

Jonathan L. Verner
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Commentationes Mathematicae Universitatis Carolinae, Vol. 54 (2013), No. 1, 105--110

Persistent URL: <http://dml.cz/dmlcz/143156>

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Lonely points revisited

JONATHAN L. VERNER

Abstract. In our previous paper, we introduced the notion of a lonely point, due to P. Simon. A point $p \in X$ is lonely if it is a limit point of a countable dense-in-itself set, it is not a limit point of a countable discrete set and all countable sets whose limit point it is form a filter. We use the space \mathcal{G}_ω from a paper of A. Dow, A.V. Gubbi and A. Szymański [*Rigid Stone spaces within ZFC*, Proc. Amer. Math. Soc. **102** (1988), no. 3, 745–748] to construct lonely points in ω^* . This answers the question of P. Simon posed in our paper *Lonely points in ω^** , Topology Appl. **155** (2008), no. 16, 1766–1771.

Keywords: $\beta\omega$, lonely point, weak P-point, irresolvable spaces

Classification: Primary 54D80, 54D40, 54G05

1. Introduction

1.1 Definition. A *topological type* in a space X is a subset $T \subseteq X$ which is invariant under homeomorphisms.

An example of a topological type are discrete points in a space X . Another more interesting type is given in the following definition. The first part is due to W. Rudin ([10]), the second to K. Kunen ([8]).

1.2 Definition (Rudin, Kunen). A point $x \in X$ is a *P-point* if a countable intersection of neighbourhoods of x is again a neighbourhood of x . It is a *weak P-point* if it is not a limit point of a countable subset of X .

Clearly any isolated point is a P-point, and a P-point is a weak P-point. However none of the implications can be reversed.

If a space contains two distinct topological types, then it is not homogeneous. The motivation for finding topological types in ω^* was given by the following surprising result of Z. Frolík ([5], [4]):

1.3 Theorem (Frolík). ω^* is not homogeneous.

His proof used a clever combinatorial argument but it gave no intrinsically topological reason for the non-homogeneity of ω^* . This motivated the question whether one can find a “topologically defined” topological type — an “honest”

The author would like to acknowledge the support of GAČR 401/09/H007 Logické základy sémantiky.

proof of nonhomogeneity. Under CH, this was answered already by W. Rudin in [10] where he proved that P-points exist in ω^* . However in ZFC the question remained open for some twenty years.

In his seminal paper [8], K. Kunen proved in ZFC that ω^* contains a weak P-point:

1.4 Theorem (Kunen). ω^* contains a weak P-point.

Since it obviously contains non weak P-points, this is an “honest” proof of nonhomogeneity. In [9], J. van Mill exploited the techniques of K. Kunen to prove, in ZFC, the existence of sixteen distinct topological types in ω^* ! One of the types he introduced is given in the following theorem:

1.5 Theorem (van Mill). *There is a point $p \in \omega^*$ which is a limit point of a countable discrete set and the countable sets whose limit point it is form a filter.*

PROOF: (Idea) Use Kunen’s result to construct a weak P-point $p \in \omega^* \subseteq \beta\omega$. Now use a theorem of P. Simon (see Theorem 2.5) to embed $\beta\omega$ into ω^* as a weak P-set. Then the image of p via the embedding will be as required, since p clearly has the property in $\beta\omega$ and the embedding does not destroy it since the image of $\beta\omega$ is a weak P-set. \square

This motivated P. Simon to define the following notion, which we have called a lonely point in [12]. We want essentially the same type of point as in the above theorem only replacing the countable discrete set whose limit point it is by a crowded set:

1.6 Definition. A point $p \in X$ is a *lonely point* provided:

- (i) p is ω -discretely untouchable, i.e. not a limit point of a countable discrete set,
- (ii) p is a limit point of a countable crowded (i.e. without isolated points) set and
- (iii) the countable sets whose limit point p is form a filter.

In the paper we were able to show that lonely points exist in some open dense subspace of ω^* . Here we prove that they actually exist in ω^* :

1.7 Theorem. ω^* contains a lonely point.

The idea is to construct a countable, perfectly disconnected space X with an \aleph_0 -bounded remainder and then embed it as a weak P-set into ω^* . Any point of X will then be a lonely point of βX and, since βX is a weak P-set in ω^* , also a lonely point of ω^* .

2. Basic definitions and theorems

2.1 Definition (Kunen). $F \subseteq X$ is a weak P-set of X if any countable $D \subseteq X$ disjoint from F has closure disjoint from F .

2.2 Observation. If $F \subseteq X$ is a weak P-set of X and $x \in F$ is a lonely point of F then it is also a lonely point of X .

2.3 Definition. A space X is *extremally disconnected* (or ED for short) if the closure of any open set is open.

The following is standard, see e.g. [3]:

2.4 Theorem. *If X is ED then so is βX .*

We shall also need the following theorem of P. Simon (see [11]):

2.5 Theorem (Simon). *The Čech-Stone compactification of any T_3 ED space of weight $\leq 2^{\aleph_0}$ can be embedded into ω^* as a closed weak P -set.*

3. Irresolvable spaces

In this section, unless otherwise stated, we assume all spaces to be crowded (i.e. without isolated points). The following definitions were introduced in [1]:

3.1 Definition (van Douwen). A crowded space X is *perfectly disconnected* if no point of X is a limit point of two disjoint subsets of X . It is irresolvable, if it contains no disjoint dense sets. It is open-hereditarily-irresolvable (OHI for short), provided each open subspace is irresolvable. A crowded space is *maximal regular* if each finer topology either contains an isolated point or is not regular.

Irresolvable spaces were constructed by E. Hewitt ([6]) and independently by M. Katětov ([7]). They were extensively studied in [1] where the following theorems may be found:

3.2 Theorem ([1, 1.7, 1.11]). *Maximal regular spaces are zero dimensional, ED and OHI.*

3.3 Theorem ([1, 1.4, 1.6]). *If A, B are disjoint crowded subspaces of a maximal regular space, then \overline{A} and \overline{B} are disjoint.*

3.4 Theorem ([1, 2.2]). *If X is ED and OHI and each nowhere dense subset of X is closed then X is perfectly disconnected.*

The following theorem is not explicitly stated in van Douwen's paper, but its proof is essentially given in his Lemma 3.2 and Example 3.3.

3.5 Theorem (van Douwen). *Any countable maximal regular space X contains an open perfectly disconnected subspace.*

PROOF: For each $Z \subseteq X$ let

$$A_Z = \{x \in Z : x \text{ is a limit point of a relatively discrete subset of } Z\}.$$

Claim $A_Z \neq Z$ for each open subset Z of X .

Assume otherwise. Enumerate Z as $\langle x_n : n < \omega \rangle$. By induction construct pairwise disjoint, relatively discrete sets $\langle D_n : n < \omega \rangle$ such that:

- (i) $\bigcup_{i < n} D_i \subseteq \overline{D_n}$ for all $n < \omega$ and
- (ii) $x_n \in \overline{D_n}$ for $n < \omega$.

This will lead to a contradiction with the irresolvability of Z (by Theorem 3.2, X is OHI, so Z is irresolvable). Indeed, $\bigcup_{n < \omega} D_{2n}$ and $\bigcup_{n < \omega} D_{2n+1}$ are disjoint dense subsets of Z . To see that the construction can be carried out let $D_0 = \{x_0\}$ and assume we have constructed D_i for $i \leq n$. Let $Y = D_n \cup Z \setminus \overline{D_n}$. Since D_n is relatively discrete, Y is open. Since Z is regular and D_n is countable and relatively discrete, there is a pairwise disjoint collection of open sets $\{U_x : x \in D_n\}$ such that $x \in U_x \subseteq Y$. Since we assume $A_Z = Z$ we can choose for each $x \in D_n$ a relatively discrete set D_x such that $D_x \subseteq U_x$ and $x \in \overline{D_x} \setminus D_x$. Let $D'_{n+1} = \bigcup_{x \in D_n} D_x$. If x_{n+1} is a limit point of D'_{n+1} let $D_{n+1} = D'_{n+1}$, otherwise let $D_{n+1} = D'_{n+1} \cup \{x_{n+1}\}$. Then D_{n+1} is as required.

Claim $\text{int } A_X = \emptyset$.

For any clopen U , $A_X \cap U = A_U$. Since X is regular and countable, it is zero dimensional. Suppose U is clopen and $U \subseteq A_X$. By the previous claim $U \setminus A_U \neq \emptyset$ but then $U \setminus A_X \neq \emptyset$, which is a contradiction.

Claim A_X is nowhere dense.

Take any open $U \subseteq X$. Then $U \setminus A_X$ is dense in U , since $\text{int } A_X = \emptyset$. Since X is OHI (by Theorem 3.2) and U is irresolvable, A_X cannot be dense in U and so $U \not\subseteq \overline{A_X}$. Thus we have that $\text{int } \overline{A_X} = \emptyset$.

Claim If $A \subseteq X$ is nowhere dense then there is a discrete $D \subseteq A$ dense in A .

Let $D = \{x \in A : x \text{ is isolated in } A\}$. Since X is regular and countable D is relatively discrete. Since A is nowhere dense, D is discrete. Let $E = A \setminus \overline{D}$. Then E has no isolated points. Also $X \setminus E$ has no isolated points. By Theorem 3.3 E must be open which contradicts that A is nowhere dense.

Let $\vartheta = \{x \in X : x \text{ is not a limit point of a nowhere dense subset of } X\}$.

By the previous claim (and by the fact that each discrete subset of X is nowhere dense) we have that

$$\vartheta = \{x \in X : x \text{ is not a limit point of a discrete set}\}$$

Then $X \setminus \vartheta \subseteq A_X$ so $X \setminus \vartheta$ is nowhere dense, so $\text{int } \vartheta$ is nonempty. We finally show that $\text{int } \vartheta$ is perfectly disconnected. By the definition of ϑ any nowhere dense subset of $\text{int } \vartheta$ is closed. Now it remains to apply Theorem 3.4 remembering that by Theorem 3.2 $\text{int } \vartheta$ is ED and OHI (any open subspace of a maximal regular space is maximal regular). □

4. Proof of the main theorem

The following definition and theorem is taken from [2]:

4.1 Definition. Let $p \in \omega^*$ be a weak P-point. The space \mathcal{G}_ω is the space $\omega^{<\omega}$ of all finite sequences of natural numbers with $G \subseteq \omega^{<\omega}$ being open precisely when for each $\sigma \in G$ the set $\{n : \sigma \frown n \in G\}$ is in p .

4.2 Theorem (Dow, Gubbi, Szymanski). *The remainder of \mathcal{G}_ω is \aleph_0 -bounded. Moreover \mathcal{G}_ω is a T_2 , zero dimensional, ED space.*

Notice that if a space X has an \aleph_0 -bounded remainder, any finer topology also has an \aleph_0 -bounded remainder:

4.3 Proposition. *If $(X, \tau)^*$ is a zero dimensional \aleph_0 -bounded space and $\sigma \supseteq \tau$ is also zero dimensional, then $(X, \sigma)^*$ is \aleph_0 -bounded.*

PROOF: Note that any $p \in (X, \tau)^*$ corresponds to a closed subset of $(X, \sigma)^*$ (denoted by $[p]$). Now given $\{q_n : n < \omega\} \subseteq (X, \sigma)^*$ we can find $\{p_n : n < \omega\} \subseteq (X, \tau)^*$ such that $\{q_n : n < \omega\} \subseteq \bigcup \{[p_n] : n < \omega\}$. Since $(X, \tau)^*$ is \aleph_0 -bounded, $\overline{\{p_n : n < \omega\}}^{\beta(X, \tau)} \cap X = \emptyset$ so also $\overline{\{q_n : n < \omega\}}^{\beta(X, \sigma)} \cap X = \emptyset$ which implies that $(X, \sigma)^*$ is \aleph_0 -bounded. \square

4.4 Theorem. *There is a countable, ED, perfectly disconnected space X with an \aleph_0 -bounded remainder.*

PROOF: Take the space \mathcal{G}_ω from Theorem 4.2, and refine the topology to a maximal regular topology. Then, by the previous proposition, this space still has an \aleph_0 -bounded remainder and so does its open perfectly disconnected subspace given by Theorem 3.5. Let X be this subspace. \square

4.5 Theorem. ω^* contains a lonely point.

PROOF: Let X be the space from the previous theorem. Since it is crowded perfectly disconnected, each of its points is a lonely point of X . Since its remainder is \aleph_0 -bounded, each of its points is also a lonely point of βX . Since it is ED, βX is also ED and since it is countable, βX has weight at most 2^{\aleph_0} . Hence, by Theorem 2.5, βX can be embedded as a weak P-set into ω^* and each point of X will be a lonely point of ω^* (by Observation 2.2). \square

Acknowledgment. The author would like to thank A. Dow for a stimulating discussion about the topic as well as members of the Prague Set Theory seminar, who gave numerous helpful comments and encouraged him to work on the topic. The final version of the paper was written while the author was visiting the Kurt Gödel Research Center for Mathematical Logic and he would like to thank its members for their hospitality.

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CHARLES UNIVERSITY, FACULTY OF ARTS, DEPARTMENT OF LOGIC,
PALACHOVO NÁM. 2, 116 38 PRAHA 1, CZECH REPUBLIC

E-mail: jonathan.verner@matfyz.cz

(Received August 16, 2012, revised December 8, 2012)