Hemar Godinho; Diego Marques; Alain Togbé
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On the Diophantine equation $x^2 + 2^{\alpha}5^{\beta}17^{\gamma} = y^n$

Hemar Godinho, Diego Marques, Alain Togbé

Abstract. In this paper, we find all solutions of the Diophantine equation $x^2 + 2^{\alpha}5^{\beta}17^{\gamma} = y^n$ in positive integers $x, y \geq 1$, $\alpha, \beta, \gamma, n \geq 3$ with $\gcd(x, y) = 1$.

1 Introduction

There are many results concerning the generalized Ramanujan-Nagell equation

$$x^2 + C = y^n,$$

where $C > 0$ is a given integer and $x, y, n$ are positive integer unknowns with $n \geq 3$. Results obtained for general superelliptic equations clearly provide effective finiteness results for this equation, too (see for example [8], [31], [32] and the references given there). The first result concerning the above equation was due to V. A. Lebesgue [23] and it goes back to 1850, where he proved that the above equation has no solutions for $C = 1$. More recently, other values of $C$ were considered. Tengely [33] solved the equation with $C = b^2$, $b$ odd and $3 \leq b \leq 501$. The case where $C = p^k$, a power of a prime number, was studied in [5], [21], [20] for $p = 2$, in [6], [4], [24] for $p = 3$, in [1], [2] for $p = 5$, and in [27] for $p = 7$. The case $C = p^{2k}$ with $2 \leq p < 100$ prime and $\gcd(x, y) = 1$ was solved by Bérczes and Pink [9]. For arbitrary primes, some advances can be found in [7]. In [13], the cases with $1 \leq C \leq 100$ were completely solved. The solutions for the cases $C = 2^a \cdot 3^b$, $C = 2^a \cdot 5^b$ and $C = 5^a \cdot 13^b$, when $x$ and $y$ are coprime, can be found in [3], [25], [26], respectively. Recent progress on the subject were made in the cases $C = 5^a \cdot 11^b$, $C = 2^a \cdot 11^b$, $C = 2^a \cdot 3^b \cdot 11^c$, $C = 2^a \cdot 5^b \cdot 13^c$ and can be found in [16], [15], [14], [18]. For related results concerning equation (1) see [10], [22], [29], [30] and the references given there. For a survey concerning equation (1) see [12].

In this paper, we are interested in solving the Diophantine equation

$$x^2 + 2^{\alpha}5^{\beta}17^{\gamma} = y^n, \quad \gcd(x, y) = 1, \quad x, y \geq 1, \quad \alpha, \beta, \gamma \geq 0, \quad n \geq 3.$$

2010 MSC: Primary 11D61, Secondary 11Y50
Key words: Diophantine equation, exponential equation, primitive divisor theorem
Our result is the following.

**Theorem 1.** The equation (2) has no solution except for:

- \( n = 3 \) the solutions given in Table 1;
- \( n = 4 \) the solutions given in Table 2;
- \( n = 5 \) \((x, y, \alpha, \beta, \gamma) = (401, 11, 1, 3, 0)\);
- \( n = 6 \) \((x, y, \alpha, \beta, \gamma) = (7, 3, 3, 1, 1), (23, 3, 3, 2, 0)\);
- \( n = 8 \) \((x, y, \alpha, \beta, \gamma) = (47, 3, 8, 0, 1), (79, 3, 6, 1, 0)\).

One can deduce from the above result the following corollary.

**Corollary 1.** The equation

\[
x^2 + 5^k 17^l = y^n, \quad x \geq 1, \quad y \geq 1, \quad \gcd(x, y) = 1, \quad n \geq 3, \quad k \geq 0, \quad l \geq 0
\]

(3)

has only the solutions

\[
(x, y, k, l, n) = (94, 21, 2, 1, 3), \quad (2034, 161, 3, 2, 3), \quad (8, 3, 0, 1, 4).
\]

Therefore, our work extends that of Pink and Rábai [28]. We will follow the standard approach to work on equation (2) but with another version of MAGMA (V2.18-6) that gives better results when we deal with the corresponding elliptic curves.

2 The case \( n = 3 \)

**Lemma 1.** When \( n = 3 \), all the solutions to equation (2) are given in Table 1.

For \( n = 6 \), we have \((x, y, \alpha, \beta, \gamma) = (7, 3, 3, 1, 1), (23, 3, 3, 2, 0)\).

Proof. Equation (2) can be rewritten as

\[
\left(\frac{x}{z^3}\right)^2 + A = \left(\frac{y}{z^2}\right)^3,
\]

(4)

where \( A \) is sixth-power free and defined implicitly by \( 2^{a_1} 5^{b_1} 17^{c_1} = Az^6 \). One can see that \( A = 2^{a_1} 5^{b_1} 17^{c_1} \) with \( a_1, \beta_1, \gamma_1, \in \{0, 1, 2, 3, 4, 5\} \). We thus get

\[
V^2 = U^3 - 2^{a_1} 5^{b_1} 17^{c_1},
\]

(5)

with \( U = y/z^2 \), \( V = x/z^3 \) and \( a_1, \beta_1, \gamma_1 \in \{0, 1, 2, 3, 4, 5\} \). We need to determine all the \( \{2, 5, 17\} \)-integral points on the above 216 elliptic curves. Recall that if \( \mathcal{S} \) is a finite set of prime numbers, then an \( \mathcal{S} \)-integer is rational number \( a/b \) with coprime integers \( a \) and \( b \), where the prime factors of \( b \) are in \( \mathcal{S} \). We use the command SIntegralPoints of MAGMA [17] to determine all the \( \{2, 5, 17\} \)-integer points on the above elliptic curves. Here are a few remarks about the computations:

1. We eliminate the solutions with \( UV = 0 \) because they yield to \( xy = 0 \).
On the Diophantine equation $x^2 + 2^\alpha 5^\beta 17^\gamma = y^n$

Table 1: Solutions for $n = 3$.

<table>
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<th>$\alpha_1$</th>
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<th>$\gamma_1$</th>
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2. We consider only solutions such that the numerators of $U$ and $V$ are coprime.

3. If $U$ and $V$ are integers then $z = 1$. So $\alpha_1 = \alpha$, $\beta_1 = \beta$, and $\gamma_1 = \gamma$.

4. If $U$ and $V$ are rational numbers which are not integers, then $z$ is determined by the denominators of $U$ and $V$. The numerators of these rational numbers give $x$ and $y$. Then $\alpha, \beta, \gamma$ are computed knowing that $2^\alpha 5^\beta 17^\gamma = Az^6$.

Therefore, we first determine $(U, V, \alpha_1, \beta_1, \gamma_1)$ and then we use the relations

$$U = \frac{y}{z^2}, \quad V = \frac{x}{z^3}, \quad 2^\alpha 5^\beta 17^\gamma = Az^6,$$

to find the solutions $(x, y, \alpha, \beta, \gamma)$ listed in Table 1.
For \( n = 6 \), equation
\[
x^2 + 2^5 5^\beta 17^\gamma = y^6
\] becomes equation
\[
x^2 + 2^5 5^\beta 17^\gamma = (y^2)^3.
\] We look in the list of solutions of Table 1 and observe that \( y \) is a perfect square only when \( y = 9 \) corresponding to two solutions. Therefore, the only solutions to equation (2) for \( n = 6 \) are the two solutions listed in Theorem 1. This completes the proof of Lemma 1. \( \square \)

**Remark 1.** Notice that with the old version of MAGMA, it was difficult to determine the rational points of certain elliptic curves when \( 2^5 5^\beta 17^\gamma \) is very high. That is the case of the following elliptic curves:
\[
V^2 = U^3 - 2^3 \cdot 5^5 \cdot 17^5, \quad V^2 = U^3 - 2^5 \cdot 5^1 \cdot 17^4.
\] We thank the team MAGMA, particularly Steve Donnelly for the new version (Magma V2.18-6) and their help.

### 3 The case \( n = 4 \)

Here, we have the following result.

**Lemma 2.** If \( n = 4 \), then the only solutions to equation (2) are given in Table 2.

If \( n = 8 \), then the only solution to equation (2) is \((x, y, \alpha, \beta, \gamma) = (47, 3, 8, 0, 1), (79, 3, 6, 1, 0)\).

<table>
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<th>( \gamma_1 )</th>
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**Proof.** Equation (2) can be written as
\[
\left(\frac{x}{z^2}\right)^2 + A = \left(\frac{y}{z}\right)^4,
\]
where $A$ is fourth-power free and defined implicitly by $2^a 5^\beta 17^\gamma = Az^4$. One can see that $A = 2^{a_1} 5^{b_1} 17^{\gamma_1}$ with $a_1, \beta_1, \gamma_1 \in \{0, 1, 2, 3\}$. Hence, the problem consists of determining the $\{2, 5, 17\}$-integer points on the totality of the 64 elliptic curves

$$V^2 = U^4 - 2^{a_1} 5^{b_1} 17^{\gamma_1}, \quad (9)$$

with $U = y/z$, $V = x/z^2$ and $a_1, \beta_1, \gamma_1 \in \{0, 1, 2, 3\}$. Here, we use the command $\text{SIntegralQuarticPoints}$ of MAGMA [17] to determine the $\{2, 5, 17\}$-integer points on the above elliptic curves. As in Section 2, we first find $(U, V, \alpha_1, \beta_1, \gamma_1)$, and then using the coprimality conditions on $x$ and $y$ and the definition of $U$ and $V$, we determine all the corresponding solutions $(x, y, \alpha, \beta, \gamma)$ listed in Table 2.

Looking in the list of solutions of equation Table 2, we observe the 2 solutions for $(x, y, \alpha, \beta, \gamma)$ listed in Table 2 whose values for $y$ are perfect squares. Thus, the only solutions to equation (2) with $n = 8$ are those listed in Theorem 1. This concludes the proof of Lemma 2.

4 The case $n \geq 5$

The aim of this section is to determine all solutions of equation (2), for $n \geq 5$ and to prove its unsolvability for $n = 7$ and $n \geq 9$. The cases when $n$ is of the form $2^a 3^b$ were treated in previous sections. So, apart from these cases, in order to prove that (2) has no solution for $n \geq 7$, it suffices to consider $n$ prime. In fact, if $(x, y, \alpha, \beta, \gamma, n)$ is a solution for (2) and $n = pk$, where $p \geq 7$ is prime and $k > 1$, then $(x, y^k, \alpha, \beta, \gamma, p)$ is also a solution. So, from now on, $n$ will denote a prime number.

Lemma 3. The Diophantine equation (2) has no solution with $n \geq 5$ prime except for

$$n = 5 \quad (x, y, \alpha, \beta, \gamma) = (401, 11, 1, 3, 0).$$

Proof. Let $(x, y, \alpha, \beta, \gamma, n)$ be a solution for (2). We claim that $y$ is odd. In fact, if $y$ is even and since $\gcd(x, y) = 1$, one has that $x$ is odd, and then $-2^a 5^\beta 17^\gamma \equiv x^2 - y^n \equiv 1 \pmod{4}$, but this contradicts the fact that $-2^a 5^\beta 17^\gamma \equiv 0, 2$ or $3 \pmod{4}$ (according to $a \geq 2, \alpha = 1$ or $\alpha = 0$, respectively). Now, write equation (2) as $x^2 + dz^2 = y^n$, where

$$d = 2^{a-2(|\alpha/2|5^\beta-2|\beta/2|17^\gamma-2|\gamma/2)},$$

and $z = 2^{|\alpha/2|5^{|\beta/2|17^{|\gamma/2|}}}$. Since $x - 2|x/2| \in \{0, 1\}$, we have

$$d \in \{1, 2, 5, 10, 17, 34, 85, 170\}.$$

We then factor the previous equation over $K = \mathbb{Q}[i\sqrt{d}] = \mathbb{Q}[\sqrt{-d}]$ as

$$(x + i\sqrt{d}z)(x - i\sqrt{d}z) = y^n.$$

Now, we claim that the ideals $(x + i\sqrt{d}z)O_K$ and $(x - i\sqrt{d}z)O_K$ are coprime. If this is not the case, there must exist a prime ideal $p$ containing these ideals. Therefore, $x \pm i\sqrt{d}z$ and $y^n$ (and so $y$) belong to $p$. Thus $2x \in p$ and hence either 2
or $x$ belongs to $p$. Since $\gcd(2, y) = \gcd(x, y) = 1$, then 1 belongs to the ideals $(2, y)$ and $(x, y)$, then $1 \in p$ leading to an absurdity of $p = O_K$. By the unique factorization of ideals, it follows that $(x + i\sqrt{d})O_K = j^n$, for some ideal $j$ of $O_K$. Using Mathematica’s command `NumberFieldClassNumber[Sqrt[-d]]`, we obtain that the class number of $K$ is either 1, 2, 4 or 12 and so coprime to $n$, then $j$ is a principal ideal yielding

$$x + i\sqrt{d}z = \varepsilon \eta^n,$$

for some $\eta \in O_K$ and $\varepsilon$ a unit of $K$. Since the group of units of $K$ is a subset of $\{\pm 1, \pm i\}$ and $n$ is odd, then $\varepsilon$ is a $n$-th power. Thus, (10) can be reduced to $x + i\sqrt{d}z = \eta^n$. Since $K$ is an imaginary quadratic field and $-d \not\equiv 1 \pmod{4}$, then $\{1, i\sqrt{d}\}$ is an integral basis and we can write $\eta = u + i\sqrt{d}v$, for some integers $u$ and $v$. We then get

$$\frac{\eta^n - \overline{\eta}^n}{\eta - \overline{\eta}} = \frac{2^{\lfloor \alpha/2 \rfloor}5^{\lfloor \beta/2 \rfloor}17^{\lfloor \gamma/2 \rfloor} \prod_{j=1}^{n-1} L_j}{v},$$

where, as usual, $\overline{w}$ denotes the complex conjugate of $w$.

Let $(L_m)_{m \geq 0}$ be the Lucas sequence given by

$$L_m = \frac{\eta^m - \overline{\eta}^m}{\eta - \overline{\eta}}, \text{ for } m \geq 0.$$

We recall that the Primitive Divisor Theorem for Lucas sequences ensures for primes $n \geq 5$, that there exists a primitive divisor for $L_n$, except for the finitely many (defective) pairs $(\eta, \overline{\eta})$ given in Table 1 of [11] (a primitive divisor of $L_n$ is a prime that divides $L_n$ but does not divide $(\eta - \overline{\eta})^2 \prod_{j=1}^{n-1} L_j$). And a helpful property of a primitive divisor $p$ is that $p \equiv \pm 1 \pmod{n}$.

For $n = 5$, we find in Table 1 in [11] that $L_5$ has a primitive divisor except for $(u, d, v) = (1, 10, 1)$ which leads to a number $\eta = 1 + i\sqrt{10} \in \mathbb{Q}[i\sqrt{10}]$ $(d = 10$ is one of the possible values of $d$ described in the beginning of this proof), which gives the solution with $n = 5$.

Apart from this case, let $p$ be a primitive divisor of $L_n$, $n \geq 7$. The identity (11) implies that $p \in \{2, 5, 17\}$ and so $p = 17$, since $p \not\equiv \pm 1 \pmod{n}$, for $p = 2, 5$. Hence, $n$ is a prime dividing $17 \pm 1$ and so $n = 2$ or 3 which contradicts the fact that $n \geq 7$. This completes the proof of Theorem 1.

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References


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