Helena Myšková

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AN ITERATIVE ALGORITHM FOR TESTING SOLVABILITY OF MAX-MIN INTERVAL SYSTEMS

Helena Myšková

This paper is dealing with solvability of interval systems of linear equations in max-min algebra. Max-min algebra is the algebraic structure in which classical addition and multiplication are replaced by $\oplus$ and $\otimes$, where $a \oplus b = \max\{a, b\}$, $a \otimes b = \min\{a, b\}$.

The notation $\boldsymbol{A} \otimes x = \boldsymbol{b}$ represents an interval system of linear equations, where $\boldsymbol{A} = [\underline{A}, \overline{A}]$ and $\boldsymbol{b} = [\underline{b}, \overline{b}]$ are given interval matrix and interval vector, respectively. We can define several types of solvability of interval systems. In this paper, we define the T4 and T5 solvability and give necessary and sufficient conditions for them.

Keywords: max-min algebra, interval system, T4-vector, T4 solvability, T5-vector, T5 solvability

Classification: 15A06, 65G30

1. INTRODUCTION

The last decades have seen a lot of attention given to studying systems of linear equations in the form $A \otimes x = b$, where A is a matrix, b and x are vectors of suitable dimensions and one or both of classical addition and multiplication operations are replaced by maximum and/or minimum. If addition and multiplication are replaced by maximum and minimum, respectively, we call this algebraic structure *max-min algebra.* If multiplication is replaced by addition, we talk about *max-plus algebra.* One of the questions, which we can deal with in max-min algebra, is solving the systems of linear equations.

Max-min (fuzzy relational) equations have found a broad area of applications in causal models, which emphasize relationships between input and output variables. They are used in diagnosis models [1, 10, 14, 15] or models of nondeterministic systems [16]. Diagnosis models are of particular interest, since they cope with uncertainty existing in many real-life case, either concerning medical diagnosis or diagnosis of technical devices. In the simplest formulation we are faced with a space of symptoms and a space of faults. The elements of faults are related with the elements of symptoms by means of a fuzzy relation. In this framework $R(x_i, y_j) = r_{ij}$ stands for the degree to which the symptom x_i is related to the fault y_j.

In the situation, when a set of symptoms is represented as a fuzzy set X, where the degree of membership $a(x_i) = a_i$ refers to the strength of evidence of ith symptom, by performing max-min composition $(a \circ R)$, we obtain the fuzzy set Y of faults which indicates degrees of faults $(b(y_j) = b_j)$. In this context we get not only an indication of the fault element in the structure but a list of elements that are fault to a certain degree. The solution of the equation $a \circ R = b$ provides a maximal set of symptoms that produce the given effect (fault).

In practice, it may happen that a given system of max-min linear equations is unsolvable. A possible method of restoring the solvability is to replace the matrix A and vector b by an interval matrix and an interval vector. The resulting systems are the so-called interval systems of linear equations. The theory of interval computations and in particular of interval systems in the classical algebra is already quite developed, see e. g. the monograph [6] or [12, 13]. Also, an interesting approach to interval computations in max-min algebra was published in [5, 11]. Interval systems of linear equations in the max-min and max-plus algebra have been studied by K. Cechlárová and R. A. Cuninghame-Green in [2, 3]. They dealt with the weak, strong and tolerance solvability. In [7, 8, 9], we studied other solvability concepts. In this paper, the T4 and T5 solvability are presented and the necessary and sufficient conditions for them are given.

There is also motivation coming from applications for the use of interval systems. One of possible applications is presented in the following example.

Example 1.1. Let us consider a situation, in which different transportation means provide transporting goods from places $P_1, P_2, \dots, P_m$ to a terminal T. We assume that the connection between P_i and T is possible only via one of the places (e. g. cities) $Q_1, Q_2, \dots, Q_n$ and the capacities of the roads between P_i and Q_j are equal to $a_{ij} > 0$. If place Q_j is linked with T by a road with a capacity x_j, the capacity of the connection between P_i and T via Q_j is equal to $\min\{a_{ij}, x_j\}$. Our task is to choose the appropriate capacities x_j, $j \in N = \{1, 2, \dots, n\}$. Moreover, it is required that the maximum capacity of the road from P_i to T is equal to a given number b_i for all $i \in M = \{1, 2, \dots, m\}$, i. e.,

$$\max_{j \in N} \min\{a_{ij}, x_j\} = b_i \tag{1}$$

for each $i \in M$.

The entries of the vector of feasible capacities $x = (x_1, \dots, x_n)$ are elements of the set of solutions of system (1).

2. PRELIMINARIES

Max-min algebra $\mathcal{B}$ is the triple $(B, \oplus, \otimes)$, where $(B, \leq)$ is a bounded linearly ordered set with binary operations *maximum* and *minimum*, denoted by $\oplus$ and $\otimes$, respectively. The least element in B will be denoted by O, the greatest one by I.

Denote by M and N the index sets $\{1, 2, \dots, m\}$ and $\{1, 2, \dots, n\}$, respectively. The set of all $m \times n$ *matrices* over B is denoted by $B(m, n)$ and the set of all column n-*vectors* over B by $B(n)$.

Operations $\oplus$ and $\otimes$ are extended to matrices and vectors in the same way as in the classical algebra. We will consider the *ordering* $\leq$ on the sets $B(m, n)$ and $B(n)$ defined as follows:

- for $A, C \in B(m,n):\ A \leq C$ if $a_{ij} \leq c_{ij}$ for all $i \in M,\ j \in N$,
- for $x, y \in B(n):\ x \leq y$ if $x_j \leq y_j$ for all $j \in N$.

We will use the *monotonicity of* $\otimes$, which means that for each $A, C \in B(m,n)$ and $x, y \in B(n)$ the implication

$$\text{if } A \leq C \text{ and } x \leq y \text{ then } A \otimes x \leq C \otimes y$$

holds true.

In max-min algebra we can rewrite the system of equations (1) in the form

$$A \otimes x = b, \tag{2}$$

which represents a *system of max-min linear equations.*

The crucial role for the solvability of system (2) is played by a *principal solution* of system (2), defined by

$$x_j^*(A,b) = \min_{i \in M}\{b_i; a_{ij} > b_i\} \tag{3}$$

for each $j \in N$, where $\min \emptyset = I$.

The following theorem describes the importance of the principal solution for the solvability of (2).

Theorem 2.1. (Cuninghame-Green [4], Zimmermann [17]) Let $A \in B(m,n)$ and $b \in B(m)$ be given.

i) If $A \otimes x = b$ for $x \in B(n)$, then $x \leq x^*(A,b)$.

ii) $A \otimes x^*(A,b) \leq b$.

iii) The system $A \otimes x = b$ is solvable, if and only if $x^*(A,b)$ is its solution.

The properties of a principal solution are expressed in the following assertions.

Lemma 2.2. (Cechlárová [2]) Let $A \in B(m,n),\ b, d \in B(m)$ be such that $b \leq d$. Then $x^*(A,b) \leq x^*(A,d)$.

Lemma 2.3. (Myšková [7]) Let $b \in B(m),\ C, D \in B(m,n)$ be such that $D \leq C$. Then $x^*(C,b) \leq x^*(D,b)$.

3. INTERVAL SYSTEMS

In practice, the capacities a_{ij} of roads in Example 1.1 may depend on external conditions, so they are from an interval of possible values, i. e., $a_{ij} \in [\underline{a}_{ij}, \overline{a}_{ij}]$ for each $i \in M,\ j \in N$. Due to this fact, we will require the maximal capacity of the road from P_i to T to be from a given interval, i. e., $b_i \in [\underline{b}_i, \overline{b}_i]$ for each $i \in M$.

Solvability concept	**Definition**
Weak solvability [2]	$(\exists x \in B(n))(\exists A \in \boldsymbol{A})(\exists b \in \boldsymbol{b}) : A \otimes x = b$
Strong solvability [3]	$(\forall A \in \boldsymbol{A})(\forall b \in \boldsymbol{b})(\exists x \in B(n)) : A \otimes x = b$
Tolerance solvability [2]	$(\exists x \in B(n))(\forall A \in \boldsymbol{A})(\exists b \in \boldsymbol{b}) : A \otimes x = b$
Weak tolerance solvability [7]	$(\forall A \in \boldsymbol{A})(\exists x \in B(n))(\exists b \in \boldsymbol{b}) : A \otimes x = b$
Control solvability [8]	$(\exists x \in B(n))(\forall b \in \boldsymbol{b})(\exists A \in \boldsymbol{A}) : A \otimes x = b$
Weak control solvability [8]	$(\forall b \in \boldsymbol{b})(\exists x \in B(n))(\exists A \in \boldsymbol{A}) : A \otimes x = b$
Universal solvability [7]	$(\exists x \in B(n))(\forall b \in \boldsymbol{b})(\forall A \in \boldsymbol{A}) : A \otimes x = b$
Weak universal solvability [8]	$(\forall b \in \boldsymbol{b})(\exists x \in B(n))(\forall A \in \boldsymbol{A}) : A \otimes x = b$
T1 solvability [9]	$(\exists A \in \boldsymbol{A})(\forall x \in B(n))(\exists b \in \boldsymbol{b}) : A \otimes x = b$
T2 solvability [9]	$(\forall x \in B(n))(\exists A \in \boldsymbol{A})(\exists b \in \boldsymbol{b}) : A \otimes x = b$
T3 solvability [9]	$(\forall x \in B(n))(\nexists b \in \boldsymbol{b})(\forall A \in \boldsymbol{A}) : A \otimes x = b$

Tab. 1. Solvability concepts of (4).

Similarly to [2, 7, 8, 11], we define an *interval matrix* $\boldsymbol{A}$ and *interval vector* $\boldsymbol{b}$ as follows:

$$\begin{aligned} \boldsymbol{A} &= [\underline{A}, \overline{A}] = \left\{ A \in B(m,n);\ \underline{A} \leq A \leq \overline{A} \right\} \\ \boldsymbol{b} &= [\underline{b}, \overline{b}] = \left\{ b \in B(n);\ \underline{b} \leq b \leq \overline{b} \right\}. \end{aligned}$$

Denote by

$$\boldsymbol{A} \otimes x = \boldsymbol{b} \tag{4}$$

the set of all systems of linear max-min equations of the form (2) such that $A \in \boldsymbol{A}$, $b \in \boldsymbol{b}$. We call (4) the *interval system of linear equations.* A system of the form (2) is called the *subsystem* of (4) if $A \in \boldsymbol{A}$, $b \in \boldsymbol{b}$.

We say, that interval system (4) has the *constant matrix*, if $\underline{A} = \overline{A}$ and has the *constant right-hand side*, if $\underline{b} = \overline{b}$. Subsystem (2) is *extremal*, if each of the equations has the form $[\underline{A} \otimes x]_i = \overline{b}_i$ or $[\overline{A} \otimes x]_i = \underline{b}_i$ and we call them an *LU equation* or an *UL equation*, respectively.

We can define several conditions, which the given interval system has to fulfill. According to them, we will define several solvability concepts. Table 1 contains the list of all up to now studied types of the solvability of (4) in max-min algebra. The solvability concepts, which lead to trivial conditions, are omitted there.

4. T4 SOLVABILITY

The notions of a T4-vector and the T4 solvability of interval system (4), are defined in the following section. We present the procedure for checking the T4 solvability.

Definition 4.1.

i) A vector $b \in B(n)$ is called a T4-vector of interval system (4), if there exists $x \in B(n)$ such that $A \otimes x = b$ for each $A \in \boldsymbol{A}$.

ii) Interval system (4) is T4 solvable, if there exists $b \in \boldsymbol{b}$ such that b is a T4-vector of (4).

To give a necessary and sufficient condition for the T4 solvability, we recall the notion of a *universal solution*, which has been studied by K. Cechlárová in [2].

Definition 4.2. A vector $x \in B(n)$ is a universal solution of interval system (4), if for each $A \in \boldsymbol{A}$ and for each $b \in \boldsymbol{b}$, the equality $A \otimes x = b$ holds.

Theorem 4.3. (Myšková [7]) Interval system (4) with the constant right-hand side $b = \underline{b} = \overline{b}$ has a universal solution, if and only if

$$\underline{A} \otimes x^*(\overline{A}, b) = b, \tag{5}$$

and in this case $x^*(\overline{A}, b)$ is the maximal universal solution.

Lemma 4.4. A vector $b \in \boldsymbol{b}$ is a T4-vector of interval system (4), if and only if it fulfills equality (5).

Proof. A vector $b \in \boldsymbol{b}$ is a T4-vector of interval system (4), if and only if interval system (4) with the constant right-hand side $\underline{b} = \overline{b} = b$ has a universal solution, which is according to Theorem 4.3 equivalent to (5). □

The last lemma does not give an efficient method for finding a T4-vector. To suggest polynomial procedure, we define a *T4-sequence* of interval system (4).

Definition 4.5. A T4-sequence of interval system (4) is the sequence $\{b^{(k)}\}_{k=0}^{\infty}$ defined as follows:

$$b^{(k)} = \begin{cases} \overline{b} & \text{for } k = 0, \\ \underline{A} \otimes x^*(\overline{A}, b^{(k-1)}) & \text{for } k \geq 1. \end{cases} \tag{6}$$

Lemma 4.6. Let $\{b^{(k)}\}_{k=0}^{\infty}$ be the T4-sequence of interval system (4). The following assertions hold true:

i) The sequence $\{b^{(k)}\}_{k=0}^{\infty}$ is decreasing.

ii) There exists $l \in \mathbb{N}_0$ such that $b^{(l+1)} = b^{(l)}$.

Proof.

i) By Lemma 2.1 and by monotonicity of $\otimes$ we have

$$b^{(k+1)} = \underline{A} \otimes x^*(\overline{A}, b^{(k)}) \leq \overline{A} \otimes x^*(\overline{A}, b^{(k)}) \leq b^{(k)}.$$

ii) From (3) and (6) it follows, that $b_i^{(k)} \in \{\overline{b}_i,\ i \in M\} \cup \{\underline{a}_{ij},\ i \in M,\ j \in N\}$ for each $i \in M,\ k \in \mathbb{N}_0$, i.e., at most $m + mn$ different values in each entry of $b^{(k)}$, can be considered. As the sequence $\{b^{(k)}\}_{k=1}^{\infty}$ is decreasing, the number of different vectors $b^{(k)}$ is bounded by $m(m + mn)$, which means that there exists $l \in \mathbb{N}_0$ $(l \leq m(m + mn))$ such that $b^{(l+1)} = b^{(l)}$.

□

Theorem 4.7. Let $b \in \boldsymbol{b}$ be a T4-vector of interval system (4). Then for each $k \in \mathbb{N}_0$ the inequality $b \leq b^{(k)}$ is satisfied.

Proof. By mathematical induction on k

1. For $k = 0$ the inequality $b \leq \overline{b} = b^{(0)}$ trivially holds.
2. Suppose that $b \leq b^{(k)}$. We get

$$b = \underline{A} \otimes x^*(\overline{A}, b) \leq \underline{A} \otimes x^*(\overline{A}, b^{(k)}) = b^{(k+1)}.$$

□

Theorem 4.8. Interval system (4) is T4 solvable if and only if there exists $l \in \mathbb{N}_0$ such that $b^{(l+1)} = b^{(l)} \in \boldsymbol{b}$.

Proof. If $b^{(l+1)} = b^{(l)} \in \boldsymbol{b}$ then $\underline{A} \otimes x^*(\overline{A}, b^{(l)}) = b^{(l)} \in \boldsymbol{b}$ which means that vector $b^{(l)}$ is a T4-vector of (4), so interval system (4) is T4 solvable.

For the converse implication suppose that interval system (4) is T4 solvable with a T4-vector $b \in \boldsymbol{b}$. According to Lemma 4.6ii) there exists $l \in \mathbb{N}_0$ such that $b^{(l+1)} = b^{(l)}$. Then

$$\underline{b} \leq b \leq b^{(l)} = b^{(l+1)} \leq \overline{b}.$$

Hence $b^{(l+1)} = b^{(l)} \in \boldsymbol{b}$. □

The above introduced assertions enable us to give the following algorithm for checking the T4 solvability.

Algorithm T4
Input: $\boldsymbol{A}, \boldsymbol{b}$
Output: 'yes' in variable $t4$, if the given interval system is T4 solvable, and 'no' in $t4$ otherwise

begin
Step 1. $b^{(0)} = \overline{b},\ k = 0$;
Step 2. $b^{(k+1)} = \underline{A} \otimes x^*(\overline{A}, b^{(k)})$;
Step 3. If $\underline{b} \not\leq b^{(k+1)}$ then $t4$:='no', go to **end**;
Step 4. If $b^{(k+1)} = b^{(k)}$ then $t4$:='yes', $b^* = b^{(k)}$, go to **end**;
Step 5. $k := k + 1$, go to *Step 2*;
end

Theorem 4.9. Let $\boldsymbol{A} = [\underline{A}, \overline{A}]$ with $\underline{A}, \overline{A} \in B(m, n)$ and $\boldsymbol{b} = [\underline{b}, \overline{b}]$ be given. Then Algorithm T4 decides whether the given interval system (4) is T4 solvable and in the positive case finds the maximal T4-vector b^* in $O(m^3n^2)$ time.

Proof. The most time-consuming is Step 2 which requires $O(mn)$ operations. The question which arises is the number of repetitions of the loop 2–5 till the algorithm gives an answer. This number is bounded by the number of different vectors $b^{(k)}$, which is maximally $m(m+mn)$ (see the proof of Lemma 4.6ii)). Consequently, the complexity of Algorithm T4 is $O(m^3n^2)$. From Theorem 4.7 it follows that in the positive case the vector b^* is the maximal T4-vector. □

Denote $X(\boldsymbol{A},\boldsymbol{b}) = \{x \in B(n);\ (\exists b \in \boldsymbol{b})(\forall A \in \boldsymbol{A}) : A \otimes x = b\}$. It is easy to see that the T4 solvability of (4) is equivalent to $X(\boldsymbol{A},\boldsymbol{b}) \neq \emptyset$.

Corollary 4.10. If interval system is T4 solvable with the maximal T4-vector b^*, then $x^*(\overline{A}, b^*) = \max X(\boldsymbol{A},\boldsymbol{b})$.

P r o o f. If $x \in X(\boldsymbol{A},\boldsymbol{b})$ then there exists a T4-vector $b \in \boldsymbol{b}$ such that for each $A \in \boldsymbol{A}$ the equality $A \otimes x = b$ holds true. Then

$$x \le x^*(\overline{A}, b) \le x^*(\overline{A}, b^*),$$

where the first inequality follows from Theorem 4.3. □

Remark 4.11. Using Algorithm T4 for the situation described in Example 1.1 we can find the maximal vector of capacities $b^* \in \boldsymbol{b}$ which can be achieved by suitable choice of the vector x (for example the maximal vector $x^*(\overline{A}, b^*)$) independently of the capacities of the roads from the places P_i to the places Q_j, if such a vector of capacities b exists.

Example 4.12. Let $B = [0,1]$ and

$$\boldsymbol{A} = \begin{pmatrix} [0.1, 0.8] & [0.4, 0.6] & [0.7, 0.7] \\ [0.5, 0.8] & [0.4, 0.5] & [0.3, 0.9] \\ [0.6, 0.9] & [0.8, 0.8] & [0.4, 0.6] \end{pmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{pmatrix} [0.4, 0.7] \\ [0.4, 0.8] \\ [0.5, 0.9] \end{pmatrix}.$$

We check the T4 solvability of interval system $\boldsymbol{A} \otimes x = \boldsymbol{b}$. We get

$$x^*(\overline{A}, b^{(0)}) = \begin{pmatrix} 0.7 \\ 1 \\ 0.8 \end{pmatrix}, \quad b^{(1)} = \begin{pmatrix} 0.7 \\ 0.5 \\ 0.8 \end{pmatrix},$$

$$x^*(\overline{A}, b^{(1)}) = \begin{pmatrix} 0.5 \\ 1 \\ 0.5 \end{pmatrix}, \quad b^{(2)} = \begin{pmatrix} 0.5 \\ 0.5 \\ 0.8 \end{pmatrix},$$

$$x^*(\overline{A}, b^{(2)}) = \begin{pmatrix} 0.5 \\ 0.5 \\ 0.5 \end{pmatrix}, \quad b^{(3)} = \begin{pmatrix} 0.5 \\ 0.5 \\ 0.5 \end{pmatrix},$$

$$x^*(\overline{A}, b^{(3)}) = x^*(\overline{A}, b^{(2)}), \quad b^{(4)} = b^{(3)}.$$

Since $b^{(4)} = \underline{A} \otimes x^*(\overline{A}, b^{(3)}) = b^{(3)} \in \boldsymbol{b}$, according to Theorem 4.3 the vector $b^{(3)}$ is a T4-vector of (4). The given interval system is T4 solvable and the vector $b^* = b^{(3)}$ is its maximal T4-vector.

5. T5 SOLVABILITY

The notions of a T5-vector and the T5 solvability of interval system (4) are defined in this section. We will prove, that the T4 and T5 solvability are equivalent in max-min algebra.

Definition 5.1.

i) A vector $b \in B(n)$ is a T5-vector of interval system (4), if for each $A \in \boldsymbol{A}$ there exists $x \in B(n)$ such, that $A \otimes x = b$.

ii) Interval system (4) is T5 solvable, if there exists a vector $b \in \boldsymbol{b}$ such, that b is a T5-vector of (4).

To give a necessary and sufficient condition for the T5 solvability we recall the notion of a *strong solvability*, which has been studied in [3].

Definition 5.2. Interval system (4) is strongly solvable, if each of its subsystems of the form (2) is solvable.

Theorem 5.3. [3] Interval system (4) is strongly solvable, if and only if all its extremal subsystems with exactly one LU equation are solvable.

For each $k = 1, 2, \ldots, m$ denote by $A^{(k)} = (a_{ij}^{(k)})$ the matrix with entries

$$a_{ij}^{(k)} = \begin{cases} \underline{a}_{ij} & \text{for} \quad i = k, \ j \in N, \\ \overline{a}_{ij} & \text{for} \quad i \neq k, \ j \in N. \end{cases}$$

Lemma 5.4. A vector $b \in \boldsymbol{b}$ is a T5-vector of interval system (4), if and only if

$$A^{(k)} \otimes x^*(A^{(k)}, b) = b \tag{7}$$

for each $k \in M$.

P r o o f . A vector $b \in \boldsymbol{b}$ is a T5-vector of interval system (4), if and only if interval system (4) with the constant right-hand side $\underline{b} = \overline{b} = b$ is strongly solvable, which is by Theorem 2.1iii) and Theorem 5.3 fulfilled, if and only if equality (7) holds true for each $k \in M$. □

Theorem 5.5. A vector $b \in \boldsymbol{b}$ is a T5-vector of interval system (4), if and only if b is a T4-vector of interval system (4).

P r o o f . It is easy to see that, if $b \in \boldsymbol{b}$ is a T4-vector of (4), then b is a T5-vector of (4). For the converse implication suppose that a vector b is not a T4-vector of (4), i. e., $\underline{A} \otimes x^*(\overline{A}, b) \neq b$. According to the inequality $\underline{A} \otimes x^*(\overline{A}, b) \leq b$, there exists an index $r \in M$ such, that $[\underline{A} \otimes x^*(\overline{A}, b)]_r < b_r$, i. e.,

$$\underline{a}_{rj} \otimes x_j^*(\overline{A}, b) < b_r \tag{8}$$

for each $j \in N$.

Denote by $N_1,\ N_2$ the sets $N_1 = \{j \in N : \underline{a}_{rj} < b_r\}$, $N_2 = \{j \in N : \underline{a}_{rj} \geq b_r\}$. For $j \in N_1$ we have $\underline{a}_{rj} \otimes x_j^*(A^{(r)}, b) < b_r$ which implies

$$\bigoplus_{j \in N_1} \left(\underline{a}_{rj} \otimes x_j^*(A^{(r)}, b)\right) < b_r.$$

For $j \in N_2$ inequality (8) implies the inequality $x_j^*(\overline{A}, b) < b_r$. We have to distinguish two cases:

i) If $\underline{a}_{rj} > b_r$ then the equality $\{b_i : \overline{a}_{ij} > b_i\} = \{b_i : a_{ij}^{(r)} > b_i\}$ implies

$$x_j^*(\overline{A}, b) = x_j^*(A^{(r)}, b).$$

ii) If $\underline{a}_{rj} = b_r$ then $x_j^*(\overline{A}, b) = \min_{i \in M}\{b_i : \overline{a}_{ij} > b_i\} = \min_{i \neq r}\{b_i : \overline{a}_{ij} > b_i\} = \min_{i \neq r}\{b_i : a_{ij}^{(r)} > b_i\} = \min_{i \in M}\{b_i : a_{ij}^{(r)} > b_i\} = x_j^*(A^{(r)}, b)$.

In both cases we have $x_j^*(A^{(r)}, b) = x_j^*(\overline{A}, b) < b_r$, which implies $\underline{a}_{rj} \otimes x_j^*(A^{(r)}, b) < b_r$ and consequently $\bigoplus_{j \in N_2} \left(\underline{a}_{rj} \otimes x_j^*(A^{(r)}, b)\right) < b_r$.

Hence

$$\left[A^{(r)} \otimes x^*(A^{(r)}, b)\right]_r = \left(\bigoplus_{j \in N_1} \underline{a}_{rj} \otimes x_j^*(A^{(r)}, b)\right) \oplus \left(\bigoplus_{j \in N_2} \underline{a}_{rj} \otimes x_j^*(A^{(r)}, b)\right) < b_r.$$

Since equality (7) does not hold for $k = r$, the vector b is not a T5-vector of interval system (4). □

Theorem 5.6. Interval system (4) is T5 solvable, if and only if it is T4 solvable.

Proof. From Theorem 5.5 it follows that the set of all T4-vectors of (4) is equal to the set of all T5-vectors of interval system (4). This means that the existence of a T4-vector of (4) is equivalent to the existence of a T5-vector of (4), i.e., interval system (4) is T4 solvable if and only if it is T5 solvable. □

Example 5.7. Let $\boldsymbol{A} \otimes x = \boldsymbol{b}$ be the interval system given in Example 4.12 and $\tilde{b} = (0.4, 0.4, 0.4)^T \in \boldsymbol{b}$. First, we check if the vector $\tilde{b}$ is a T5-vector of the given interval system. We have

$$A^{(1)} = \begin{pmatrix} 0.1 & 0.4 & 0.7 \\ 0.8 & 0.5 & 0.9 \\ 0.9 & 0.8 & 0.6 \end{pmatrix}, \ A^{(2)} = \begin{pmatrix} 0.8 & 0.6 & 0.7 \\ 0.5 & 0.4 & 0.3 \\ 0.9 & 0.8 & 0.6 \end{pmatrix}, \ A^{(3)} = \begin{pmatrix} 0.8 & 0.6 & 0.7 \\ 0.8 & 0.5 & 0.9 \\ 0.6 & 0.8 & 0.4 \end{pmatrix}.$$

We compute

$$x^*(A^{(1)}, \tilde{b}) = \begin{pmatrix} 0.4 \\ 0.4 \\ 0.4 \end{pmatrix}, \quad A^{(1)} \otimes x^*(A^{(1)}, \tilde{b}) = \begin{pmatrix} 0.4 \\ 0.4 \\ 0.4 \end{pmatrix} = \tilde{b},$$

$$x^*(A^{(2)},\tilde{b}) = \begin{pmatrix} 0.4 \\ 0.4 \\ 0.4 \end{pmatrix}, \quad A^{(2)} \otimes x^*(A^{(2)},\tilde{b}) = \begin{pmatrix} 0.4 \\ 0.4 \\ 0.4 \end{pmatrix} = \tilde{b},$$

$$x^*(A^{(3)},\tilde{b}) = \begin{pmatrix} 0.4 \\ 0.4 \\ 0.4 \end{pmatrix}, \quad A^{(3)} \otimes x^*(A^{(3)},\tilde{b}) = \begin{pmatrix} 0.4 \\ 0.4 \\ 0.4 \end{pmatrix} = \tilde{b}.$$

According to Lemma 5.4 the vector $\tilde{b}$ is a T5-vector of the given interval system. We check if $\tilde{b}$ is a T4-vector. Since $\underline{A} \otimes x^*(\overline{A},\tilde{b}) = \underline{A} \otimes (0.4, 0.4, 0.4)^T = (0.4, 0.4, 0.4)^T = \tilde{b}$, vector $\tilde{b}$ is a T4-vector of the given interval system.

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Helena Myšková, Department of Mathematics and Theoretical Informatics, Faculty of Electrical Engineering and Informatics, Technical University in Košice, Němcovej 32, 042 00 Košice. Slovak Republic.

e-mail: helena.myskova@tuke.sk