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A New Variational Characterization Of Compact Conformally Flat 4-Manifolds

Faen Wu, Xinnuan Zhao

Abstract. In this paper, we give a new variational characterization of certain 4-manifolds. More precisely, let R and Ric denote the scalar curvature and Ricci curvature respectively of a Riemannian metric, we prove that if (M^4, g) is compact and locally conformally flat and g is the critical point of the functional

$$F(g) = \int_{M^4} (aR^2 + b|Ric|^2)\,\mathrm{d}v_g\,,$$

where

$$(a, b) \in \mathbb{R}^2 \setminus L_1 \cup L_2$$

$$L_1\colon 3a + b = 0\,; \quad L_2\colon 6a - b + 1 = 0\,,$$

then (M^4, g) is either scalar flat or a space form.

1 Introduction

Let (M^n, g) be an n-dimensional compact smooth manifold. Denote by $\mathcal{M}$ and $\mathcal{G}$ the space of smooth Riemannian metric and the diffeomorphism group of M respectively. We call a functional $F\colon \mathcal{M} \to R$ Riemannian if F is invariant under the action of $\mathcal{G}$, i. e. $F(\varphi^* g) = F(g)$ for $\varphi \in \mathcal{G}$ and $g \in \mathcal{M}$.

By letting $S_2(M)$ denote the bundle of symmetric $(0, 2)$ tensors on M^n, we say that F has a gradient ∇F at $g \in \mathcal{M}$ if for $h \in S_2(M)$

$$\frac{\mathrm{d}}{\mathrm{d}t} F(g + th)|_{t=0} = \int_M \langle h, \nabla F\rangle_g \,\mathrm{d}v_g$$

In [6], Gursky and Viaclovsky studied the functional

$$F(g) = \int_{M^3} \sigma_k(C_g)\,\mathrm{d}v_g$$

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where $\sigma_k(C_g)$ is the k-th elementary symmetric function of the eigenvalues of the Schonten tensor $C_g = Ric - \frac{1}{2}\frac{R}{n-1}g$. They proved that

Theorem 1. *[6] Let M be a compact 3-manifold, then a metric g with $F_2(g) \geq 0$ is critical for $F_2|_{\mathcal{M}_1}$ if and only if g has constant sectional curvature,where $\mathcal{M}_1 = \{g \in \mathcal{M} \mid \mathrm{Vol}(g) = 1\}$*

This gives a new variational characterization of three-dimensional space forms. In [7], Hu and Li generalized the above result to the case $n \geq 5$. There are many deep on going results about the 4-manifolds. M. J. Gursky considered in [5] 4-manifolds with harmonic self-dual Weyl tensor and obtained a lower bound of the L^2 norm of the self-dual part of Weyl tensor. S.-Y. A. Chang, M. J. Gursky and P. Yang obtained in [3] some sufficient geometric conditions for a 4-manifold to have certain conformal class of metric and consequently to have finite fundamental group. C. LeBrun and B. Maskit [9] completely determined compact simply connected and oriented 4-manifolds up to homomorphism which admit scalar flat, anti-self-dual Riemannian metrics. There is a rich literature concerning results related to the variation of curvature functional [1], [4], [10], [11], [12].

Early in 1938, before the higher dimensional Gauss-Bonnet formula were discovered, C. Lanczos [8] studied the functional

$$\phi_{a,b,c}(g) = \int_{M^4} \left(a|Rie|^2 + b|Ric|^2 + cR^2\right) \, \mathrm{d}v_g$$

on 4-manifolds. He found that the functional $\phi_{1,-4,1}$ has a gradient which is identically zero. In fact this establishes that this integral is a differential invariant of the manifold M. It is even a topological invariant, namely $32\pi^2\chi(M)$, where $\chi(M)$ the Euler-Poincare characteristic of M, i. e.

$$32\pi^2\chi(M) = \int_{M^4} \left(|Rie|^2 - 4|Ric|^2 + R^2\right) \, \mathrm{d}v_g \tag{1}$$

Taking this Gauss-Bonnet formula into account, we naturally study the functional

$$F(g) = \int_{M^4} \left(aR_g^2 + b|Ric_g|^2\right) \, \mathrm{d}v_g \tag{2}$$

We obtain a new variational characterization of 4-manifolds as follow

Theorem 2. *Suppose that (M^4, g) is compact and locally conformally flat. If g is a critical point of the functional (2) with any pairs (a, b) in the real plane with two fixed lines deleted, that is*

$$(a, b) \in \mathbb{R}^2 \setminus L_1 \cup L_2\,; \quad L_1\colon 3a + b = 0\,; \quad L_2\colon 6a - b + 1 = 0\,,$$

then (M^4, g) is either scalar flat or a space form.

2 Preliminaries

Recently, the first author [13] studied the variation formulas of a metric by the moving frame method. He obtained the first and the second variation formulas for the Riemannian curvature tensor, Ricci tensor and scalar curvature of a metric in another formalism which should be equivalent to the classical ones. He also obtained some interesting applications of these formulas. We believe that these formulas are more convenient in the computations of calculus of variation, especially in the computations where the second variation of a metric is involved. We follow the notations as in [13]. Classical variational formulas of metric can be found in [2] and [12].

Suppose that

$$g(t) = \sum_{i=1}^{n} \theta_i^2(t)$$

is a variation of a given metric g. For the sake of simplicity, from now on we use Einstein summation convention; i. e., the repeated indices imply summation. The indices $i, j, k, \dots$ are from 1 to n unless otherwise stated. Let $\theta_{ij}(t)$ and $\Omega_{ij}(t)$ are connection one-forms and curvature two-forms determined respectively by

$$\mathrm{d}\theta_i(t) = \theta_{ij}(t) \wedge \theta_j(t)$$

$$\Omega_{ij}(t) = \mathrm{d}\theta_{ij}(t) - \theta_{ik}(t) \wedge \theta_{kj}(t) = -\frac{1}{2} R_{ijkl}(t)\theta_k(t) \wedge \theta_l(t)$$

where d is the exterior differential operator on the manifold. These equations are known as the structural equation of the Levi-Civita connection of the metric. R_{ijkl} are the components of the $(0, 4)$ type Riemannian curvature tensor. Assume that

$$\theta_i(t) = \theta_i + \omega_i t + o(t) \qquad R_{ijkl}(t) = R_{ijkl} + r_{ijkl} t + o(t)$$

where $\theta_i = \theta_i(t)|_{t=0}$, $\omega_i = \left.\frac{\mathrm{d}\theta_i(t)}{\mathrm{d}t}\right|_{t=0} = a_{ij}\theta_j$, $R_{ijkl} = R_{ijkl}(t)|_{t=0}$, $r_{ijkl} = \left.\frac{\mathrm{d}R_{ijkl}(t)}{\mathrm{d}t}\right|_{t=0}$.

By a crucial lemma proved in [13], there exists an isometry of $g(t)$, such that a_{ij} are symmetric. So we may always assume $a_{ij} = a_{ji}$ without loss of generality. With these preparation we have [13]

$$r_{ijkl} = -(a_{ik,jl} - a_{il,jk} + a_{jl,ik} - a_{jk,il} + R_{ijkm}a_{ml} + R_{ijml}a_{mk}) \tag{3}$$

where $a_{ij,kl}$ is defined by

$$a_{ij,kl}\theta_l = da_{ij,k} + a_{lj,k}\theta_{li} + a_{il,k}\theta_{lj} + a_{ij,l}\theta_{lk}$$

and $a_{ij,k}$ is defined by

$$a_{ij,k}\theta_k = da_{ij} + a_{kj}\theta_{ki} + a_{ik}\theta_{kj} ,$$

$$\theta_{ij} = \theta_{ij}(t)|_{t=0}$$

$a_{ij,k}$ and $a_{ij,kl}$ are the first and the second covariant derivatives of a_{ij} with respect to the initial metric g.

Defined the Ricci curvature

$$R_{ij}(t) = \sum_{k=1}^{n} R_{ikjk}(t) = R_{ij} + r_{ij}t + o(t)$$

and the Scalar curvature

$$R(t) = \sum_{i=1}^{n} R_{ii}(t) = R + rt + o(t)$$

of $g(t)$ respectively the above two formulas, then by making contraction from (3) one obtain immediately

$$\frac{\partial R_{ij}(t)}{\partial t}\Big|_{t=0} = r_{ij} = -\Delta a_{ij} - a_{kk,ij} + a_{ik,kj} + a_{kj,ik} - R_{ik}a_{kj} - R_{ikjl}a_{kl} \tag{4}$$

$$\frac{\partial R(t)}{\partial t}\Big|_{t=0} = r = 2(a_{ij,ij} - \Delta a_{ii} - a_{ij}R_{ij}) \tag{5}$$

where Δa_{ij} denotes the Laplacian of a_{ij} with respect to the original metric g. For more details see [13].

3 Proof of the theorem 2

By (4) and (5) we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}F(t)\big|_{t=0} &= \int_{M^4} \Big\{2\Big(aR(t)\frac{dR(t)}{dt} + bR_{ij}\frac{dR_{ij}(t)}{dt}\Big) + (aR^2 + bR_{ij}^2)a_{mm}\Big\}\,\mathrm{d}v_g\big|_{t=0}\\ &= \int_{M^4} \{2aR\cdot 2(a_{ij,ij} - \Delta a_{ii} - a_{ij}R_{ij})\\ &\quad + 2bR_{ij}(-\Delta a_{ij} - a_{kk,ij} + a_{ik,kj} + a_{kj,ik} - R_{im}a_{mj} - R_{ikjl}a_{kl})\\ &\quad + (aR^2 + bR_{ij}^2)a_{mm}\}\,\mathrm{d}v_g\\ &= \int_{M^4} a_{ij}(\nabla F)_{ij}\,\mathrm{d}v_g\end{aligned}$$

where

$$\begin{aligned}(\nabla F)_{ij} = {} & 4aR_{,ij} - 4a\Delta R\delta_{ij} - 4aRR_{ij} - 2b\Delta R_{ij}\\ & - 2bR_{kl,kl}\delta_{ij} + 2bR_{ik,kj} + 2bR_{kj,ik} - 2bR_{im}R_{mj}\\ & - 2bR_{kl}R_{ik,jl} + (aR^2 + bR_{kl}^2)\delta_{ij}\,.\end{aligned} \tag{6}$$

Since g is a critical point of the functional (2), we have

$$(\nabla F)_{ij} = 0\,. \tag{7}$$

Taking trace of (7) and making use of the following identities which are obtained from the second Bianchi identity and the Ricci identity respectively

$$2R_{ij,i} = R_{,j},$$

$$2R_{ij,ij} = \Delta R,$$
$$R_{ij,kl} - R_{ij,lk} = R_{mj}R_{mikl} + R_{im}R_{mjkl},$$
$$R_{kj,ik} = \frac{1}{2}R_{,ij} + R_{ik}R_{kj} + R_{kl}R_{iklj},$$

then we have

$$4a\Delta R - 4\cdot 4a\Delta R - 4aR^2 - 2b\Delta R - 4b\Delta R + b\Delta R + b\Delta R \\ - 2bR_{ij}^2 - 2bR_{ij}^2 + 4(aR^2 + bR_{ij}^2) = 0$$

or after simplifying we arrive at

$$(3a+b)\Delta R = 0\,.$$

By the assumptions of the theorem, $3a+b \neq 0$. This gives $\Delta R = 0$. Since M^4 is compact, R must be a constant. In this case, from (7) and (6) we have

$$-4aRR_{ij} - 2b\Delta R_{ij} + 2b(R_{in}R_{nj} + R_{kl}R_{iklj}) \\ - 2bR_{im}R_{mj} + 2bR_{kl}R_{iklj} + (aR^2 + bR_{kl}^2)\delta_{ij} = 0\,. \quad (8)$$

If (M^4, g) is locally conformally flat, then

$$R_{ijkl} = \frac{1}{2}(R_{ik}\delta_{jl} - R_{il}\delta_{jk} + \delta_{ik}R_{jl} - \delta_{il}R_{jk}) - \frac{1}{6}R(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})\,.$$

Substituting this expression into (8) we have

$$\left(4a + \frac{2}{3}b\right)RE_{ij} + 2b\Delta E_{ij} - 4bE_{ik}E_{kj} + bE_{kl}^2\delta_{ij} = 0 \quad (9)$$

where

$$E_{ij} = R_{ij} - \frac{1}{4}R\delta_{ij},$$

is the traceless part of the Ricci tensor. If $b \neq 0$, then

$$\Delta E_{ij} = 2E_{ik}E_{kj} - \frac{1}{b}\left(2a + \frac{b}{3}\right)RE_{ij} - \frac{1}{2}E_{kl}^2\delta_{ij}\,. \quad (10)$$

Comparing the standard result in [7]

$$\Delta E_{ij} = 2E_{ik}E_{kj} - \frac{1}{3}RE_{ij} - \frac{1}{2}E_{kl}^2\delta_{ij}$$

on a locally conformally flat 4-manifold. We have

$$-\frac{1}{b}\left(2a + \frac{1}{3}\right)RE_{ij} = -\frac{1}{3}RE_{ij}$$

or equivalently

$$(6a - b + 1)RE_{ij} = 0\,.$$

Again by the assumption of the theorem, $6a - b + 1 \neq 0$, then

$$RE_{ij} = 0\,. \tag{11}$$

So $R = 0$ or $E_{ij} = 0$. In the first case, (M^4, g) is scalar flat and in the second case, considering g is also locally conformally flat we see that (M^4, g) has constant sectional curvature. If $b = 0$, then $a \neq 0$ by the assumption. From (9) we still have $RE_{ij} = 0$, and the same conclusion remains true. This completes the proof of theorem 2.

Remark 1.

1. If $3a + b = 0$ and $6a - b + 1 = 0$, then $(a, b) = (-\frac{1}{9}, \frac{1}{3})$. It can be checked that

$$R^2_{ijkl} - 4R^2_{ij} + R^2 = -6\big(-\frac{1}{9}R^2 + \frac{1}{3}R^2_{ij}\big)$$

   that is, the integrand of our functional is a multiple of the integrand of the Gauss-Bonnet formula. In this case, the variation is identically zero.

2. All points (a, b) considered in our functional fall into four regions. It would be interesting to study further property of the functional.

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