')^{-1}\left[\mathbf{B}\hat{\boldsymbol{\beta}}(\mathbf{Y}_1)+\mathbf{b}\right],\\ \hat{\boldsymbol{\beta}}(\mathbf{Y}_2) &= \left[\mathbf{D}'(\mathbf{M}_{X_2}\boldsymbol{\Sigma}_2\mathbf{M}_{X_2})^+\mathbf{D}\right]^{-1}\mathbf{D}'(\mathbf{M}_{X_2}\boldsymbol{\Sigma}_2\mathbf{M}_{X_2})^+\mathbf{Y}_2,\\ \hat{\hat{\boldsymbol{\beta}}}(\mathbf{Y}_2) &= \hat{\boldsymbol{\beta}}(\mathbf{Y}_2) - \mathbf{W}_2\mathbf{B}'(\mathbf{B}\mathbf{W}_2\mathbf{B}')^{-1}\left[\mathbf{B}\hat{\boldsymbol{\beta}}(\mathbf{Y}_2)+\mathbf{b}\right],\\ \mathbf{V}_1 &= \mathbf{C}_1^{-1} - \mathbf{C}_1^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}_1^{-1}\mathbf{B}')^{-1}\mathbf{B}\mathbf{C}_1^{-1},\\ \mathbf{V}_2 &= \mathbf{W}_2 - \mathbf{W}_2\mathbf{B}'(\mathbf{B}\mathbf{W}_2\mathbf{B}')^{-1}\mathbf{B}\mathbf{W}_2,\\ \mathbf{W}_2 &= \left[\mathbf{D}'(\mathbf{M}_{X_2}\boldsymbol{\Sigma}_2\mathbf{M}_{X_2})^+\mathbf{D}\right]^{-1}\end{aligned}$$

and

$$\mathbf{V} = \mathrm{Var}\left[\hat{\hat{\boldsymbol{\beta}}}(\mathbf{Y}_1,\mathbf{Y}_2)\right] = \mathrm{Var}\left[\hat{\hat{\boldsymbol{\beta}}}(\mathbf{Y}_1)\right] - \mathbf{V}_1(\mathbf{V}_1+\mathbf{V}_2)^+\mathbf{V}_1.$$

Proof In the model

$$\begin{pmatrix}\hat{\hat{\boldsymbol{\beta}}}(\mathbf{Y}_1)\\ \hat{\hat{\boldsymbol{\beta}}}(\mathbf{Y}_2)\end{pmatrix} \sim_{2k} \left[\begin{pmatrix}\mathbf{I}\\ \mathbf{I}\end{pmatrix}\boldsymbol{\beta}, \begin{pmatrix}\mathbf{V}_1, & \mathbf{0}\\ \mathbf{0}, & \mathbf{V}_2\end{pmatrix}\right],$$

$$\mathbf{V}_1 = \mathbf{C}_1^{-1} - \mathbf{C}_1^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}_1^{-1}\mathbf{B}')^{-1}\mathbf{B}\mathbf{C}_1^{-1},$$

$$\mathbf{V}_2 = \mathbf{W}_2 - \mathbf{W}_2\mathbf{B}'(\mathbf{B}\mathbf{W}_2\mathbf{B}')^{-1}\mathbf{B}\mathbf{W}_2,$$

$$\mathbf{W}_2 = \left[\mathbf{D}'(\mathbf{M}_{X_2}\boldsymbol{\Sigma}_2\mathbf{M}_{X_2})^+\mathbf{D}\right]^{-1},$$

the BLUE of $\boldsymbol{\beta}$ is

$$\hat{\hat{\boldsymbol{\beta}}}(\mathbf{Y}_1,\mathbf{Y}_2) = \left[(\mathbf{I},\mathbf{I})^-_{m\begin{pmatrix}V_1, & 0\\ 0, & V_2\end{pmatrix}}\right]'\begin{pmatrix}\hat{\hat{\boldsymbol{\beta}}}(\mathbf{Y}_1)\\ \hat{\hat{\boldsymbol{\beta}}}(\mathbf{Y}_2)\end{pmatrix}$$

$$= \left[(\mathbf{I},\mathbf{I})\begin{pmatrix}\mathbf{V}_1+\mathbf{I}, & \mathbf{I}\\ \mathbf{I}, & \mathbf{V}_2+\mathbf{I}\end{pmatrix}^+\begin{pmatrix}\mathbf{I}\\ \mathbf{I}\end{pmatrix}\right]^{-1}(\mathbf{I},\mathbf{I})\begin{pmatrix}\mathbf{V}_1+\mathbf{I}, & \mathbf{I}\\ \mathbf{I}, & \mathbf{V}_2+\mathbf{I}\end{pmatrix}^+\begin{pmatrix}\hat{\hat{\boldsymbol{\beta}}}(\mathbf{Y}_1)\\ \hat{\hat{\boldsymbol{\beta}}}(\mathbf{Y}_2)\end{pmatrix}.$$

Here

$$\left[(\mathbf{I},\mathbf{I})\begin{pmatrix}\mathbf{V}_1+\mathbf{I}, & \mathbf{I}\\ \mathbf{I}, & \mathbf{V}_2+\mathbf{I}\end{pmatrix}^{+}\begin{pmatrix}\mathbf{I}\\ \mathbf{I}\end{pmatrix}\right]^{-1} =$$

$$= \Big\{(\mathbf{V}_1+\mathbf{I})^{-1} + \big[\mathbf{I}-(\mathbf{V}_1+\mathbf{I})^{-1}\big]\big[\mathbf{V}_2+\mathbf{I}-(\mathbf{V}_1+\mathbf{I})^{-1}\big]^{+}\big[\mathbf{I}-(\mathbf{V}_1+\mathbf{I})^{-1}\big]\Big\}^{-1}$$

$$= \mathbf{V}_1+\mathbf{I}-(\mathbf{V}_1+\mathbf{I})\big[\mathbf{I}-(\mathbf{V}_1+\mathbf{I})^{-1}\big]$$

$$\times \Big\{\mathbf{V}_2+\mathbf{I}-(\mathbf{V}_1+\mathbf{I})^{-1} + \big[\mathbf{I}-(\mathbf{V}_1+\mathbf{I})^{-1}\big](\mathbf{V}_1+\mathbf{I})\big[\mathbf{I}-(\mathbf{V}_1+\mathbf{I})^{-1}\big]\Big\}^{+}$$

$$\times \big[\mathbf{I}-(\mathbf{V}_1+\mathbf{I})^{-1}\big](\mathbf{V}_1+\mathbf{I})$$

$$= \mathbf{V}_1+\mathbf{I}-(\mathbf{V}_1+\mathbf{I})\big[\mathbf{I}-(\mathbf{V}_1+\mathbf{I})^{-1}\big](\mathbf{V}_1+\mathbf{V}_2)^{+}\big[\mathbf{I}-(\mathbf{V}_1+\mathbf{I})^{-1}\big](\mathbf{V}_1+\mathbf{I})$$

$$= \mathbf{V}_1+\mathbf{I}-\mathbf{V}_1(\mathbf{V}_1+\mathbf{V}_2)^{+}\mathbf{V}_1$$

and

$$(\mathbf{I},\mathbf{I})\begin{pmatrix}\mathbf{V}_1+\mathbf{I}, & \mathbf{I}\\ \mathbf{I}, & \mathbf{V}_2+\mathbf{I}\end{pmatrix}^{+}\begin{pmatrix}\widehat{\widehat{\boldsymbol{\beta}}}(\mathbf{Y}_1)\\ \widehat{\widehat{\boldsymbol{\beta}}}(\mathbf{Y}_2)\end{pmatrix} =$$

$$= \Big\{(\mathbf{V}_1+\mathbf{I})^{-1} - \big[\mathbf{I}-(\mathbf{V}_1+\mathbf{I})^{-1}\big]\big[\mathbf{V}_2+\mathbf{I}-(\mathbf{V}_1+\mathbf{I})^{-1}\big]^{+}(\mathbf{V}_1+\mathbf{I})^{-1}\Big\}\widehat{\widehat{\boldsymbol{\beta}}}(\mathbf{Y}_1)$$

$$+ \big[\mathbf{I}-(\mathbf{V}_1+\mathbf{I})^{-1}\big]\big[\mathbf{V}_2+\mathbf{I}-(\mathbf{V}_1+\mathbf{I})^{-1}\big]^{+}\widehat{\widehat{\boldsymbol{\beta}}}(\mathbf{Y}_2).$$

Thus

$$\widehat{\widehat{\boldsymbol{\beta}}}(\mathbf{Y}_1,\mathbf{Y}_2) = \widehat{\widehat{\boldsymbol{\beta}}}(\mathbf{Y}_1) - (\mathbf{V}_1+\mathbf{I})\big[\mathbf{I}-(\mathbf{V}_1+\mathbf{I})^{-1}\big]\Big\{\mathbf{V}_2+\mathbf{I}-(\mathbf{V}_1+\mathbf{I})^{-1}$$

$$+\big[\mathbf{I}-(\mathbf{V}_1+\mathbf{I})^{-1}\big](\mathbf{V}_1+\mathbf{I})\big[\mathbf{I}-(\mathbf{V}_1+\mathbf{I})^{-1}\big]\Big\}^{+}\big[\mathbf{I}-(\mathbf{V}_1+\mathbf{I})^{-1}\big]\widehat{\widehat{\boldsymbol{\beta}}}(\mathbf{Y}_1)$$

$$-\Big(\mathbf{V}_1+\mathbf{I}-(\mathbf{V}_1+\mathbf{I})\big[\mathbf{I}-(\mathbf{V}_1+\mathbf{I})^{-1}\big](\mathbf{V}_1+\mathbf{V}_2)^{+}\big[\mathbf{I}-(\mathbf{V}_1+\mathbf{I})^{-1}\big](\mathbf{V}_1+\mathbf{I})\Big)$$

$$\times \big[\mathbf{I}-(\mathbf{V}_1+\mathbf{I})^{-1}\big]\big[\mathbf{V}_2+\mathbf{I}-(\mathbf{V}_1+\mathbf{I})^{-1}\big]^{+}\big[(\mathbf{V}_1+\mathbf{I})^{-1}\widehat{\widehat{\boldsymbol{\beta}}}(\mathbf{Y}_1)-\widehat{\widehat{\boldsymbol{\beta}}}(\mathbf{Y}_2)\big].$$

Since

$$(\mathbf{V}_1+\mathbf{I})\big[\mathbf{I}-(\mathbf{V}_1+\mathbf{I})^{-1}\big](\mathbf{V}_1+\mathbf{V}_2)^{+}\big[\mathbf{I}-(\mathbf{V}_1+\mathbf{I})^{-1}\big]\widehat{\widehat{\boldsymbol{\beta}}}(\mathbf{Y}_1)$$

$$= \Big((\mathbf{V}_1+\mathbf{I})^{-1} + \big[\mathbf{I}-(\mathbf{V}_1+\mathbf{I})^{-1}\big]\big[\mathbf{V}_2+\mathbf{I}-(\mathbf{V}_1+\mathbf{I})^{-1}\big]^{+}\big[\mathbf{I}-(\mathbf{V}_1+\mathbf{I})^{-1}\big]\Big)^{+}$$

$$\times \big[\mathbf{I}-(\mathbf{V}_1+\mathbf{I})^{-1}\big]\big[\mathbf{V}_2+\mathbf{I}-(\mathbf{V}_1+\mathbf{I})^{-1}\big]^{+}\big[\mathbf{I}-(\mathbf{V}_1+\mathbf{I})^{-1}\big]\widehat{\widehat{\boldsymbol{\beta}}}(\mathbf{Y}_1),$$

the statement concerning the estimator is obvious.

As far as $\text{Var}\,[\widehat{\widehat{\boldsymbol\beta}}(\mathbf{Y}_1,\mathbf{Y}_2)]$ is concerned it is valid that

$$\text{Var}\,[\widehat{\widehat{\boldsymbol\beta}}(\mathbf{Y}_1,\mathbf{Y}_2)] = \left[(\mathbf{I},\mathbf{I})\begin{pmatrix}\mathbf{V}_1+\mathbf{I}, & \mathbf{I}\\ \mathbf{I}, & \mathbf{V}_2+\mathbf{I}\end{pmatrix}^{+}\begin{pmatrix}\mathbf{I}\\ \mathbf{I}\end{pmatrix}\right]^{-1} - \mathbf{I}$$
$$= \mathbf{V}_1 - \mathbf{V}_1(\mathbf{V}_1+\mathbf{V}_2)^{+}\mathbf{V}_1,$$

what was already shown. □

Theorem 9 *If*

$$\begin{pmatrix}\mathbf{Y}_1\\ \mathbf{Y}_2\end{pmatrix} \sim_{n+m} \left[\begin{pmatrix}\mathbf{X}_1, & \mathbf{0}\\ \mathbf{D}, & \mathbf{X}_2\end{pmatrix}\begin{pmatrix}\boldsymbol\beta\\ \boldsymbol\gamma\end{pmatrix}, \begin{pmatrix}\boldsymbol\Sigma_1, & \mathbf{0}\\ \mathbf{0}, & \boldsymbol\Sigma_2\end{pmatrix}\right],$$
$$\mathbf{b}+\mathbf{B}\boldsymbol\beta = \mathbf{0}, \quad \mathbf{g}+\mathbf{G}\boldsymbol\beta = \mathbf{0},$$

then

$$\widehat{\widehat{\widehat{\boldsymbol\beta}}}(\mathbf{Y}_1,\mathbf{Y}_2) = \widehat{\widehat{\boldsymbol\beta}}(\mathbf{Y}_1) - \mathbf{A}\big[\widehat{\widehat{\boldsymbol\beta}}(\mathbf{Y}_1) - \widehat{\widehat{\boldsymbol\beta}}(\mathbf{Y}_2)\big] - \big[(\mathbf{V}+\mathbf{M}_{G'})^{+} + \mathbf{G}'\mathbf{G}\big]^{-1}\mathbf{G}'$$
$$\times\left\{\mathbf{G}\big[(\mathbf{V}+\mathbf{M}_{G'})^{+} + \mathbf{G}'\mathbf{G}\big]^{-1}\mathbf{G}'\right\}^{+}\big[\mathbf{G}\widehat{\widehat{\boldsymbol\beta}}(\mathbf{Y}_1,\mathbf{Y}_2)+\mathbf{g}\big]$$

where $\mathbf{V}$ *is given in Theorem 8,*

$$\mathbf{A} = \Big(\mathbf{V}_1+\mathbf{I}-\mathbf{V}_1(\mathbf{V}_1+\mathbf{V}_2)^{+}\mathbf{V}_1\Big)\big[\mathbf{I}-(\mathbf{V}_1+\mathbf{I})^{-1}\big]\big[\mathbf{V}_2+\mathbf{I}-(\mathbf{V}_1+\mathbf{I})^{-1}\big]^{+}$$

and

$$\text{Var}\,[\widehat{\widehat{\widehat{\boldsymbol\beta}}}(\mathbf{Y}_1,\mathbf{Y}_2)] = \big[(\mathbf{V}+\mathbf{M}_{G'})^{+} + \mathbf{G}'\mathbf{G}\big]^{-1} - \big[(\mathbf{V}+\mathbf{M}_{G'})^{+} + \mathbf{G}'\mathbf{G}\big]^{-1}\mathbf{G}'$$
$$\times\left\{\mathbf{G}\big[(\mathbf{V}+\mathbf{M}_{G'})^{+} + \mathbf{G}'\mathbf{G}\big]^{-1}\mathbf{G}'\right\}^{+}\mathbf{G}\big[(\mathbf{V}+\mathbf{M}_{G'})^{+} + \mathbf{G}'\mathbf{G}\big]^{-1} - \mathbf{M}_{G'}.$$

Here

$$\begin{aligned}
\mathbf{V} &= \text{Var}\,[\widehat{\widehat{\boldsymbol\beta}}(\mathbf{Y}_1,\mathbf{Y}_2)] = \mathbf{V}_1 - \mathbf{V}_1(\mathbf{V}_1+\mathbf{V}_2)^{+}\mathbf{V}_1,\\
\mathbf{V}_1 &= \text{Var}\,[\widehat{\widehat{\boldsymbol\beta}}(\mathbf{Y}_1)] = \mathbf{C}_1^{-1} - \mathbf{C}_1^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}_1^{-1}\mathbf{B}')^{-1}\mathbf{B}\mathbf{C}_1^{-1},\\
\mathbf{V}_2 &= \text{Var}\,[\widehat{\widehat{\boldsymbol\beta}}(\mathbf{Y}_2)] = \mathbf{W} - \mathbf{W}\mathbf{B}'(\mathbf{B}\mathbf{W}\mathbf{B}')^{-1}\mathbf{B}\mathbf{W},\\
\mathbf{W} &= \big[\mathbf{D}'(\mathbf{M}_{X_2}\boldsymbol\Sigma_2\mathbf{M}_{X_2})^{+}\mathbf{D}\big]^{-1}.
\end{aligned}$$

Proof With respect to Lemma 3 it is valid that

$$\widehat{\widehat{\widehat{\boldsymbol\beta}}}(\mathbf{Y}_1,\mathbf{Y}_2) = \widehat{\widehat{\boldsymbol\beta}}(\mathbf{Y}_1,\mathbf{Y}_2) - \big[(\mathbf{V}+\mathbf{M}_{G'})^{+} + \mathbf{G}'\mathbf{G}\big]^{-1}\mathbf{G}'$$
$$\times\left\{\mathbf{G}\big[(\mathbf{V}+\mathbf{M}_{G'})^{+} + \mathbf{G}'\mathbf{G}\big]^{-1}\mathbf{G}'\right\}^{+}\big[\mathbf{G}\widehat{\widehat{\boldsymbol\beta}}(\mathbf{Y}_1,\mathbf{Y}_2)+\mathbf{g}\big],$$

where (see Theorem 8)

$$\widehat{\widehat{\boldsymbol{\beta}}}(\mathbf{Y}_1,\mathbf{Y}_2) = \widehat{\widehat{\boldsymbol{\beta}}}(\mathbf{Y}_1) - \mathbf{A}\big[\widehat{\widehat{\boldsymbol{\beta}}}(\mathbf{Y}_1) - \widehat{\widehat{\boldsymbol{\beta}}}(\mathbf{Y}_2)\big],$$
$$\mathbf{A} = \Big(\mathbf{V}_1 + \mathbf{I} - \mathbf{V}_1(\mathbf{V}_1+\mathbf{V}_2)^{+}\mathbf{V}_1\Big)\big[\mathbf{I} - (\mathbf{V}_1+\mathbf{I})^{-1}\big]\big[\mathbf{V}_2 + \mathbf{I} - (\mathbf{V}_1+\mathbf{I})^{-1}\big]^{+}.$$

□

If

$$\begin{pmatrix}\mathbf{Y}_1\\ \mathbf{Y}_2\end{pmatrix} \sim_{n+m} \left[\begin{pmatrix}\mathbf{X}_1, & \mathbf{0}\\ \mathbf{D}, & \mathbf{X}_2\end{pmatrix}\begin{pmatrix}\boldsymbol{\beta}\\ \boldsymbol{\gamma}\end{pmatrix}, \begin{pmatrix}\boldsymbol{\Sigma}_1, & \mathbf{0}\\ \mathbf{0}, & \boldsymbol{\Sigma}_2\end{pmatrix}\right],$$
$$\mathbf{b} + \mathbf{B}\boldsymbol{\beta} = \mathbf{0}, \quad \mathbf{g} + \mathbf{G}_1\boldsymbol{\beta} + \mathbf{G}_2\boldsymbol{\gamma} = \mathbf{0},$$

then the explicit expression for $\mathbf{k}$ in the relationship

$$\widehat{\widehat{\widehat{\boldsymbol{\beta}}}}(\mathbf{Y}_1,\mathbf{Y}_2) = \widehat{\widehat{\boldsymbol{\beta}}}(\mathbf{Y}_1,\mathbf{Y}_2) + \mathbf{k}$$

is rather complicated. Therefore a sequence of relationships is given which enables us to obtain the vector $\mathbf{k}$ in the actual situation at least. It is assumed that $r(\mathbf{X}_1) = k < m$, $r(\mathbf{D},\mathbf{X}_2) = k + l < m$, $\boldsymbol{\Sigma}_1$ and $\boldsymbol{\Sigma}_2$ are positive definite and $r(\mathbf{B}) = q < k$, $r(\mathbf{G}_1,\mathbf{G}_2) = r < k + l$, $r(\mathbf{G}_2) = l < r$. The notation

$$\begin{pmatrix}\mathbf{T}_{1,1}, & \mathbf{T}_{1,2}\\ \mathbf{T}_{2,1}, & \mathbf{T}_{2,2}\end{pmatrix} = \left\{\left[\begin{pmatrix}\mathbf{W}_{1,1}, & \mathbf{W}_{1,2}\\ \mathbf{W}_{2,1}, & \mathbf{W}_{2,2}\end{pmatrix} + \mathbf{M}_{\begin{pmatrix}G_1'\\ G_2'\end{pmatrix}}\right]^{+} + \begin{pmatrix}\mathbf{G}_1'\\ \mathbf{G}_2'\end{pmatrix}(\mathbf{G}_1,\mathbf{G}_2)\right\}^{-1}.$$

will be used.

It is valid that

$$\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1,\mathbf{Y}_2) = \widehat{\boldsymbol{\beta}}(\mathbf{Y}_1) - \mathbf{C}_1^{-1}\mathbf{D}'\big[\mathbf{M}_{X_2}(\boldsymbol{\Sigma}_2 + \mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}')\mathbf{M}_{X_2}\big]^{+}\mathbf{D}\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1)$$
$$+ \big[\mathbf{C}_1 + \mathbf{D}'(\mathbf{M}_{X_2}\boldsymbol{\Sigma}_2\mathbf{M}_{X_2})^{+}\mathbf{D}\big]^{-1}\mathbf{D}'(\mathbf{M}_{X_2}\boldsymbol{\Sigma}_2\mathbf{M}_{X_2})^{+}\mathbf{Y}_2,$$

$$\widehat{\boldsymbol{\gamma}}(\mathbf{Y}_1,\mathbf{Y}_2) = \big[\mathbf{X}_2'(\boldsymbol{\Sigma}_2+\mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}')^{-1}\mathbf{X}_2\big]^{-1}\mathbf{X}_2'(\boldsymbol{\Sigma}_2+\mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}')^{-1}\big[\mathbf{Y}_2 - \mathbf{D}\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1)\big],$$

$$\mathbf{V}_{1,1} = \mathrm{Var}\big[\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1,\mathbf{Y}_2)\big] = \mathbf{C}_1^{-1} - \mathbf{C}_1^{-1}\mathbf{D}'\big[\mathbf{M}_{X_2}(\boldsymbol{\Sigma}_2 + \mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}')\mathbf{M}_{X_2}\big]^{+}\mathbf{D}\mathbf{C}_1^{-1},$$
$$\mathbf{V}_{2,1} = -\big[\mathbf{X}_2'(\boldsymbol{\Sigma}_2 + \mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}')^{-1}\mathbf{X}_2\big]^{-1}\mathbf{X}_2'(\boldsymbol{\Sigma}_2 + \mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}')^{-1}\mathbf{D}\mathbf{C}_1^{-1},$$
$$\mathbf{V}_{2,2} = \big[\mathbf{X}_2'(\boldsymbol{\Sigma}_2 + \mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}')^{-1}\mathbf{X}_2\big]^{-1},$$

$$\widehat{\widehat{\boldsymbol{\beta}}}(\mathbf{Y}_1) = \widehat{\boldsymbol{\beta}}(\mathbf{Y}_1) - \mathbf{C}_1^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}_1^{-1}\mathbf{B}')^{-1}\big[\mathbf{B}\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1) + \mathbf{b}\big],$$

$$\widehat{\widehat{\boldsymbol{\beta}}}(\mathbf{Y}_1,\mathbf{Y}_2) = \widehat{\boldsymbol{\beta}}(\mathbf{Y}_1,\mathbf{Y}_2) - \mathbf{V}_{1,1}\mathbf{B}'\Big\{\mathbf{B}\mathbf{V}_{1,1}\mathbf{B}'\Big\}^{-1}\big[\mathbf{B}\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1,\mathbf{Y}_2) + \mathbf{b}\big],$$

$$\begin{pmatrix} \widehat{\widehat{\boldsymbol{\beta}}}(\mathbf{Y}_1,\mathbf{Y}_2) \\ \widehat{\widehat{\gamma}}(\mathbf{Y}_1,\mathbf{Y}_2) \end{pmatrix} = \begin{pmatrix} \widehat{\boldsymbol{\beta}}(\mathbf{Y}_1,\mathbf{Y}_2) \\ \widehat{\gamma}(\mathbf{Y}_1,\mathbf{Y}_2) \end{pmatrix} - \begin{pmatrix} \mathbf{V}_{1,1}, & \mathbf{V}_{1,2} \\ \mathbf{V}_{2,1}, & \mathbf{V}_{2,2} \end{pmatrix} \begin{pmatrix} \mathbf{B}' \\ \mathbf{0} \end{pmatrix}$$
$$\times \left[(\mathbf{B},\mathbf{0}) \begin{pmatrix} \mathbf{V}_{1,1}, & \mathbf{V}_{1,2} \\ \mathbf{V}_{2,1}, & \mathbf{V}_{2,2} \end{pmatrix} \begin{pmatrix} \mathbf{B}' \\ \mathbf{0} \end{pmatrix} \right]^{-1} \left[\mathbf{B}\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1,\mathbf{Y}_2) + \mathbf{b} \right],$$

$$\operatorname{Var}\begin{pmatrix} \widehat{\widehat{\boldsymbol{\beta}}}(\mathbf{Y}_1,\mathbf{Y}_2) \\ \widehat{\widehat{\gamma}}(\mathbf{Y}_1,\mathbf{Y}_2) \end{pmatrix} = \begin{pmatrix} \mathbf{W}_{1,1}, & \mathbf{W}_{1,2} \\ \mathbf{W}_{2,1}, & \mathbf{W}_{2,2} \end{pmatrix} = \begin{pmatrix} \mathbf{V}_{1,1}, & \mathbf{V}_{1,2} \\ \mathbf{V}_{2,1}, & \mathbf{V}_{2,2} \end{pmatrix}$$
$$- \begin{pmatrix} \mathbf{V}_{1,1}, & \mathbf{V}_{1,2} \\ \mathbf{V}_{2,1}, & \mathbf{V}_{2,2} \end{pmatrix} \begin{pmatrix} \mathbf{B}' \\ \mathbf{0} \end{pmatrix} (\mathbf{B}\mathbf{V}_{1,1}\mathbf{B}')^{-1} (\mathbf{B},\mathbf{0}) \begin{pmatrix} \mathbf{V}_{1,1}, & \mathbf{V}_{1,2} \\ \mathbf{V}_{2,1}, & \mathbf{V}_{2,2} \end{pmatrix},$$

$$\begin{pmatrix} \widehat{\widehat{\widehat{\boldsymbol{\beta}}}}(\mathbf{Y}_1,\mathbf{Y}_2) \\ \widehat{\widehat{\widehat{\gamma}}}(\mathbf{Y}_1,\mathbf{Y}_2) \end{pmatrix} = \begin{pmatrix} \widehat{\widehat{\boldsymbol{\beta}}}(\mathbf{Y}_1,\mathbf{Y}_2) \\ \widehat{\widehat{\gamma}}(\mathbf{Y}_1,\mathbf{Y}_2) \end{pmatrix} - \begin{pmatrix} \mathbf{T}_{1,1}, & \mathbf{T}_{1,2} \\ \mathbf{T}_{2,1}, & \mathbf{T}_{2,2} \end{pmatrix} \begin{pmatrix} \mathbf{G}_1' \\ \mathbf{G}_2' \end{pmatrix}$$
$$\times \left[(\mathbf{G}_1,\mathbf{G}_2) \begin{pmatrix} \mathbf{T}_{1,1}, & \mathbf{T}_{1,2} \\ \mathbf{T}_{2,1}, & \mathbf{T}_{2,2} \end{pmatrix} \begin{pmatrix} \mathbf{G}_1' \\ \mathbf{G}_2' \end{pmatrix} \right]^{-1} \left[(\mathbf{G}_1,\mathbf{G}_2) \begin{pmatrix} \widehat{\widehat{\boldsymbol{\beta}}}(\mathbf{Y}_1,\mathbf{Y}_2) \\ \widehat{\widehat{\gamma}}(\mathbf{Y}_1,\mathbf{Y}_2) \end{pmatrix} + \mathbf{g} \right],$$

$$\operatorname{Var}\begin{pmatrix} \widehat{\widehat{\widehat{\boldsymbol{\beta}}}}(\mathbf{Y}_1,\mathbf{Y}_2) \\ \widehat{\widehat{\widehat{\gamma}}}(\mathbf{Y}_1,\mathbf{Y}_2) \end{pmatrix} = \begin{pmatrix} \mathbf{T}_{1,1}, & \mathbf{T}_{1,2} \\ \mathbf{T}_{2,1}, & \mathbf{T}_{2,2} \end{pmatrix} - \begin{pmatrix} \mathbf{T}_{1,1}, & \mathbf{T}_{1,2} \\ \mathbf{T}_{2,1}, & \mathbf{T}_{2,2} \end{pmatrix} \begin{pmatrix} \mathbf{G}_1' \\ \mathbf{G}_2' \end{pmatrix}$$
$$\times \left[(\mathbf{G}_1,\mathbf{G}_2) \begin{pmatrix} \mathbf{T}_{1,1}, & \mathbf{T}_{1,2} \\ \mathbf{T}_{2,1}, & \mathbf{T}_{2,2} \end{pmatrix} \begin{pmatrix} \mathbf{G}_1' \\ \mathbf{G}_2' \end{pmatrix} \right]^{-1} \times (\mathbf{G}_1,\mathbf{G}_2) \begin{pmatrix} \mathbf{T}_{1,1}, & \mathbf{T}_{1,2} \\ \mathbf{T}_{2,1}, & \mathbf{T}_{2,2} \end{pmatrix} - \mathbf{M}_{\begin{pmatrix} G_1' \\ G_2' \end{pmatrix}}.$$

4 Numerical examples

Many examples can be found when levelling networks in geodesy are designed.

Example 1

$$\mathbf{Y}_1 \sim \left[\begin{pmatrix} 1, & 0, & 0 \\ -1, & 1, & 0 \\ 0, & -1, & 1 \\ 0, & 0, & -1 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}, \sigma_1^2 \mathbf{I}_4 \right],$$
$$\mathbf{Y}_2 \sim \left[\begin{pmatrix} 1, & 0, & -1 \\ -1, & 1, & 0 \\ 0, & -1, & 1 \\ 0, & 1, & 0 \end{pmatrix} \begin{pmatrix} \beta_1, \\ \beta_2 \\ \beta_3 \end{pmatrix}, \sigma_2^2 \mathbf{I}_4 \right],$$
$$\sigma_1^2 = (0.01\,\mathrm{m})^2, \quad \sigma_2^2 = (0.001\,\mathrm{m})^2.$$

Here β_1, β_2, β_3 are heights of points P_1, P_2, P_3 and $\{\mathbf{Y}_1\}_1$ means a measurement of a difference of the heights between the point P_1 and a reference point A with the height equal to zero. Analogously $\{\mathbf{Y}_1\}_2$ means a measurement of the difference between the heights of the point P_1 and P_2, etc.

$$\mathbf{C}_1 = (0.01)^{-2}\begin{pmatrix} 2, & -1, & 0 \\ -1, & 2, & -1 \\ 0, & -1, & 2 \end{pmatrix}, \quad \mathbf{C}_2 = (0.001)^{-2}\begin{pmatrix} 2, & -1, & -1 \\ -1, & 3, & -1 \\ -1, & -1, & 2 \end{pmatrix},$$

$$\mathbf{C}_1^{-1} = (0.01)^2\frac{1}{4}\begin{pmatrix} 3, & 2, & 1 \\ 2, & 4, & 2 \\ 1, & 2, & 3 \end{pmatrix}, \quad \mathbf{C}_2^{-1} = (0.001)^2\frac{1}{3}\begin{pmatrix} 5, & 3, & 4 \\ 3, & 3, & 3 \\ 4, & 3, & 5 \end{pmatrix},$$

$$\mathbf{Y}_1 = (5.18\,\mathrm{m},\ -0.35\,\mathrm{m},\ 2.48\,\mathrm{m},\ -7.29\,\mathrm{m})',$$

$$\mathbf{Y}_2 = (-2.134\,\mathrm{m},\ -0.351\,\mathrm{m},\ 2.485\,\mathrm{m},\ 4.825\,\mathrm{m})',$$

$$\begin{pmatrix} \widehat{\beta}_1(\mathbf{Y}_1) \\ \widehat{\beta}_2(\mathbf{Y}_1) \\ \widehat{\beta}_3(\mathbf{Y}_1) \end{pmatrix} = \mathbf{C}_1^{-1}\mathbf{X}_1'\boldsymbol{\Sigma}_1^{-1}\mathbf{Y}_1 = \begin{pmatrix} 5.175\,\mathrm{m} \\ 4.820\,\mathrm{m} \\ 7.295\,\mathrm{m} \end{pmatrix}$$

$$(\mathbf{C}_1 + \mathbf{C}_2)^{-1} = 10^{-6}\begin{pmatrix} 1.617, & 0.971, & 1.286 \\ 0.971, & 0.981, & 0.971 \\ 1.286, & 0.971, & 1.617 \end{pmatrix}, \quad \mathbf{C}_1^{-1} = 10^{-6}\begin{pmatrix} 75, & 50, & 25 \\ 50, & 100, & 50 \\ 25, & 50, & 75 \end{pmatrix}$$

$$\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1, \mathbf{Y}_2) = \widehat{\boldsymbol{\beta}}(\mathbf{Y}_1) + (\mathbf{C}_1 + \mathbf{C}_2)^{-1}\mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\big[\mathbf{Y}_2 - \mathbf{X}_2\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1)\big] = \begin{pmatrix} 5.176\,\mathrm{m} \\ 4.825\,\mathrm{m} \\ 7.310\,\mathrm{m} \end{pmatrix},$$

$$\mathbf{Y}_2 - \mathbf{X}_2\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1) = \begin{pmatrix} -0.014\,\mathrm{m} \\ 0.004\,\mathrm{m} \\ 0.010\,\mathrm{m} \\ 0.005\,\mathrm{m} \end{pmatrix} \quad \text{(a discrepancy between the original and the additional experiment.)}$$

The improvement of the estimator is obvious.

Example 2 The original model describes the measurement of heights of three points P_1, P_2, P_3. The additional model involves new point P_4 of the height γ and it describes the measurement of height differences between the points P_4P_3, P_1P_4, and P_2P_1, respectively.

$$\mathbf{Y}_1 \sim_3 (\mathbf{I}_3\boldsymbol{\beta}, \sigma_1^2\mathbf{I}_3), \quad \mathbf{Y}_2 \sim_3 \left[(\mathbf{D}, \mathbf{X}_2)\begin{pmatrix} \boldsymbol{\beta} \\ \gamma \end{pmatrix}, \sigma_2^2\mathbf{I}_3\right],$$

$$\sigma_1^2 = (0.01\,\mathrm{m})^2, \quad \sigma_2^2 = (0.001\,\mathrm{m})^2,$$

$$\mathbf{D} = \begin{pmatrix} 0, & 0, & -1 \\ 1, & 0, & 0 \\ -1, & 1, & 0 \end{pmatrix}, \quad \mathbf{X}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix},$$

$$\mathbf{Y}_1 = (5.18\,\mathrm{m}, 4.82\,\mathrm{m}, 7.30\,\mathrm{m})', \quad \mathbf{Y}_2 = (-4.360\,\mathrm{m}, 2.226\,\mathrm{m}, -0.351)',$$

$$\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1) = \mathbf{Y}_1,$$
$$\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1,\mathbf{Y}_2) = \widehat{\boldsymbol{\beta}}(\mathbf{Y}_1) - \mathbf{C}_1^{-1}\mathbf{D}'\left[\mathbf{M}_{X_2}(\boldsymbol{\Sigma}_2 + \mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}')\mathbf{M}_{X_2}\right]^{+}\mathbf{D}\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1)$$
$$+ \left[\mathbf{C}_1 + \mathbf{D}'(\mathbf{M}_{X_2}\boldsymbol{\Sigma}_2\mathbf{M}_{X_2})^{+}\mathbf{D}\right]^{-1}\mathbf{D}'(\mathbf{M}_{X_2}\boldsymbol{\Sigma}_2\mathbf{M}_{X_2})^{+}\mathbf{Y}_2,$$
$$= \begin{pmatrix} 5.18 \\ 4.82 \\ 7.30 \end{pmatrix} - \begin{pmatrix} -0.5797 \\ -0.9304 \\ 1.5101 \end{pmatrix} + \begin{pmatrix} -0.5873 \\ -0.9290 \\ 1.5164 \end{pmatrix} = \begin{pmatrix} 5.1724\,\mathrm{m} \\ 4.8214\,\mathrm{m} \\ 7.3063m \end{pmatrix},$$
$$\mathrm{Var}\left[\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1,\mathbf{Y}_2)\right] = \mathrm{Var}\left[\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1)\right] - \mathbf{C}_1^{-1}\mathbf{D}'\left[\mathbf{M}_{X_2}(\boldsymbol{\Sigma}_2 + \mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}')\mathbf{M}_{X_2}\right]^{+}\mathbf{D}\mathbf{C}_1^{-1}$$
$$= 10^{-4}\begin{pmatrix} 0.337, & 0.333, & 0.330 \\ 0.333, & 0.340, & 0.327 \\ 0.330, & 0.327, & 0.343 \end{pmatrix}.$$

Example 3 A free levelling traverse consists of points P_1, P_2, P_3, P_4. The original model describes the measurement of the height differences $\beta_1 \sim P_2P_1$, $\beta_2 \sim P_3P_2$, $\beta_3 \sim P_4P_3$. In the additional experiment the height difference $\gamma \sim P_1P_4$ is measured. After the measurement of the additional experiment the fact that the levelling traverse P_1, P_2, P_3, P_4, P_1, is closed must be taken into account.

$$\mathbf{Y}_1 \sim_3 (\mathbf{I}_3\boldsymbol{\beta}, \sigma_1^2\mathbf{I}_3), \quad Y_2 \sim_1 \left[(\mathbf{0}',1)\begin{pmatrix}\boldsymbol{\beta} \\ \gamma\end{pmatrix}, \sigma_2^2\right],$$
$$g + \mathbf{G}_1\boldsymbol{\beta} + G_2\gamma = 0, \quad g = 0, \quad \mathbf{G}_1 = (1,1,1), \quad G_2 = 1,$$
$$\sigma_1^2 = (0.01\,\mathrm{m})^2, \quad \sigma_2^2 = (0.001\,\mathrm{m})^2,$$
$$\mathbf{Y}_1 = (3.51\,\mathrm{m}, 2.70\,\mathrm{m}, 1.32\,\mathrm{m})', \quad Y_2 = -7.516\,\mathrm{m},$$
$$\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1) = \mathbf{Y}_1, \quad \widehat{\gamma}(Y_2) = Y_2,$$
$$\widehat{\widehat{\boldsymbol{\beta}}}(\mathbf{Y}_1,\mathbf{Y}_2) = \widehat{\boldsymbol{\beta}}(\mathbf{Y}_1) - (\mathbf{C}_1^{-1},\mathbf{0})\begin{pmatrix}\mathbf{1} \\ 1\end{pmatrix}\left[(\mathbf{1}',1)\begin{pmatrix}\mathbf{C}_1^{-1}, & \mathbf{0} \\ \mathbf{0}, & \mathbf{C}_2^{-1}\end{pmatrix}\begin{pmatrix}\mathbf{1} \\ 1\end{pmatrix}\right]^{-1}$$
$$\times (\mathbf{1}',1)\begin{pmatrix}\mathbf{Y}_1 \\ Y_2\end{pmatrix} = \begin{pmatrix} 3.512 \\ 2.70 \\ 1.32 \end{pmatrix} - \frac{1}{301}\begin{pmatrix} 0.014 \\ 0.014 \\ 0.014 \end{pmatrix} = \begin{pmatrix} 3.5054\,\mathrm{m} \\ 2.6954\,\mathrm{m} \\ 1.3154\,\mathrm{m} \end{pmatrix},$$
$$\mathrm{Var}\left[\widehat{\widehat{\boldsymbol{\beta}}}(\mathbf{Y}_1,Y_2)\right] = \mathbf{C}_1^{-1} - \mathbf{C}_1^{-1}\mathbf{1}\left[(\mathbf{1}',1)\begin{pmatrix}\mathbf{C}_1^{-1}, & \mathbf{0} \\ \mathbf{0}, & \mathbf{C}_2^{-1}\end{pmatrix}\begin{pmatrix}\mathbf{1} \\ 1\end{pmatrix}\right]^{-1}\mathbf{1}'\mathbf{C}_1^{-1}$$
$$= 10^{-4}\,(\mathbf{I}_3 - 0.332 \times \mathbf{1}\mathbf{1}') = 10^{-4}\begin{pmatrix} 0.668, & -0.332, & -0.332 \\ -0.332, & 0.668, & -0.332 \\ -0.332, & -0.332, & 0.668 \end{pmatrix}.$$

Ackowledgement Author would like to express the deepest thank to the reviewer for thorough reading the manuscript. It was a great help in eliminating many errors and mistakes. Non eliminated errors are to the debit of the author.

References

[1] Fišerová, E., Kubáček, L., Kunderová, P.: Linear Statistical Models: Regularity and Singularities.. *Academia*, Praha, 2007

[2] Kubáček, L.: Foundations of Estimation Theory. *Elsevier*, Amsterdam–Oxford–New York–Tokyo, 1988.

[3] Rao, C. R., Mitra, S. K.: Generalized Inverse of Matrices and its Applications. *Wiley*, New York–London–Sydney–Toronto, 1971.