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UNBOUNDED SOLUTIONS OF BVP FOR SECOND ORDER ODE WITH p-LAPLACIAN ON THE HALF LINE

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Abstract. By applying the Leggett-Williams fixed point theorem in a suitably constructed cone, we obtain the existence of at least three unbounded positive solutions for a boundary value problem on the half line. Our result improves and complements some of the work in the literature.

Keywords: second order differential equation on a half line, non-homogeneous boundary value problem, Leggett-Williams fixed point theorem

MSC 2010: 34B10, 34B15, 35B10

1. Introduction

Recently there has been increasing interest in the existence of positive solutions of boundary value problems (BVP) for differential equations on the half lines, see the references [1–7], [9–30]. Fixed point theorems have been useful in establishing the existence of positive solutions. To apply a fixed point theorem, one needs to define a Banach space, a cone, and a completely continuous operator.

Liu [19] applied the fixed point theorem of cone expansion and compression of norm type to establish the existence of single and multiple positive solutions of the boundary value problem

$$\begin{cases} x''(t) + f(t, x(t)) = 0, \quad t \in (0, \infty), \\ x(0) = 0, \\ \lim\limits_{t\to\infty} x'(t) = x_\infty \geqslant 0. \end{cases}$$

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Motivated by [19], in this paper we consider the following non-homogeneous boundary value problem for the differential equation on the half line whose boundary conditions are of integral form:

$$(1.1)\qquad \begin{cases} [\varrho(t)\varphi(x'(t))]' + f(t,x(t)) = 0, \quad t \in (0,\infty),\\ x(0) = \int_0^\infty g(s)x(s)\,\mathrm{d}s + a,\\ \lim\limits_{t\to\infty} \varphi^{-1}(\varrho(t))x'(t) = b. \end{cases}$$

Note that here we do not have the boundary condition $x'(\infty) = 0$ as in [9], [11], [14], [15], [17], [23], but $\lim\limits_{t\to\infty} \varphi^{-1}(\varrho(t))x'(t) = b$ contains $x'(\infty) = 0$ and $\lim\limits_{t\to\infty} x'(t) = x_\infty \geqslant 0$ as special cases. In (1.1) it is assumed that $a, b \geqslant 0$, $g\colon [0,\infty) \to [0,\infty)$ is continuous with $\int_0^\infty g(s)\,\mathrm{d}s < 1$, $f\colon (0,\infty)\times[0,\infty) \to [0,\infty)$, $\varrho\colon (0,\infty) \to (0,\infty)$ is continuous (f and ϱ may be singular at $t = 0$), and $\varphi(x) = |x|^{p-2}x$ with $p > 1$, whose inverse is denoted by φ^{-1} with $\varphi^{-1}(x) = |x|^{q-2}x$, where $1/p + 1/q = 1$.

We say $x\colon [0,\infty) \to (0,\infty)$ is a *positive solution* of (1.1) if $x \in C^1[0,\infty)$, $[\varrho\varphi(x')]' \in L^1(0,\infty)$ and x satisfies (1.1).

The aim of this paper is to establish existence results for at least three unbounded positive solutions of (1.1) by applying the Leggett-Williams fixed point theorem. In our derivation, the cone needed has to be very technically constructed – this is so since the boundary value problem involves the nonlinear operator $[\varrho\varphi(x')]'$ and the possible solutions are not concave if $\varrho \not\equiv 1$, hence the cone cannot be constructed by using the concavity of x or even the Green function. Our result improves and complements the work of [1–7], [9–30]. The paper is organized as follows. Section 2 contains some preliminary lemmas and the Leggett-Williams fixed point theorem. The main result is given in Section 3. Finally, in Section 4 we present an example to illustrate the result obtained.

2. Preliminary results

In this section, we present some background definitions and some preliminary lemmas.

Definition 2.1. A function $f\colon (0,\infty)\times\mathbb{R} \to \mathbb{R}$ is called an *S-Carathéodory function* if

(i) for each $u \in \mathbb{R}$, $t \mapsto f(t,u)$ is measurable on $(0,\infty)$;
(ii) for *a.e.* $t \in (0,\infty)$, $u \mapsto f(t,u)$ is continuous on $\mathbb{R}$;
(iii) for each $r > 0$, there exists $B_r \in L^1(0,\infty)$ satisfying $B_r(t) > 0$, $t \in (0,\infty)$, and $\int_0^\infty B_r(s)\,\mathrm{d}s < \infty$ such that $|u| \leqslant r$ implies

$$|f(t,(1+t)u)| \leqslant B_r(t), \ \textit{a.e. } t \in (0,\infty).$$

Let X be a real Banach space. A nonempty convex closed subset P of X is called a *cone* in X if (i) $ax \in P$ for all $x \in P$ and $a \geqslant 0$; (ii) $x \in X$ and $-x \in X$ imply $x = 0$. A map $\psi\colon P \to [0,\infty)$ is a *nonnegative continuous concave* (*convex*) *functional map* provided ψ is nonnegative, continuous and satisfies

$$\psi(tx + (1-t)y) \geqslant (\leqslant)\, t\psi(x) + (1-t)\psi(y) \quad \text{for all } x, y \in P,\ t \in [0,1].$$

An operator $T\colon X \to X$ is *completely continuous* if it is continuous and maps bounded sets into relatively compact sets.

Let ψ be a nonnegative functional on a cone P of a real Banach space X. We define the sets

$$P_r = \{y \in P\colon \|y\| < r\},$$
$$P(\psi; a, b) = \{y \in P\colon a \leqslant \psi(y),\ \|y\| < b\}.$$

Theorem 2.1 [8] (Leggett-Williams Fixed-Point Theorem). *Let $A < B < D < C$ be positive numbers, $T\colon \overline{P}_C \to \overline{P}_C$ a completely continuous operator, and ψ a nonnegative continuous concave functional on P such that $\psi(y) \leqslant \|y\|$ for all $y \in \overline{P}_C$. Suppose that*

(E1) $\{y \in P(\psi; B, D)\colon \psi(y) > B\} \neq \emptyset$ *and* $\psi(Ty) > B$ *for* $y \in P(\psi; B, D)$;
(E2) $\|Ty\| < A$ *for* $y \in P$ *with* $\|y\| \leqslant A$;
(E3) $\psi(Ty) > B$ *for* $y \in P(\psi; B, C)$ *with* $\|Ty\| > D$.

Then T has at least three fixed points y_1, y_2 and y_3 such that $\|y_1\| < A$, $\psi(y_2) > B$ and $\|y_3\| > A$ with $\psi(y_3) < B$.

For easy referencing, we list the conditions needed as follows:

(A1) ϱ and g satisfy

$$\int_0^1 \varphi^{-1}\Big(\frac{1}{\varrho(s)}\Big)\,\mathrm{d}s < \infty,\ \int_0^\infty \varphi^{-1}\Big(\frac{1}{\varrho(s)}\Big)\,\mathrm{d}s = \infty,$$
$$\int_0^\infty g(t)\int_0^t \varphi^{-1}\Big(\frac{1}{\varrho(s)}\Big)\,\mathrm{d}s\,\mathrm{d}t < \infty,$$
$$\lim_{t\to\infty}\int_0^t \varphi^{-1}\Big(\frac{1}{\varrho(s)}\Big)\varphi^{-1}\Big(\int_s^\infty f(u,1)\,\mathrm{d}u\Big)\,\mathrm{d}s = \infty,$$

and there exists the limit

$$\lim_{t\to\infty}\frac{1+\tau(t)}{1+t}, \quad \text{where } \tau = \tau(t) = \int_0^t \varphi^{-1}\Big(\frac{1}{\varrho(s)}\Big)\,\mathrm{d}s.$$

(A2) $f\colon (0,\infty)\times[0,\infty)\to[0,\infty)$ is an S-Carathéodory function with $f(t,0)\not\equiv 0$ on each sub-interval of $[0,\infty)$.

(A3) There exist real numbers $\alpha<0<\beta$ and $\sigma_2>\sigma_1>0$ such that

$$f(t,cx)\geqslant c^{\alpha}f(t,x)\quad \text{for } c\geqslant\sigma_2,\ \text{sufficiently large } t \text{ and sufficiently small } x$$

and

$$f(t,c)\geqslant c^{\beta}f(t,1) \text{ for } 0<c\leqslant\sigma_1 \text{ and sufficiently large } t.$$

Choose k (>1) large enough such that

$$\int_0^{1/k}\varphi^{-1}\Big(\frac{1}{\varrho(s)}\Big)\,\mathrm{d}s=\tau\Big(\frac{1}{k}\Big)<1.$$

Let

$$\mu=\frac{1}{1+k}\int_0^{1/k}\varphi^{-1}\Big(\frac{1}{\varrho(s)}\Big)\,\mathrm{d}s\bigg[\inf_{t\in[0,\infty)}\frac{1+t}{1+\int_0^t\varphi^{-1}\big(1/\varrho(s)\big)\,\mathrm{d}s}\bigg].$$

Noting that $\sup\limits_{t\in[0,\infty)}\frac{1+\tau(t)}{1+t}<\infty$, it is clear that

$$0<\mu\leqslant\frac{1}{1+k}\int_0^{1/k}\varphi^{-1}\Big(\frac{1}{\varrho(s)}\Big)\,\mathrm{d}s\bigg[\frac{1+1/k}{1+\int_0^{1/k}\varphi^{-1}\big(1/\varrho(s)\big)\,\mathrm{d}s}\bigg]<\frac{1}{k}<1.$$

Let the Banach space

(2.1) $$X=\Big\{x\in C^0[0,\infty)\colon \text{there exists the limit }\lim_{t\to\infty}\frac{x(t)}{1+t}\Big\}$$

be equipped with the norm

(2.2) $$\|x\|=\sup_{t\in[0,\infty)}\frac{|x(t)|}{1+t}\quad\text{for } x\in X.$$

Define the cone P in X by

(2.3) $$P=\left\{x\in X\colon \begin{array}{l} x(t)\geqslant 0 \text{ on } [0,\infty)\\ x(t)\text{ is non-decreasing on } [0,\infty),\\ \min\limits_{t\in[1/k,k]}\dfrac{x(t)}{1+t}\geqslant\mu\sup\limits_{t\in[0,\infty)}\dfrac{x(t)}{1+t}\end{array}\right\}.$$

Define the functional $\psi\colon P\to\mathbb{R}$ by

(2.4) $$\psi(y)=\min_{t\in[1/k,k]}\frac{y(t)}{1+t},\quad y\in P.$$

It is easy to see that ψ is a nonnegative continuous concave functional on P such that $\psi(y) \leqslant \|y\|$ for all $y \in P$.

Now, to study (1.1), for $x \in X$ we consider the boundary value problem

$$\begin{cases} [\varrho(t)\varphi(y'(t))]' + f(t, x(t)) = 0, \quad t \in (0, \infty), \\ y(0) = \int_0^\infty g(s) y(s) \, \mathrm{d}s + a, \\ \lim_{t\to\infty} \varphi^{-1}(\varrho(t)) y'(t) = b. \end{cases} \tag{2.5}$$

Lemma 2.1. *Suppose that* (A1) *and* (A2) *hold and y is a solution of* (2.5) *for $x \in X$. Then y can be expressed as*

$$y(t) = \frac{1}{1 - \int_0^\infty g(s)\,\mathrm{d}s} \int_0^\infty g(t) \int_0^t \varphi^{-1}\Big(\frac{1}{\varrho(s)}\varphi(b) + \frac{1}{\varrho(s)} \int_s^\infty f(u, x(u))\,\mathrm{d}u\Big)\,\mathrm{d}s\,\mathrm{d}t \\ + \int_0^t \varphi^{-1}\Big(\frac{1}{\varrho(s)}\varphi(b) + \frac{1}{\varrho(s)} \int_s^\infty f(u, x(u))\,\mathrm{d}u\Big)\,\mathrm{d}s + \frac{a}{1 - \int_0^\infty g(s)\,\mathrm{d}s}.$$

P r o o f. Since $x \in X$ and f is an S-Carathéodory function, we get

$$\int_0^\infty f(s, x(s))\,\mathrm{d}s < \infty.$$

Because y is a solution of BVP (2.5), we get

$$y'(t) = \varphi^{-1}\Big(\frac{1}{\varrho(t)}\varphi(b) + \frac{1}{\varrho(t)} \int_t^\infty f(u, x(u))\,\mathrm{d}u\Big), \quad t \geqslant 0.$$

Integrating gives

$$y(t) = y(0) + \int_0^t \varphi^{-1}\Big(\frac{1}{\varrho(s)}\varphi(b) + \frac{1}{\varrho(s)} \int_s^\infty f(u, x(u))\,\mathrm{d}u\Big)\,\mathrm{d}s, \quad t \geqslant 0. \tag{2.6}$$

The boundary conditions in (2.5) imply that

$$y(0) = y(0) \int_0^\infty g(s)\,\mathrm{d}s + \int_0^\infty g(t) \int_0^t \varphi^{-1}\Big(\frac{1}{\varrho(s)}\varphi(b) + \frac{1}{\varrho(s)} \int_s^\infty f(u, x(u))\,\mathrm{d}u\Big)\,\mathrm{d}s\,\mathrm{d}t + a.$$

It follows that

$$y(0) = \frac{\int_0^\infty g(t) \int_0^t \varphi^{-1}\big(\varrho(s)^{-1}\varphi(b) + \varrho(s)^{-1} \int_s^\infty f(u, x(u))\,\mathrm{d}u\big)\,\mathrm{d}s\,\mathrm{d}t + a}{1 - \int_0^\infty g(s)\,\mathrm{d}s}. \tag{2.7}$$

Substituting (2.7) into (2.6) completes the proof. □

Lemma 2.2. *Suppose that* (A1) *and* (A2) *hold and* y *is a solution of* (2.5) *for* $x \in X$. *Then* $y'(t) \geqslant 0$ *and* $y(t) \geqslant 0$ *for all* $t \in [0, \infty)$, *and* $y(t)$ *is concave with respect to* τ *on* $[0, \infty)$, *where*

$$\tau = \int_0^t \varphi^{-1}\Big(\frac{1}{\varrho(s)}\Big)\,\mathrm{d}s.$$

Proof. First, we shall prove that y' is positive on $[0, \infty)$. Since y is a solution of (2.5), (A2) implies that $[\varrho(t)\varphi(y'(t))]' \leqslant 0$ for all $t \in [0, \infty)$. Then

$$\varphi(b) - \varrho(t)\varphi(y'(t)) \leqslant 0, \quad t \in [0, \infty).$$

Since $b \geqslant 0$, we have $\varrho(t)\varphi(y'(t)) \geqslant 0$. Thus $y'(t) \geqslant 0$ for all $t \in [0, \infty)$.

Next, we shall prove that $y(t) \geqslant 0$ for $t \in [0, \infty)$. Since $y'(t) \geqslant 0$ for all $t \in [0, \infty)$, it suffices to show that $y(0) \geqslant 0$. The boundary conditions in (2.5) imply that

$$y(0) = \int_0^\infty g(s)y(s)\,\mathrm{d}s + a \geqslant y(0)\int_0^\infty g(s)\,\mathrm{d}s.$$

Since $\int_0^\infty g(s)\,\mathrm{d}s < 1$, we get $y(0) \geqslant 0$. Hence, $y(t) \geqslant 0$ for $t \in [0, \infty)$.

Finally, we shall prove that y is concave with respect to τ on $[0, \infty)$. From (A1) we have $\int_0^\infty \varphi^{-1}\big(1/\varrho(s)\big)\,\mathrm{d}s = \infty$. So $\tau \in C([0, \infty), [0, \infty))$ and

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} = \varphi^{-1}\Big(\frac{1}{\varrho(t)}\Big) > 0.$$

Thus

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}\tau}\frac{\mathrm{d}\tau}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}\tau}\varphi^{-1}\Big(\frac{1}{\varrho(t)}\Big). \tag{2.8}$$

It follows that

$$\frac{\mathrm{d}y}{\mathrm{d}\tau} = \frac{\mathrm{d}y}{\mathrm{d}t}\frac{1}{\varphi^{-1}\big(1/\varrho(t)\big)} \geqslant 0.$$

Moreover, since

$$\varrho(t)\varphi\Big(\frac{\mathrm{d}y}{\mathrm{d}t}\Big) = \varphi\Big(\frac{\mathrm{d}y}{\mathrm{d}\tau}\Big),$$

we get

$$\Big[\varrho(t)\varphi\Big(\frac{\mathrm{d}y}{\mathrm{d}t}\Big)\Big]' = \varphi'\Big(\frac{\mathrm{d}y}{\mathrm{d}\tau}\Big)\frac{\mathrm{d}^2y}{\mathrm{d}\tau^2}\frac{\mathrm{d}\tau}{\mathrm{d}t}.$$

So

$$\frac{\mathrm{d}^2y}{\mathrm{d}\tau^2} = \frac{\big[\varrho(t)\varphi\big(\mathrm{d}y/\mathrm{d}t\big)\big]'}{\varphi'(\mathrm{d}y/\mathrm{d}\tau)\mathrm{d}\tau/\mathrm{d}t}.$$

Since $[\varrho(t)\varphi(y'(t))]' \leqslant 0$, $\varphi'(y) > 0$ $(y > 0)$ and $\mathrm{d}\tau/\mathrm{d}t > 0$, we obtain $\mathrm{d}^2y/\mathrm{d}\tau^2 \leqslant 0$. Hence, $y(t)$ is concave with respect to τ on $[0, \infty)$. The proof is complete. □

Define the nonlinear operator $T\colon\ P\to X$ by

(2.9)

$$(Tx)(t)=\frac{1}{1-\int_0^\infty g(s)\,\mathrm{d}s}\int_0^\infty g(t)\int_0^t\varphi^{-1}\Big(\frac{1}{\varrho(s)}\varphi(b)+\frac{1}{\varrho(s)}\int_s^\infty f(u,x(u))\,\mathrm{d}u\Big)\,\mathrm{d}s\,\mathrm{d}t$$
$$+\int_0^t\varphi^{-1}\Big(\frac{1}{\varrho(s)}\varphi(b)+\frac{1}{\varrho(s)}\int_s^\infty f(u,x(u))\,\mathrm{d}u\Big)\,\mathrm{d}s+\frac{a}{1-\int_0^\infty g(s)\,\mathrm{d}s}.$$

Lemma 2.3. *Suppose that* (A1) *and* (A2) *hold. Then the following assertions hold:*

(i) *For $x\in P$, Tx satisfies*

$$\begin{cases}[\varrho(t)\varphi((Tx)'(t))]'+f(t,x(t))=0,\ t\in(0,\infty),\\ (Tx)(0)=\int_0^\infty g(s)(Tx)(s)\,\mathrm{d}s+a,\\ \lim\limits_{t\to\infty}\varphi^{-1}(\varrho(t))(Tx)'(t)=b;\end{cases}\tag{2.10}$$

(ii) *$Tx\in P$ for each $x\in P$;*

(iii) *x is a positive solution of* BVP (1.1) *if and only if x is a solution of the operator equation $x=Tx$ in P.*

Proof. The proofs of (i) and (iii) follow from the definition of T and are omitted.

To show (ii), we note from (i) that Tx is a solution of (2.5). Then, Lemma 2.2 implies that $(Tx)(t)\geqslant 0$ and $(Tx)'(t)\geqslant 0$ for all $t\in[0,\infty)$, and $(Tx)(t)$ is concave with respect to $\tau=\int_0^t\varphi^{-1}\big(1/\varrho(s)\big)\,\mathrm{d}s$. To complete the proof of $TP\subseteq P$, it suffices to prove that for $x\in P$ we have $Tx\in X$ and

$$\min_{t\in[1/k,k]}\frac{(Tx)(t)}{1+t}\geqslant\mu\sup_{t\in[0,\infty)}\frac{(Tx)(t)}{1+t}.\tag{2.11}$$

First, we shall show that $Tx\in X$ for $x\in P$. To begin, we shall prove that

$$\lim_{t\to\infty}\frac{(Tx)(t)}{1+\tau(t)}=b.\tag{2.12}$$

Note from (2.10) that $\lim\limits_{t\to\infty}\varphi^{-1}(\varrho(t))(Tx)'(t)=b$. We consider two cases: $b=0$ and $b\neq 0$.

Suppose that $b=0$. Then, for any $\varepsilon>0$, there exists $H>0$ such that

$$|\varphi^{-1}(\varrho(t))(Tx)'(t)|<\frac{\varepsilon}{2},\quad t\geqslant H.$$

It follows that

$$\begin{aligned}\frac{|(Tx)(t)|}{1+\tau(t)} &\leqslant \frac{|(Tx)(H)| + \int_H^t |(Tx)'(s)|\,\mathrm{d}s}{1+\tau(t)}\\ &\leqslant \frac{|(Tx)(H)|}{1+\tau(t)} + \frac{\varepsilon}{2}\frac{\int_H^t \varphi^{-1}\big(1/p(s)\big)\,\mathrm{d}s}{1+\tau(t)}\\ &\leqslant \frac{|(Tx)(H)|}{1+\tau(t)} + \frac{\varepsilon}{2}, \quad t \geqslant H.\end{aligned}$$

Since $\lim\limits_{t\to\infty} \tau(t) = \infty$, we can choose $H' > H$ large enough so that

$$\frac{|(Tx)(t)|}{1+\tau(t)} \leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \quad t \geqslant H',$$

which implies that

$$\lim_{t\to\infty} \frac{(Tx)(t)}{1+\tau(t)} = 0 = b.$$

Suppose that $b \neq 0$. Since $\lim\limits_{t\to\infty} (\varphi^{-1}(\varrho(t))(Tx)'(t) - b) = 0$, it follows that

$$\lim_{t\to\infty} \varphi^{-1}(\varrho(t))\Big[(Tx)(t) - b\int_0^t \varphi^{-1}\Big(\frac{1}{\varrho(s)}\Big)\,\mathrm{d}s\Big]' = 0.$$

By a similar argument as above, we get

$$\lim_{t\to\infty} \frac{(Tx)(t) - b\int_0^t \varphi^{-1}\big(1/\varrho(s)\big)\,\mathrm{d}s}{1+\tau(t)} = 0.$$

It follows that

$$\lim_{t\to\infty} \frac{(Tx)(t)}{1+\tau(t)} = b.$$

Hence, (2.12) is proved.

Now, knowing from (A1) that $\sup\limits_{t\in[0,\infty)} (1+\tau(t))/(1+t) < \infty$ leads to

$$\frac{(Tx)(t)}{1+t} = \frac{1+\tau(t)}{1+t}\frac{(Tx)(t)}{1+\tau(t)} \quad \text{is bounded on } [0,\infty).$$

Thus $Tx \in X$.

Next, we shall prove (2.11). We consider two cases. First, suppose $(Tx)(t)/(1+t)$ achieves its maximum at $\sigma \in [0,\infty)$. Noting that

$$\tau(t) = \int_0^t \varphi^{-1}\Big(\frac{1}{\varrho(s)}\Big)\,\mathrm{d}s,$$

and the inverse function of $\tau = \tau(t)$ is denoted by $t = t(\tau)$, one sees that for $t \in [1/k, k]$,

$$\begin{aligned}
\frac{(Tx)(t)}{1+t} &\geqslant \frac{(Tx)(1/k)}{1+k} \\
&= \frac{(Tx)(t(\tau(1/k)))}{1+k} \\
&= \frac{(Tx)\big(t\big(\frac{1-\tau(1/k)+\tau(\sigma)}{1+\tau(\sigma)}\,\frac{\tau(1/k)}{1-\tau(1/k)+\tau(\sigma)} + \frac{\tau(1/k)}{1+\tau(\sigma)}\tau(\sigma)\big)\big)}{1+k}.
\end{aligned}$$

Noting that $\tau(1/k) < 1$ and $(Tx)(t)$ is concave with respect to τ, we find for $t \in [1/k, k]$,

$$\begin{aligned}
\frac{(Tx)(t)}{1+t} &\geqslant \frac{\frac{1-\tau(1/k)+\tau(\sigma)}{1+\tau(\sigma)}(Tx)\big(t\big(\frac{\tau(1/k)}{1-\tau(1/k)+\tau(\sigma)}\big)\big) + \frac{\tau(1/k)}{1+\tau(\sigma)}(Tx)(t(\tau(\sigma)))}{1+k} \\
&\geqslant \frac{1}{1+k}\frac{\tau(1/k)}{1+\tau(\sigma)}(Tx)(t(\tau(\sigma))) \\
&= \frac{1}{1+k}\int_0^{1/k}\varphi^{-1}\Big(\frac{1}{\varrho(s)}\Big)\,\mathrm{d}s\,\frac{1}{1+\tau(\sigma)}\;(Tx)(\sigma) \\
&= \frac{1}{1+k}\int_0^{1/k}\varphi^{-1}\Big(\frac{1}{\varrho(s)}\Big)\,\mathrm{d}s\,\frac{1+\sigma}{1+\tau(\sigma)}\;\frac{(Tx)(\sigma)}{1+\sigma} \\
&\geqslant \frac{1}{1+k}\int_0^{1/k}\varphi^{-1}\Big(\frac{1}{\varrho(s)}\Big)\,\mathrm{d}s\Big[\inf_{t\in[0,\infty)}\frac{1+t}{1+\int_0^t\varphi^{-1}\big(1/\varrho(s)\big)\,\mathrm{d}s}\Big]\frac{(Tx)(\sigma)}{1+\sigma} \\
&= \mu\sup_{t\in[0,\infty)}\frac{(Tx)(t)}{1+t}.
\end{aligned}$$

Next, suppose $(Tx)(t)/(1+t)$ achieves its supremum at ∞. Choose $\sigma' \in [0, \infty)$. Similarly to the above discussion, we get for $t \in [1/k, k]$ that

$$\frac{(Tx)(t)}{1+t} \geqslant \mu\frac{(Tx)(\sigma')}{1+\sigma'}.$$

Let $\sigma' \to \infty$, we get for $t \in [1/k, k]$ that

$$\frac{(Tx)(t)}{1+t} \geqslant \mu\sup_{t\in[0,\infty)}\frac{(Tx)(t)}{1+t}.$$

It follows that (2.11) holds. Hence $Tx \in P$. The proof is complete. □

Lemma 2.4 [18]. *Let $V = \{x \in X \colon \|x\| < l\}$ $(l > 0)$. If $\bigl\{x(t)/(1+t)\colon x \in V\bigr\}$ is equicontinuous on any compact interval of $[0,\infty)$ and equiconvergent at infinity, then V is relatively compact on X.*

Note that $\bigl\{x(t)/(1+t)\colon x \in V\bigr\}$ is said to be *equiconvergent at infinity* if and only if for all $\varepsilon > 0$ there exists $N = N(\varepsilon) > 0$ such that for all $x \in V$ we have

$$\Bigl|\frac{x(t_1)}{1+t_1} - \frac{x(t_2)}{1+t_2}\Bigr| < \varepsilon, \quad t_1, t_2 > N.$$

Lemma 2.5. *$T\colon P \to P$ is completely continuous.*

Proof. It is easy to verify that $T\colon P \to P$ is well defined. We shall prove that T is continuous and maps bounded sets into relatively compact sets.

Let $x_n \to x_0$ as $n \to \infty$ in P, then there exists r_0 such that $\sup\limits_{n\geqslant 0} \|x_n\| < r_0$. Set

$$B_{r_0}(t) = \sup_{|u|\in[0,r_0]} f(t,(1+t)u).$$

Then we have

$$\int_0^\infty |f(s,x_n(s)) - f(s,x_0(s))| \,\mathrm{d}s \leqslant 2\int_0^\infty B_{r_0}(s)\,\mathrm{d}s.$$

Therefore, by the Lebesgue dominated convergence theorem, we obtain

$$\int_t^\infty f(u,x_n(u))\,\mathrm{d}u \to \int_t^\infty f(u,x_0(u))\,\mathrm{d}u \text{ uniformly as } n \to \infty.$$

So for any $\varepsilon > 0$, there exists $N > 0$ such that

$$\Bigl|\int_t^\infty f(u,x_n(u))\,\mathrm{d}u - \int_t^\infty f(u,x_0(u))\,\mathrm{d}u\Bigr| < \varepsilon,\ n > N,\ t \in [0,\infty).$$

One sees that, for all n,

$$\varphi(b) + \int_s^\infty f(u,x_n(u))\,\mathrm{d}u \leqslant \varphi(b) + \int_0^\infty B_{r_0}(u)\,\mathrm{d}u \equiv r.$$

Since φ^{-1} is uniformly continuous on $[-r,r]$, we get that there exists $\delta > 0$ such that

$$|\varphi^{-1}(u) - \varphi^{-1}(v)| \to 0 \quad \text{as } u,v \in [-r,r] \text{ and } u \to v.$$

Then there exists $N > 0$ such that

$$\Bigl|\varphi^{-1}\Bigl(\varphi(b) + \int_s^\infty f(u,x_n(u))\,\mathrm{d}u\Bigr) - \varphi^{-1}\Bigl(\varphi(b) + \int_s^\infty f(u,x_0(u))\,\mathrm{d}u\Bigr)\Bigr| < \varepsilon \quad \text{uniformly as } n > N.$$

Thus, we get for $t \in [0,\infty)$ and $n > N$ that

$$
\begin{aligned}
0 &\leqslant \Big|\frac{[(Tx_n)-(Tx_0)](t)}{1+t}\Big| \\
&= \Big|\frac{1}{1+t}\frac{1}{1-\int_0^\infty g(s)\,\mathrm{d}s}\int_0^\infty g(t)\int_0^t \Big[\varphi^{-1}\Big(\frac{1}{\varrho(s)}\varphi(b)+\frac{1}{\varrho(s)}\int_s^\infty f(u,x_n(u))\,\mathrm{d}u\Big) \\
&\qquad - \varphi^{-1}\Big(\frac{1}{\varrho(s)}\varphi(b)+\frac{1}{\varrho(s)}\int_s^\infty f(u,x_0(u))\,\mathrm{d}u\Big)\Big]\,\mathrm{d}s\,\mathrm{d}t \\
&\quad + \frac{1}{1+t}\int_0^t \Big[\varphi^{-1}\Big(\frac{1}{\varrho(s)}\varphi(b)+\frac{1}{\varrho(s)}\int_s^\infty f(u,x_n(u))\,\mathrm{d}u\Big) \\
&\qquad - \varphi^{-1}\Big(\frac{1}{\varrho(s)}\varphi(b)+\frac{1}{\varrho(s)}\int_s^\infty f(u,x_0(u))\,\mathrm{d}u\Big)\Big]\,\mathrm{d}s\Big| \\
&\leqslant \frac{1}{1+t}\frac{1}{1-\int_0^\infty g(s)\,\mathrm{d}s}\int_0^\infty g(t)\int_0^t \Big|\varphi^{-1}\Big(\frac{1}{\varrho(s)}\varphi(b)+\frac{1}{\varrho(s)}\int_s^\infty f(u,x_n(u))\,\mathrm{d}u\Big) \\
&\qquad - \varphi^{-1}\Big(\frac{1}{\varrho(s)}\varphi(b)+\frac{1}{\varrho(s)}\int_s^\infty f(u,x_0(u))\,\mathrm{d}u\Big)\Big|\,\mathrm{d}s\,\mathrm{d}t \\
&\quad + \frac{1}{1+t}\int_0^t \Big|\varphi^{-1}\Big(\frac{1}{\varrho(s)}\varphi(b)+\frac{1}{\varrho(s)}\int_s^\infty f(u,x_n(u))\,\mathrm{d}u\Big) \\
&\qquad - \varphi^{-1}\Big(\frac{1}{\varrho(s)}\varphi(b)+\frac{1}{\varrho(s)}\int_s^\infty f(u,x_0(u))\,\mathrm{d}u\Big)\Big|\,\mathrm{d}s \\
&\leqslant \frac{1}{1+t}\frac{1}{1-\int_0^\infty g(s)\,\mathrm{d}s}\int_0^\infty g(t)\int_0^t \varphi^{-1}\Big(\frac{1}{\varrho(s)}\Big) \\
&\qquad \times \Big|\int_s^\infty f(u,x_n(u))\,\mathrm{d}u - \int_s^\infty f(u,x_0(u))\,\mathrm{d}u\Big|\,\mathrm{d}s\,\mathrm{d}t \\
&\quad + \frac{1}{1+t}\int_0^t \varphi^{-1}\Big(\frac{1}{\varrho(s)}\Big)\Big|\int_s^\infty f(u,x_n(u))\,\mathrm{d}u - \int_s^\infty f(u,x_0(u))\,\mathrm{d}u\Big|\,\mathrm{d}s \\
&\leqslant \Big[\frac{1}{1-\int_0^\infty g(s)\,\mathrm{d}s}\int_0^\infty g(t)\int_0^t \varphi^{-1}\Big(\frac{1}{\varrho(s)}\Big)\,\mathrm{d}s\,\mathrm{d}t \\
&\qquad + \varepsilon \sup_{t\in[0,\infty)}\frac{1}{1+t}\int_0^t \varphi^{-1}\Big(\frac{1}{\varrho(s)}\Big)\,\mathrm{d}s\Big]\varepsilon.
\end{aligned}
$$

It follows that

$$\|Tx_n - Tx_0\| \to 0$$

uniformly as $n \to \infty$. So T is continuous.

Let Ω be any bounded subset of P. First, we shall prove that $T\Omega$ is bounded. Since Ω is bounded, there exists $r > 0$ such that $\|x\| \leqslant r$ for all $x \in \Omega$. Denote

$$B_r(t) = \sup_{|u|\in[0,r]} f(t,(1+t)u).$$

Obviously, we have

$$
\begin{aligned}
0 &\leqslant \frac{(Tx)(t)}{1+t} \\
&= \frac{1}{1+t}\frac{1}{1-\int_0^\infty g(s)\,\mathrm{d}s}\int_0^\infty g(t)\int_0^t \varphi^{-1}\Big(\frac{1}{\varrho(s)}\varphi(b)+\frac{1}{\varrho(s)}\int_s^\infty f(u,x(u))\,\mathrm{d}u\Big)\,\mathrm{d}s\,\mathrm{d}t \\
&\quad + \frac{1}{1+t}\int_0^t \varphi^{-1}\Big(\frac{1}{\varrho(s)}\varphi(b)+\frac{1}{\varrho(s)}\int_s^\infty f(u,x(u))\,\mathrm{d}u\Big)\,\mathrm{d}s + \frac{1}{1+t}\frac{a}{1-\int_0^\infty g(s)\,\mathrm{d}s} \\
&\leqslant \frac{1}{1+t}\frac{1}{1-\int_0^\infty g(s)\,\mathrm{d}s}\int_0^\infty g(t)\int_0^t \varphi^{-1}\Big(\frac{1}{\varrho(s)}\Big)\,\mathrm{d}s\,\mathrm{d}t\,\varphi^{-1}\Big(\varphi(b)+\int_0^\infty B_r(u)\,\mathrm{d}u\Big) \\
&\quad + \frac{1}{1+t}\int_0^t \varphi^{-1}\Big(\frac{1}{\varrho(s)}\Big)\,\mathrm{d}s\,\varphi^{-1}\Big(\varphi(b)+\int_0^\infty B_r(u)\,\mathrm{d}u\Big) + \frac{a}{1-\int_0^\infty g(s)\,\mathrm{d}s} \\
&\leqslant \frac{1}{1-\int_0^\infty g(s)\,\mathrm{d}s}\int_0^\infty g(t)\int_0^t \varphi^{-1}\Big(\frac{1}{\varrho(s)}\Big)\,\mathrm{d}s\,\mathrm{d}t\,\varphi^{-1}\Big(\varphi(b)+\int_0^\infty B_r(u)\,\mathrm{d}u\Big) \\
&\quad + \sup_{t\in[0,\infty)}\frac{\int_0^\infty \varphi^{-1}\big(1/\varrho(s)\big)\,\mathrm{d}s}{1+t}\varphi^{-1}\Big(\varphi(b)+\int_0^\infty B_r(u)\,\mathrm{d}u\Big) + \frac{a}{1-\int_0^\infty g(s)\,\mathrm{d}s} \\
&< \infty.
\end{aligned}
$$

So $T\Omega$ is bounded.

Next, for any $N \in (0,\infty)$ and $t_1, t_2 \in [0,N]$, one has

$$
\begin{aligned}
&\Big|\frac{(Tx)(t_1)}{1+t_1}-\frac{(Tx)(t_2)}{1+t_2}\Big| \leqslant \Big|\frac{(Tx)(t_2)-(Tx)(t_1)}{1+t_1}\Big| + \Big|\frac{1}{1+t_1}-\frac{1}{1+t_2}\Big||(Tx)(t_2)| \\
&\leqslant \frac{1}{1+t_1}\Big|\int_{t_1}^{t_2}(Tx)'(s)\,\mathrm{d}s\Big| + \Big|\frac{1}{1+t_1}-\frac{1}{1+t_2}\Big| \\
&\quad\times\Big|\frac{1}{1-\int_0^\infty g(s)\,\mathrm{d}s}\int_0^\infty g(t)\int_0^t \varphi^{-1}\Big(\Big(\frac{1}{\varrho(s)}\Big)\Big(\varphi(b)+\int_s^\infty f(u,x(u))\,\mathrm{d}u\Big)\Big)\,\mathrm{d}s\,\mathrm{d}t \\
&\qquad + \int_0^{t_2}\varphi^{-1}\Big(\frac{1}{\varrho(s)}\Big)\varphi^{-1}\Big(\varphi(b)+\int_s^\infty f(u,x(u))\,\mathrm{d}u\Big)\,\mathrm{d}s\Big| \\
&\quad + \Big|\frac{1}{1+t_1}-\frac{1}{1+t_2}\Big|\frac{a}{1-\int_0^\infty g(s)\,\mathrm{d}s} \\
&\leqslant \frac{1}{1+t_1}\Big|\int_{t_1}^{t_2}\varphi^{-1}\Big(\frac{1}{\varrho(s)}\Big)\varphi^{-1}\Big(\varphi(b)+\int_s^\infty f(u,x(u))\,\mathrm{d}u\Big)\,\mathrm{d}s\Big| \\
&\quad + \frac{\big|\frac{1}{1+t_1}-\frac{1}{1+t_2}\big|}{1-\int_0^\infty g(s)\,\mathrm{d}s}\int_0^\infty g(t)\int_0^t \varphi^{-1}\Big(\Big(\frac{1}{\varrho(s)}\Big)\Big(\varphi(b)+\int_s^\infty f(u,x(u))\,\mathrm{d}u\Big)\Big)\,\mathrm{d}s\,\mathrm{d}t \\
&\quad + \Big|\frac{1}{1+t_1}-\frac{1}{1+t_2}\Big|\Big|\int_0^{t_2}\varphi^{-1}\Big(\frac{1}{\varrho(s)}\Big)\varphi^{-1}\Big(\varphi(b)+\int_s^\infty f(u,x(u))\,\mathrm{d}u\Big)\,\mathrm{d}s\Big| \\
&\quad + \Big|\frac{1}{1+t_1}-\frac{1}{1+t_2}\Big|\frac{a}{1-\int_0^\infty g(s)\,\mathrm{d}s}
\end{aligned}
$$

$$\leqslant \frac{1}{1+t_1}\Big|\int_{t_1}^{t_2}\varphi^{-1}\Big(\frac{1}{\varrho(s)}\Big)\,\mathrm{d}s\Big|\varphi^{-1}\Big(\varphi(b)+\int_0^\infty B_r(u)\,\mathrm{d}u\Big)$$
$$+\frac{\big|\frac{1}{1+t_1}-\frac{1}{1+t_2}\big|}{1-\int_0^\infty g(s)\,\mathrm{d}s}\int_0^\infty g(t)\int_0^t\varphi^{-1}\Big(\frac{1}{\varrho(s)}\Big)\,\mathrm{d}s\,\mathrm{d}t\,\varphi^{-1}\Big(\varphi(b)+\int_0^\infty B_r(u)\,\mathrm{d}u\Big)$$
$$+\Big|\frac{1}{1+t_1}-\frac{1}{1+t_2}\Big|\int_0^N\varphi^{-1}\Big(\frac{1}{\varrho(s)}\Big)\,\mathrm{d}s\,\varphi^{-1}\Big(\varphi(b)+\int_0^\infty B_r(u)\,\mathrm{d}u\Big)$$
$$+\Big|\frac{1}{1+t_1}-\frac{1}{1+t_2}\Big|\frac{a}{1-\int_0^\infty g(s)\,\mathrm{d}s}\to 0\quad\text{uniformly as } t_1\to t_2$$

for all $x\in\Omega$. So $\{(Tx)(t)/(1+t)\colon\ x\in\Omega\}$ is equicontinuous on any compact interval of $[0,\infty)$.

Finally, we shall prove that $\{(Tx)(t)/(1+t)\colon\ x\in\Omega\}$ is equiconvergent at infinity. One sees that for any $\varepsilon>0$ there exists $N_{1,\varepsilon}>0$ such that

$$\Big|\int_{t_1}^\infty f(u,x(u))\,\mathrm{d}u-\int_{t_2}^\infty f(u,x(u))\,\mathrm{d}u\Big|<\varepsilon,\quad t_1,t_2>N_{1,\varepsilon}.$$

Since

$$0\leqslant\varphi(b)+\int_{t_1}^\infty f(u,x(u))\,\mathrm{d}u\leqslant\varphi(b)+\int_0^\infty B_r(s)\,\mathrm{d}s\equiv r,$$

we have by the assumption on φ that

$$\Big|\varphi^{-1}\Big(\varphi(b)+\int_{t_1}^\infty f(u,x(u))\,\mathrm{d}u\Big)-\varphi^{-1}\Big(\varphi(b)+\int_{t_2}^\infty f(u,x(u))\,\mathrm{d}u\Big)\Big|<\varepsilon,\ t_1,t_2>N_{1,\varepsilon}.$$

It follows that

$$\varphi^{-1}\Big(\varphi(b)+\int_t^\infty f(u,x(u))\,\mathrm{d}u\Big)\to c\quad\text{uniformly as } t\to\infty,\tag{2.13}$$

where c is a constant. We claim that

$$\frac{\int_0^t\varphi^{-1}\big(\varrho(s)^{-1}\varphi(b)+p(s)^{-1}\int_s^\infty f(u,x(u))\,\mathrm{d}u\big)\,\mathrm{d}s}{1+\tau(t)}\to c\text{ uniformly as } t\to\infty.\tag{2.14}$$

In fact, for any $\eta>0$, from (2.13) there exists $M>0$ such that

$$\Big|\varphi^{-1}\Big(\varphi(b)+\int_t^\infty f(u,x(u))\,\mathrm{d}u\Big)-c\Big|<\eta,\quad t>M,\ x\in\Omega.$$

Therefore,

$$\begin{aligned}
&\left|\frac{\int_0^t \varphi^{-1}\big(\varrho(s)^{-1}\varphi(b)+\varrho(s)^{-1}\int_s^\infty f(u,x(u))\,\mathrm{d}u\big)ds}{1+\tau(t)}-c\right| \\
&\quad=\left|\frac{\int_0^t \varphi^{-1}\big(1/\varrho(s)\big)\big[\varphi^{-1}\big(\varphi(b)+\int_s^\infty f(u,x(u))\,\mathrm{d}u\big)-c\big]\,\mathrm{d}s-c}{1+\tau(t)}\right| \\
&\quad\leqslant\frac{\eta\int_0^t \varphi^{-1}\big(1/\varrho(s)\big)\,\mathrm{d}s+c}{1+\tau(t)} \\
&\quad<\eta+\frac{c}{1+\tau(t)},\quad t>M,\ x\in\Omega.
\end{aligned}$$

Since $\lim\limits_{t\to\infty}\tau(t)=\infty$, there exists $M_1>M$ such that

$$\left|\frac{\int_0^t \varphi^{-1}\big(\varrho(s)^{-1}\varphi(b)+\varrho(s)^{-1}\int_s^\infty f(u,x(u))\,\mathrm{d}u\big)\,\mathrm{d}s}{1+\tau(t)}-c\right|<2\eta,\quad t>M_1,\ x\in\Omega.$$

So (2.14) holds.

Now, since $\sup\limits_{t\in[0,\infty)}(1+\tau(t))/(1+t)<\infty$, we get

$$\frac{1+\tau(t)}{1+t}\,\frac{\int_0^t \varphi^{-1}\big(\varrho(s)^{-1}\varphi(b)+\varrho(s)^{-1}\int_s^\infty f(u,x(u))\,\mathrm{d}u\big)\,\mathrm{d}s}{1+\tau(t)}\to c' \text{ uniformly as } t\to\infty,$$

where c' is a constant. It follows that

$$\frac{\int_0^t \varphi^{-1}\big(\varrho(s)^{-1}\varphi(b)+\varrho(s)^{-1}\int_s^\infty f(u,x(u))\,\mathrm{d}u\big)\,\mathrm{d}s}{1+t}\to c' \text{ uniformly as } t\to\infty. \tag{2.15}$$

Note that

$$\begin{aligned}
&\frac{(Tx)(t_1)}{1+t_1}-\frac{(Tx)(t_2)}{1+t_2} \\
&=\frac{1}{1+t_1}\bigg[\frac{1}{1-\int_0^\infty g(s)\,\mathrm{d}s}\int_0^\infty g(t)\int_0^t\varphi^{-1}\bigg(\frac{1}{\varrho(s)}\varphi(b)+\frac{1}{\varrho(s)}\int_s^\infty f(u,x(u))\,\mathrm{d}u\bigg)\,\mathrm{d}s\,\mathrm{d}t \\
&\quad+\int_0^{t_1}\varphi^{-1}\bigg(\frac{1}{\varrho(s)}\varphi(b)+\frac{1}{\varrho(s)}\int_s^\infty f(u,x(u))\,\mathrm{d}u\bigg)\,\mathrm{d}s+\frac{a}{1-\int_0^\infty g(s)\,\mathrm{d}s}\bigg] \\
&\quad-\frac{1}{1+t_2}\bigg[\frac{1}{1-\int_0^\infty g(s)\,\mathrm{d}s}\int_0^\infty g(t)\int_0^t\varphi^{-1}\bigg(\frac{1}{\varrho(s)}\varphi(b)+\frac{1}{\varrho(s)}\int_s^\infty f(u,x(u))\,\mathrm{d}u\bigg)\,\mathrm{d}s\,\mathrm{d}t \\
&\quad+\int_0^{t_2}\varphi^{-1}\bigg(\frac{1}{\varrho(s)}\varphi(b)+\frac{1}{\varrho(s)}\int_s^\infty f(u,x(u))\,\mathrm{d}u\bigg)\,\mathrm{d}s+\frac{a}{1-\int_0^\infty g(s)\,\mathrm{d}s}\bigg]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{1+t_1}\frac{1}{1-\int_0^\infty g(s)\,\mathrm{d}s}\int_0^\infty g(t)\int_0^t \varphi^{-1}\Big(\frac{1}{\varrho(s)}\varphi(b)+\frac{1}{\varrho(s)}\int_s^\infty f(u,x(u))\,\mathrm{d}u\Big)\,\mathrm{d}s\,\mathrm{d}t\\
&\quad - \frac{1}{1+t_2}\frac{1}{1-\int_0^\infty g(s)\,\mathrm{d}s}\int_0^\infty g(t)\int_0^t \varphi^{-1}\Big(\frac{1}{\varrho(s)}\varphi(b)+\frac{1}{\varrho(s)}\int_s^\infty f(u,x(u))\,\mathrm{d}u\Big)\,\mathrm{d}s\,\mathrm{d}t\\
&\quad + \frac{1}{1+t_1}\frac{a}{1-\int_0^\infty g(s)\,\mathrm{d}s} - \frac{1}{1+t_2}\frac{a}{1-\int_0^\infty g(s)\,\mathrm{d}s}\\
&\quad + \frac{1}{1+t_1}\int_0^{t_1}\varphi^{-1}\Big(\frac{1}{\varrho(s)}\varphi(b)+\frac{1}{\varrho(s)}\int_s^\infty f(u,x(u))\,\mathrm{d}u\Big)\,\mathrm{d}s\\
&\quad - \frac{1}{1+t_2}\int_0^{t_2}\varphi^{-1}\Big(\frac{1}{\varrho(s)}\varphi(b)+\frac{1}{\varrho(s)}\int_s^\infty f(u,x(u))\,\mathrm{d}u\Big)\,\mathrm{d}s.
\end{aligned}
$$

In view of (2.15), it is easy to see that there exists $N_\varepsilon > 0$ such that

$$
\Big|\frac{(Tx)(t_1)}{1+t_1} - \frac{(Tx)(t_2)}{1+t_2}\Big| < \varepsilon, \quad t_1, t_2 > N_\varepsilon,\ x \in \Omega.
$$

So $\big\{(Tx)(t)/(1+t)\colon\ x \in \Omega\big\}$ is equiconvergent at infinity. By Lemma 2.4 we obtain that $\big\{(Tx)(t)/(1+t)\colon\ x \in \Omega\big\}$ is pre-compact. Hence, $T\colon\ P \to P$ is completely continuous. □

3. Unbounded solutions of BVP (1.1)

In this section we shall establish the existence of at least three unbounded positive solutions of BVP (1.1).

Choose $k > 1$ sufficiently large such that $\tau(1/k) < 1$. For positive numbers e_1, e_2, and C, let $P_C = \{x \in P\colon\ \|x\| < C\}$ and M, M_1, L be defined by

(3.1)
$$
M = C\Big[\varphi\Big(\frac{\big(1-\int_0^\infty g(s)\,\mathrm{d}s\big)C - a}{\int_0^\infty g(t)\int_0^t \varphi^{-1}\big(\frac{1}{\varrho(s)}\big)\,\mathrm{d}s\,\mathrm{d}t + \big(1-\int_0^\infty g(s)\,\mathrm{d}s\big)\sup\limits_{t\in[0,\infty)}\frac{1+\tau(t)}{1+t}}\Big) - \varphi(b)\Big]^{-1},
$$

(3.2)
$$
M_1 = e_1\Big[\varphi\Big(\frac{\big(1-\int_0^\infty g(s)\,\mathrm{d}s\big)e_1 - a}{\int_0^\infty g(t)\int_0^t \varphi^{-1}\big(\frac{1}{\varrho(s)}\big)\,\mathrm{d}s\,\mathrm{d}t + \big(1-\int_0^\infty g(s)\,\mathrm{d}s\big)\sup\limits_{t\in[0,\infty)}\frac{1+\tau(t)}{1+t}}\Big) - \varphi(b)\Big]^{-1},
$$

and

$$
L = \mu(k-1)e_2\Big[\varphi\Big(\frac{\mu(1+k)^2 e_2\big(1-\int_0^\infty g(s)\,\mathrm{d}s\big) - a}{\big(1-\int_0^\infty g(s)\,\mathrm{d}s\big)\int_0^{1/k}\varphi^{-1}\big(1/\varrho(s)\big)\,\mathrm{d}s}\Big) - \varphi(b)\Big]^{-1}. \tag{3.3}
$$

Theorem 3.1. *Suppose that* (A1), (A2) *and* (A3) *hold and there exist constants* e_1, e_2 *and* C *such that*

$$0 < e_1 < \mu(1+k)e_2 < (1+k)e_2 < C, \quad LC > M\mu(1+k)e_2 > 0$$

and

(C1) $f(t,(1+t)x) \leqslant C/M(1+t)^2$ *for* $t \in [0,\infty)$ *and* $x \in [0,C]$;
(C2) $f(t,(1+t)x) \leqslant e_1/M_1(1+t)^2$ *for* $t \in [0,\infty)$ *and* $x \in [0,e_1]$;
(C3) $f(t,(1+t)x) \geqslant \mu(1+k)e_2/L(1+t)^2$ *for* $t \in [1/k,k]$ *and* $x \in [\mu(1+k)e_2, (1+k)e_2]$.

Then, BVP (1.1) *has at least three unbounded positive solutions* x_1, x_2 *and* x_3 *satisfying*

$$\sup_{t\in[0,\infty)} \frac{x_1(t)}{1+t} < e_1, \qquad \min_{t\in[1/k,k]} \frac{x_2(t)}{1+t} > \mu(1+k)e_2$$

and

$$\sup_{t\in[0,\infty)} \frac{x_3(t)}{1+t} > e_1, \qquad \min_{t\in[1/k,k]} \frac{x_3(t)}{1+t} < \mu(1+k)e_2.$$

Proof. We will apply Theorem 2.1 with T, P and ψ defined in (2.9), (2.3) and (2.4), respectively. To recap, a fixed point of T is a solution of (1.1) (Lemma 2.3), $T\colon P \to P$ is completely continuous (Lemma 2.5), and ψ is a nonnegative continuous concave functional on the cone P with $\psi(y) \leqslant \|y\|$ for all $y \in P$. Further, corresponding to Theorem 2.1, we choose

$$D = (1+k)e_2, \quad B = \mu(1+k)e_2, \quad A = e_1.$$

Then $0 < A < B < D < C$. We divide the remainder of the proof into four steps.

Step 1. We shall prove that $T(\overline{P_C}) \subset \overline{P_C}$. Let $x \in \overline{P_C}$, then $\|x\| \leqslant C$, so

$$0 \leqslant \frac{x(t)}{1+t} \leqslant C, \quad t \in [0,\infty).$$

It follows from (C1) that

$$f(t,x(t)) = f\Big(t,(1+t)\frac{x(t)}{1+t}\Big) \leqslant \frac{C}{M(1+t)^2}, \quad t \in [0,\infty).$$

Then,

$$
\begin{aligned}
\|Tx\| &= \sup_{t\in[0,\infty)} \frac{(Tx)(t)}{1+t}\\
&= \sup_{t\in[0,\infty)} \biggl[\frac{1}{1+t}\frac{1}{1-\int_0^\infty g(s)\,\mathrm{d}s}\\
&\qquad\qquad \times \int_0^\infty g(t)\int_0^t \varphi^{-1}\biggl(\frac{1}{\varrho(s)}\varphi(b)+\frac{1}{\varrho(s)}\int_s^\infty f(u,x(u))\,\mathrm{d}u\biggr)\,\mathrm{d}s\,\mathrm{d}t\\
&\quad + \frac{1}{1+t}\int_0^t \varphi^{-1}\biggl(\frac{1}{\varrho(s)}\varphi(b)+\frac{1}{\varrho(s)}\int_s^\infty f(u,x(u))\,\mathrm{d}u\biggr)\,\mathrm{d}s + \frac{1}{1+t}\frac{a}{1-\int_0^\infty g(s)\,\mathrm{d}s}\biggr]\\
&\leqslant \frac{1}{1-\int_0^\infty g(s)\,\mathrm{d}s}\int_0^\infty g(t)\int_0^t \varphi^{-1}\Bigl(\frac{1}{\varrho(s)}\Bigr)\,\mathrm{d}s\,\mathrm{d}t\,\varphi^{-1}\biggl(\varphi(b)+\int_0^\infty f(u,x(u))\,\mathrm{d}u\biggr)\\
&\quad + \sup_{t\in[0,\infty)} \frac{1}{1+t}\int_0^t \varphi^{-1}\Bigl(\frac{1}{\varrho(s)}\Bigr)\,\mathrm{d}s\,\varphi^{-1}\biggl(\varphi(b)+\int_0^\infty f(u,x(u))\,\mathrm{d}u\biggr)\\
&\quad + \frac{a}{1-\int_0^\infty g(s)\,\mathrm{d}s}\\
&\leqslant \frac{1}{1-\int_0^\infty g(s)\,\mathrm{d}s}\int_0^\infty g(t)\int_0^t \varphi^{-1}\Bigl(\frac{1}{\varrho(s)}\Bigr)\,\mathrm{d}s\,\mathrm{d}t\,\varphi^{-1}\biggl(\varphi(b)+\int_0^\infty \frac{C}{M(1+u)^2}\,\mathrm{d}u\biggr)\\
&\quad + \sup_{t\in[0,\infty)} \frac{1+\tau(t)}{1+t}\varphi^{-1}\biggl(\varphi(b)+\int_0^\infty \frac{C}{M(1+u)^2}\,\mathrm{d}u\biggr) + \frac{a}{1-\int_0^\infty g(s)\,\mathrm{d}s}\\
&= \frac{1}{1-\int_0^\infty g(s)\,\mathrm{d}s}\int_0^\infty g(t)\int_0^t \varphi^{-1}\Bigl(\frac{1}{\varrho(s)}\Bigr)\,\mathrm{d}s\,\mathrm{d}t\,\varphi^{-1}\Bigl(\varphi(b)+\frac{C}{M}\Bigr)\\
&\quad + \sup_{t\in[0,\infty)} \frac{1+\tau(t)}{1+t}\varphi^{-1}\Bigl(\varphi(b)+\frac{C}{M}\Bigr) + \frac{a}{1-\int_0^\infty g(s)\,\mathrm{d}s}\\
&= C,
\end{aligned}
$$

where the last equality follows from the definition of M in (3.1). Hence, $Tx \in \overline{P_C}$. This shows that $T(\overline{P_C}) \subset \overline{P_C}$.

Step 2. We shall show that (E1) of Theorem 2.1 holds, i.e.,

$$
\begin{aligned}
&\{y \in P(\psi; B, D)\colon\ \psi(y) > B\}\\
&\qquad = \{y \in P(\psi; \mu(1+k)e_2, (1+k)e_2)\colon\ \psi(y) > \mu(1+k)e_2\} \neq \emptyset
\end{aligned}
$$

and $\psi(Ty) > B = \mu(1+k)e_2$ for $y \in P(\psi; \mu(1+k)e_2, (1+k)e_2)$.

To prove that $\{y \in P(\psi; \mu(1+k)e_2, (1+k)e_2)\colon\ \psi(y) > \mu(1+k)e_2\} \neq \emptyset$, we choose $\lambda > 0$ and let

$$
y_0(t) = \begin{cases} \lambda - k^2\lambda\bigl(t-1/k\bigr)^2, & t \in \bigl[0, 1/k\bigr],\\ \lambda, & t \geqslant 1/k. \end{cases}
$$

It is easy to see that

$$\min_{t\in[1/k,k]}\frac{y_0(t)}{1+t}=\frac{\lambda}{1+k}$$

and

$$\sup_{t\in[0,\infty)}\frac{y_0(t)}{1+t}\leqslant\frac{k\lambda}{1+k}.$$

Since $\mu k<1$, we get $\min\limits_{t\in[1/k,k]} y_0(t)/(1+t)\geqslant\mu\sup\limits_{t\in[0,\infty)} y_0(t)/(1+t)$. It is easy to see that $y_0\in\{y\in P(\psi;B,D)\colon\ \psi(y)>B\}$ if $\lambda\in\big(\mu(1+k)^2e_2,((1+k)^2/k)e_2\big)$.

Next, let $y\in P(\psi;\mu(1+k)e_2,(1+k)e_2)$, then $\psi(y)\geqslant\mu(1+k)e_2$ and $\|y\|\leqslant(1+k)e_2$. So

$$\min_{t\in[1/k,k]}\frac{y(t)}{1+t}\geqslant\mu(1+k)e_2,\qquad\sup_{t\in[0,\infty)}\frac{y(t)}{1+t}\leqslant(1+k)e_2.$$

Hence,

$$\mu(1+k)e_2\leqslant\frac{y(t)}{1+t}\leqslant(1+k)e_2,\quad t\in[1/k,k].$$

It follows from (C3) that

$$f(t,y(t))=f\Big(t,(1+t)\frac{y(t)}{1+t}\Big)\geqslant\frac{\mu(1+k)e_2}{L(1+t)^2},\ t\in[1/k,k].$$

We find

$$\begin{aligned}
\psi(Ty)&=\min_{t\in[1/k,k]}\frac{(Ty)(t)}{1+t}>\frac{1}{1+k}(Ty)\Big(\frac{1}{k}\Big)\\
&=\frac{1}{1+k}\bigg[\frac{1}{1-\int_0^\infty g(s)\,\mathrm{d}s}\int_0^\infty g(t)\int_0^t\varphi^{-1}\Big(\frac{1}{\varrho(s)}\varphi(b)+\frac{1}{\varrho(s)}\int_s^\infty f(u,y(u))\,\mathrm{d}u\Big)\,\mathrm{d}s\,\mathrm{d}t\\
&\quad+\int_0^{1/k}\varphi^{-1}\Big(\frac{1}{\varrho(s)}\varphi(b)+\frac{1}{\varrho(s)}\int_s^\infty f(u,y(u))\,\mathrm{d}u\Big)\,\mathrm{d}s+\frac{a}{1-\int_0^\infty g(s)\,\mathrm{d}s}\bigg]\\
&\geqslant\frac{1}{1+k}\bigg[\int_0^{1/k}\varphi^{-1}\Big(\frac{1}{\varrho(s)}\varphi(b)+\frac{1}{\varrho(s)}\int_{1/k}^k f(u,y(u))\,\mathrm{d}u\Big)\,\mathrm{d}s+\frac{a}{1-\int_0^\infty g(s)\,\mathrm{d}s}\bigg]\\
&\geqslant\frac{1}{1+k}\bigg[\int_0^{1/k}\varphi^{-1}\Big(\frac{1}{\varrho(s)}\Big)\,\mathrm{d}s\,\varphi^{-1}\Big(\varphi(b)+\int_{1/k}^k\frac{\mu(1+k)e_2}{L(1+u)^2}\,\mathrm{d}u\Big)+\frac{a}{1-\int_0^\infty g(s)\,\mathrm{d}s}\bigg]\\
&=\frac{1}{1+k}\bigg[\int_0^{1/k}\varphi^{-1}\Big(\frac{1}{\varrho(s)}\Big)\,\mathrm{d}s\,\varphi^{-1}\Big(\varphi(b)+\frac{\mu(k-1)e_2}{L}\Big)+\frac{a}{1-\int_0^\infty g(s)\,\mathrm{d}s}\bigg]\\
&=B,
\end{aligned}$$

where the last equality follows from the definition of L in (3.3). This completes the proof of step 2.

Step 3. We shall prove that (E2) of Theorem 2.1 holds, i.e., $\|Ty\| < A$ for $y \in P$ with $\|y\| \leqslant A$. Let $y \in P$ with $\|y\| \leqslant A = e_1$, then

$$\sup_{t\in[0,\infty)} \frac{y(t)}{1+t} \leqslant e_1.$$

It follows from (C2) that

$$f(t,y(t)) = f\Big(t,(1+t)\frac{y(t)}{1+t}\Big) \leqslant \frac{e_1}{M_1(1+t)^2},\ t \in [0,\infty).$$

We find

$$\begin{aligned}
&\|Ty\| = \sup_{t\in[0,\infty)} \frac{(Ty)(t)}{1+t}\\
&= \sup_{t\in[0,\infty)} \bigg[\frac{1/(1+t)}{1-\int_0^\infty g(s)\,\mathrm{d}s}\int_0^\infty g(t)\int_0^t \varphi^{-1}\Big(\frac{1}{\varrho(s)}\varphi(b)+\frac{1}{\varrho(s)}\int_s^\infty f(u,y(u))\,\mathrm{d}u\Big)\,\mathrm{d}s\,\mathrm{d}t\\
&\quad+\frac{1}{1+t}\int_0^t \varphi^{-1}\Big(\frac{1}{\varrho(s)}\varphi(b)+\frac{1}{\varrho(s)}\int_s^\infty f(u,y(u))\,\mathrm{d}u\Big)\,\mathrm{d}s+\frac{1}{1+t}\frac{a}{1-\int_0^\infty g(s)\,\mathrm{d}s}\bigg]\\
&< \frac{1}{1-\int_0^\infty g(s)\,\mathrm{d}s}\int_0^\infty g(t)\int_0^t \varphi^{-1}\Big(\frac{1}{\varrho(s)}\Big)\,\mathrm{d}s\,\mathrm{d}t\,\varphi^{-1}\Big(\varphi(b)+\int_0^\infty f(u,y(u))\,\mathrm{d}u\Big)\\
&\quad+\sup_{t\in[0,\infty)}\frac{1}{1+t}\int_0^t \varphi^{-1}\Big(\frac{1}{\varrho(s)}\Big)\,\mathrm{d}s\,\varphi^{-1}\Big(\varphi(b)+\int_0^\infty f(u,y(u))\,\mathrm{d}u\Big)+\frac{a}{1-\int_0^\infty g(s)\,\mathrm{d}s}\\
&\leqslant \frac{1}{1-\int_0^\infty g(s)\,\mathrm{d}s}\int_0^\infty g(t)\int_0^t \varphi^{-1}\Big(\frac{1}{\varrho(s)}\Big)\,\mathrm{d}s\,\mathrm{d}(t)\,\varphi^{-1}\Big(\varphi(b)+\int_0^\infty \frac{e_1}{M_1(1+u)^2}\,\mathrm{d}u\Big)\\
&\quad+\sup_{t\in[0,\infty)}\frac{1+\tau(t)}{1+t}\varphi^{-1}\Big(\varphi(b)+\int_0^\infty \frac{e_1}{M_1(1+u)^2}\,\mathrm{d}u\Big)+\frac{a}{1-\int_0^\infty g(s)\,\mathrm{d}s}\\
&= \frac{1}{1-\int_0^\infty g(s)\,\mathrm{d}s}\int_0^\infty g(t)\int_0^t \varphi^{-1}\Big(\frac{1}{\varrho(s)}\Big)\,\mathrm{d}s\,\mathrm{d}(t)\,\varphi^{-1}\Big(\varphi(b)+\frac{e_1}{M_1}\Big)\\
&\quad+\sup_{t\in[0,\infty)}\frac{1+\tau(t)}{1+t}\varphi^{-1}\Big(\varphi(b)+\frac{e_1}{M_1}\Big)+\frac{a}{1-\int_0^\infty g(s)\,\mathrm{d}s}\\
&= e_1,
\end{aligned}$$

where the last equality follows from the definition of M_1 in (3.2). Thus, $\|Ty\| < e_1$ for $y \in P$ with $\|y\| \leqslant e_1$. This completes the proof of step 3.

Step 4. We shall show that (E3) of Theorem 2.1 holds, i.e., $\psi(Ty) > B$ for $y \in P(\psi;B,C)$ with $\|Ty\| > D$. Let $y \in P(\psi;B,C) = P(\psi;\mu(1+k)e_2,C)$ with $\|Ty\| > D = (1+k)e_2$, then

$$\sup_{t\in[0,\infty)} \frac{(Ty)(t)}{1+t} \geqslant (1+k)e_2 \quad \text{and} \quad \|y\| = \sup_{t\in[0,\infty)} \frac{y(t)}{1+t} \leqslant C.$$

Noting that $Ty \in P$, we get

$$\psi(Ty) = \min_{t\in[1/k,k]} \frac{(Ty)(t)}{1+t} \geqslant \mu \sup_{t\in[0,\infty)} \frac{(Ty)(t)}{1+t} \geqslant \mu(1+k)e_2 = B.$$

This completes the proof of step 4.

We have shown that all the conditions of Theorem 2.1 are satisfied. Hence, by Theorem 2.1 the operator T has three fixed points x_1, x_2 and $x_3 \in \overline{P_C}$ such that

$$\|x_1\| < A,\ \psi(x_2) > B,\ \|x_3\| > A \quad \text{with } \psi(x_3) < B,$$

i.e., x_1, x_2 and x_3 satisfy

$$\sup_{t\in[0,\infty)} \frac{x_1(t)}{1+t} < e_1, \qquad \min_{t\in[1/k,k]} \frac{x_2(t)}{1+t} > \mu(1+k)e_2 \tag{3.4}$$

and

$$\sup_{t\in[0,\infty)} \frac{x_3(t)}{1+t} > e_1, \qquad \min_{t\in[1/k,k]} \frac{x_3(t)}{1+t} < \mu(1+k)e_2. \tag{3.5}$$

Hence, BVP (1.1) has at least three positive solutions x_1, x_2 and x_3 satisfying (3.4) and (3.5).

Finally, we shall show that the solutions x_i, $i = 1,2,3$ are unbounded. If x_i, $i \in \{1,2,3\}$ is bounded, then in view of (A3) there exists $r > 0$ such that

$$0 \leqslant x_i^{-\alpha}(t) \leqslant r,\ t \in [0,\infty). \tag{3.6}$$

Moreover, by the assumption on φ we have

$$\frac{1}{\varphi^{-1}(x_i^{\alpha}(t))} = \varphi^{-1}(x_i^{-\alpha}(t)) - \varphi^{-1}(0) \leqslant M_r x_i^{-\alpha}(t). \tag{3.7}$$

We claim that there exists $\sigma_0 > 0$ such that $x_i(\sigma_0) \geqslant \sup_{t\in[0,\infty)} x_i(t)/(1+t)$. In fact, if $x_i(t) < \sup_{s\in[0,\infty)} x_i(s)/(1+s)$ for all $t \in [0,\infty)$, we get

$$0 \leqslant \frac{x_i(t)}{1+t} < \frac{\sup_{s\in[0,\infty)} x_i(s)/(1+s)}{1+t}.$$

Taking limit then gives

$$\lim_{t\to\infty} \frac{x_i(t)}{1+t} = 0.$$

Hence, there exists $\sigma_0 > 0$ such that

$$\sup_{t\in[0,\infty)} \frac{x_i(t)}{1+t} = \frac{x_i(\sigma_0)}{1+\sigma_0}.$$

It follows that

$$x_i(\sigma_0) \geqslant \sup_{t\in[0,\infty)} \frac{x_i(t)}{1+t} \tag{3.8}$$

and our claim is justified.

Now, choose $c > 0$ sufficiently large such that $c\|x_i\| \geqslant \sigma_2$ and $\frac{1}{c} \leqslant \sigma_1$. By virtue of (3.8), the condition $c\|x_i\| \geqslant \sigma_2$ leads to $cx_i(\sigma_0) \geqslant \sigma_2$. Since x_i is nondecreasing, we have

$$cx_i(u) \geqslant \sigma_2, \ u \geqslant \sigma_0. \tag{3.9}$$

Using (A3) and (3.9), we get for sufficiently large $t > \sigma_0$,

$$\begin{aligned}
x_i(t) &= \frac{1}{1-\int_0^\infty g(s)\,\mathrm{d}s} \int_0^\infty g(t) \int_0^t \varphi^{-1}\Big(\frac{1}{\varrho(s)}\varphi(b) + \frac{1}{\varrho(s)} \int_s^\infty f(u, x_i(u))\,\mathrm{d}u\Big)\,\mathrm{d}s\,\mathrm{d}t \\
&\quad + \int_0^t \varphi^{-1}\Big(\frac{1}{\varrho(s)}\varphi(b) + \frac{1}{\varrho(s)} \int_s^\infty f(u, x_i(u))\,\mathrm{d}u\Big)\,\mathrm{d}s + \frac{a}{1-\int_0^\infty g(s)\,\mathrm{d}s} \\
&\geqslant \int_0^t \varphi^{-1}\Big(\frac{1}{\varrho(s)} \int_s^\infty f(u, x_i(u))\,\mathrm{d}u\Big)\,\mathrm{d}s \\
&\geqslant \int_{\sigma_0}^t \varphi^{-1}\Big(\frac{1}{\varrho(s)} \int_s^\infty f\Big(u, \frac{1}{c} c x_i(u)\Big)\,\mathrm{d}u\Big)\,\mathrm{d}s \\
&\geqslant \int_{\sigma_0}^t \varphi^{-1}\Big(\frac{1}{\varrho(s)}\Big)\varphi^{-1}\Big(\int_s^\infty c^\alpha x_i^\alpha(u) f\Big(u, \frac{1}{c}\Big)\,\mathrm{d}u\Big)\,\mathrm{d}s \\
&\geqslant \int_{\sigma_0}^t \varphi^{-1}\Big(\frac{1}{\varrho(s)}\Big)\varphi^{-1}\Big(\int_s^\infty f\Big(u, \frac{1}{c}\Big)\,\mathrm{d}u\Big)\,\mathrm{d}(s)\, \varphi^{-1}(c^\alpha)\varphi^{-1}(x_i^\alpha(t)) \\
&\geqslant \int_{\sigma_0}^t \varphi^{-1}\Big(\frac{1}{\varrho(s)}\Big)\varphi^{-1}\Big(\int_s^\infty \frac{1}{c^\beta} f(u, 1)\,\mathrm{d}u\Big)\,\mathrm{d}(s)\, \varphi^{-1}(c^\alpha)\varphi^{-1}(x_i^\alpha(t)) \\
&= \varphi^{-1}(c^{\alpha-\beta})\varphi^{-1}(x_i^\alpha(t)) \int_{\sigma_0}^t \varphi^{-1}\Big(\frac{1}{\varrho(s)}\Big)\varphi^{-1}\Big(\int_s^\infty f(u, 1)\,\mathrm{d}u\Big)\,\mathrm{d}s.
\end{aligned}$$

Thus,

$$\frac{x_i(t)}{\varphi^{-1}(x_i^\alpha(t))} \geqslant \varphi^{-1}(c^{\alpha-\beta}) \int_{\sigma_0}^t \varphi^{-1}\Big(\frac{1}{\varrho(s)}\Big)\varphi^{-1}\Big(\int_s^\infty f(u, 1)\,\mathrm{d}u\Big)\,\mathrm{d}s. \tag{3.10}$$

Applying (3.6) and (3.7) to (3.10), we find

$$
\begin{aligned}
(3.11)\qquad r^{-(1-\alpha)/\alpha}M_r \geqslant x_i^{1-\alpha}(t)M_r &\geqslant \frac{x_i(t)}{\varphi^{-1}(x_i^{\alpha}(t))}\\
&\geqslant \varphi^{-1}(c^{\alpha-\beta})\int_{\sigma_0}^{t}\varphi^{-1}\Big(\frac{1}{\varrho(s)}\Big)\varphi^{-1}\Big(\int_s^{\infty} f(u,1)\,\mathrm{d}u\Big)\,\mathrm{d}s.
\end{aligned}
$$

Let $t\to\infty$ in (3.11), it follows from (A1) that $r^{-(1-\alpha)/\alpha}M_r \geqslant \infty$, a contradiction. Hence, x_1, x_2 and x_3 are unbounded. The proof is complete. □

4. An example

To illustrate the usefulness of our main result, we present an example that our result can readily apply, whereas the known results in the literature are not applicable.

Example 4.1. Consider the boundary value problem

$$
(4.1)\qquad \begin{cases} \big[[x'(t)]^3\big]' + f(t,x(t)) = 0, \quad t\in(0,\infty),\\ x(0) = \frac{1}{2}\int_0^{\infty} \mathrm{e}^{-s}x(s)\,\mathrm{d}s + 2,\\ \lim\limits_{t\to\infty} x'(t) = 1, \end{cases}
$$

where

$$
f(t,x) = \frac{t}{10^{39}(1+t)^3} + \frac{1}{(1+t)^2}f_0\Big(\frac{x}{1+t}\Big),
$$

and f_0 is defined by

$$
f_0(x) = \begin{cases} 13,\ x\in[0,10],\\ 13 + (x-10)\dfrac{\frac{1}{2}\big(\frac{101}{99}(1009600^3-1) + (51\times 10^{14}-2)^3 - 1\big) - 13}{100-10},\\ \qquad\qquad x\in[10,100],\\ \dfrac{\frac{101}{99}(1009600^3-1) + (51\times 10^{14}-2)^3 - 1}{2}, \quad x\in[100, 102\times 10^{14}],\\ \dfrac{\frac{101}{99}(1009600^3-1) + (51\times 10^{14}-2)^3 - 1}{2} + x - 102\times 10^{14},\\ \qquad\qquad x\geqslant 102\times 10^{14}. \end{cases}
$$

Corresponding to BVP (1.1), we have $a=2$, $b=1$, $\varphi(x)=x^3$, $\varrho(t)=1$ and $g(t)=\frac{1}{2}\mathrm{e}^{-t}$. Then, $\varphi^{-1}(x)=x^{\frac{1}{3}}$ and $\tau(t)=t$.

Choose $k = 100$, $e_1 = 10$, $e_2 = 10000$ and $C = 102 \times 10^{14}$. One finds

$$\mu = \frac{1}{1+k} \int_0^{1/k} \varphi^{-1}\Big(\frac{1}{\varrho(s)}\Big)\,\mathrm{d}s \inf_{t\in[0,\infty)} \frac{1+t}{1+\tau(t)} = \frac{1}{10100},$$

$$M = C\bigg[\varphi\bigg(\frac{\big(1-\int_0^\infty g(s)\,\mathrm{d}s\big)C - a}{\int_0^\infty g(t)\tau(t)\,\mathrm{d}t + \big(1-\int_0^\infty g(s)\,\mathrm{d}s\big) \sup\limits_{t\in[0,\infty)} (1+\tau(t))/(1+t)}\bigg) - \varphi(b)\bigg]^{-1}$$

$$= \frac{102\times 10^{14}}{(51\times 10^{14}-2)^3 - 1},$$

$$M_1 = e_1\bigg[\varphi\bigg(\frac{\big(1-\int_0^\infty g(s)\,\mathrm{d}s\big)e_1 - a}{\int_0^\infty g(t)\tau(t)\,\mathrm{d}t + \big(1-\int_0^\infty g(s)\,\mathrm{d}s\big) \sup\limits_{t\in[0,\infty)} (1+\tau(t))/(1+t)}\bigg) - \varphi(b)\bigg]^{-1}$$

$$= \frac{5}{13},$$

$$L = \mu(k-1)e_2\bigg[\varphi\bigg(\frac{\mu(1+k)^2 e_2\big(1-\int_0^\infty g(s)\,\mathrm{d}s\big) - a}{\big(1-\int_0^\infty g(s)\,\mathrm{d}s\big)\int_0^{1/k}\varphi^{-1}\big(1/\varrho(s)\big)\,\mathrm{d}s}\bigg) - \varphi(b)\bigg]^{-1}$$

$$= \frac{9900}{101} \times \frac{1}{1009600^3 - 1}.$$

Thus, we have

$$D = (1+k)e_2 = 1010000, \quad B = \mu(1+k)e_2 = 100, \quad A = e_1 = 10,$$

and

$$0 < e_1 < \mu(1+k)e_2 < (1+k)e_2 < C, \quad LC > M\mu(1+k)e_2 > 0.$$

On the other hand, one sees that

$$f(t,cx) = \frac{t}{10^{39}(1+t)^3} + \frac{1}{(1+t)^2} f_0\Big(\frac{cx}{1+t}\Big),$$

$$f(t,c) = \frac{t}{10^{39}(1+t)^3} + \frac{1}{(1+t)^2} f_0\Big(\frac{c}{1+t}\Big),$$

$$f(t,(1+t)x) = \frac{t}{10^{39}(1+t)^3} + \frac{1}{(1+t)^2} f_0(x),$$

$$f(t,1) = \frac{t}{10^{39}(1+t)^3} + \frac{1}{(1+t)^2} f_0\Big(\frac{1}{1+t}\Big)$$

$$= \frac{t}{10^{39}(1+t)^3} + \frac{1}{(1+t)^2}\,\frac{5\times 51^2\times 10^{18}}{2}.$$

It is easy to check that conditions (A1)–(A3) and (C1)–(C3) are satisfied. Indeed, we have

(A1) ϱ and g satisfy

$$\int_0^1 \varphi^{-1}\Big(\frac{1}{\varrho(s)}\Big)\,\mathrm{d}s < \infty, \quad \int_0^\infty \varphi^{-1}\Big(\frac{1}{\varrho(s)}\Big)\,\mathrm{d}s = \infty,$$
$$\int_0^\infty g(t)\int_0^t \varphi^{-1}\Big(\frac{1}{\varrho(s)}\Big)\,\mathrm{d}s\,\mathrm{d}t < \infty,$$
$$\lim_{t\to\infty}\int_0^t \varphi^{-1}\Big(\frac{1}{\varrho(s)}\Big)\varphi^{-1}\Big(\int_s^\infty f(u,1)\,\mathrm{d}u\Big)\,\mathrm{d}s = \infty,$$

and

$$\sup_{t\in[0,\infty)} \frac{1+\tau(t)}{1+t} = 1 < \infty;$$

(A2) $f\colon (0,\infty)\times[0,\infty)\to[0,\infty)$ is an S-Carathéodory function with $f(t,0)\not\equiv 0$ on each sub-interval of $[0,\infty)$;

(A3) since $f(t,x)$ is increasing in x, there exist real numbers $\alpha < 0 < \beta$ and $\sigma_2 > \sigma_1 > 0$ such that

$$f(t,cx) \geqslant c^\alpha f(t,x) \quad \text{for } c \geqslant \sigma_2, \text{ sufficiently large } t \text{ and sufficiently small } x,$$

and

$$f(t,c) \geqslant c^\beta f(t,1) \quad \text{for } 0 < c \leqslant \sigma_1 \text{ and sufficiently large } t;$$

(C1) $f(t,(1+t)x) \leqslant (1+t)^{-2}((51\times 10^{14}-2)^3-1)$ for $t\in(0,\infty)$ and $x\in[0,102\times 10^{14}]$;

(C2) $f(t,(1+t)x) \leqslant 26(1+t)^{-2}$ for $t\in(0,\infty)$ and $x\in[0,10]$;

(C3) $f(t,(1+t)x) \geqslant (1+t)^{-2}\cdot\frac{101}{99}(1009600^3-1)$ for $t\in[0.01,100]$ and $x\in[100,1010000]$.

Hence, it follows from Theorem 3.1 that BVP (4.1) has at least three unbounded positive solutions x_1, x_2 and x_3 such that

$$\sup_{t\in[0,\infty)}\frac{x_1(t)}{1+t} < 10, \quad \min_{t\in[0.01,100]}\frac{x_2(t)}{1+t} > 100$$

and

$$\sup_{t\in[0,\infty)}\frac{x_3(t)}{1+t} > 10, \quad \min_{[0.01,100]}\frac{x_3(t)}{1+t} < 100.$$

Remark 4.1. We note that Example 4.1 cannot be covered by the theorems in [5], [10], [14], [15], [17–21], [23–26], [29] since the nonlinear operator $[x']^3$ appears in (4.1), the first boundary condition in (4.1) is of integral type and both the conditions in (4.1) are non-homogeneous boundary conditions while $x'(\infty)=0$ is contained in

[5], [10], [14], [15], [17], [18], [20], [21], [23–26], [29], $x(0) = 0$ and $x'(\infty) = x_\infty \geqslant 0$ in [19]. Further, it is evident from Example 4.1 that

(i) in (4.1), if $f(t,x) = g(t) + (1+t)^{-2} f_0(x)$ and $g(t)$ is nonnegative and sufficiently small, then there is a large number of functions f_0 that satisfy the conditions of Theorem 3.1;

(ii) the conditions of Theorem 3.1 are easy to check;

(iii) provided the differential equation in (4.1) is replaced by

$$\left[\frac{1}{t}[x'(t)]^3\right]' + f(t, x(t)) = 0, \quad t \in (0, \infty),$$

in this case $\varrho(t) = 1/t$ is singular at $t = 0$. The existence result can be established similarly.

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