Jung-Jo Lee
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CONGRUENCES FOR CERTAIN BINOMIAL SUMS

JUNG-JO LEE, Seoul

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Abstract. We exploit the properties of Legendre polynomials defined by the contour integral $\mathbf{P}_n(z) = (2\pi \mathrm{i})^{-1} \oint (1-2tz+t^2)^{-1/2} t^{-n-1}\,\mathrm{d}t$, where the contour encloses the origin and is traversed in the counterclockwise direction, to obtain congruences of certain sums of central binomial coefficients. More explicitly, by comparing various expressions of the values of Legendre polynomials, it can be proved that for any positive integer r, a prime $p \geqslant 5$ and $n = rp^2 - 1$, we have $\sum\limits_{k=0}^{\lfloor n/2 \rfloor} \binom{2k}{k} \equiv 0, 1$ or $-1 \pmod{p^2}$, depending on the value of $r \pmod 6$.

Keywords: central binomial coefficient, Legendre polynomial

MSC 2010: 05A10, 11B65

1. INTRODUCTION

In 2007, David Callan proposed in American Mathematical Monthly, the following problem: if p is a prime and $p \geqslant 5$, then p^2 divides the binomial sum $\sum\limits_{k=1}^{p^2-1} \binom{2k}{k}$.

In this paper, we prove the following generalization of this result.

Theorem 1. *Let p be a prime such that $p \geqslant 5$. Let $n = rp^2 - 1$ with $r \geqslant 1$ an integer. Then*

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{2k}{k} \equiv \begin{cases} 1 \pmod{p^2}, & \text{if } r \equiv 1 \text{ or } 2 \pmod 6; \\ 0 \pmod{p^2}, & \text{if } r \equiv 0 \pmod 3; \\ -1 \pmod{p^2}, & \text{if } r \equiv 4 \text{ or } 5 \pmod 6. \end{cases}$$

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Observe that when $r = 2$, this reduces to Callan's problem. A solution submitted by Robin Chapman to Callan's problem was published in 2009 [2]. His method of proof does not give our theorem. We apply a different method and exploit properties of Legendre polynomials.

Studying such binomial sums has its own interests, and there are studies on it. For example, in [4], S. Mattarei studies asymptotic behaviour of $\sum\limits_{k=0}^{n} \binom{2k}{k}$ and a corresponding property of Catalan numbers.

2. Preliminary lemmas

Lemma 2. *Let $r \geqslant 1$ be a natural number, and p a prime number. Then*

$$\binom{rp^2-1}{k} \equiv (-1)^k \quad \textit{and} \quad \binom{rp^2-1}{2k} \equiv 1 \pmod{p^2}. \tag{2.1}$$

P r o o f. We make an observation that

$$\binom{rp^2-1}{k-1} = \frac{(rp^2-1)!}{(k-1)!(rp^2-k)!} = \frac{\{(rp^2-1)\dots(rp^2-k+1)\}(rp^2-k)!}{(k-1)!(rp^2-k)!} \equiv (-1)^{k-1} \pmod{p^2}.$$

Thus, the first congruence follows. The second congruence is obtained from the first by replacing k by $2k$. □

To prove the theorem given in the introduction, we will use formulas for the Legendre polynomials. The Legendre polynomial $\mathbf{P}_n(z)$ can be defined by the contour integral

$$\mathbf{P}_n(z) = \frac{1}{2\pi \mathrm{i}} \oint (1-2tz+t^2)^{-1/2} t^{-n-1}\,\mathrm{d}t,$$

where the contour encloses the origin and is traversed in the counterclockwise direction [3].

We have a formula for $\mathbf{P}_n(x)$ as

$$\mathbf{P}_n(x) = x^n p_n\Big(\frac{x^2-1}{4x^2}\Big) \tag{2.2}$$

where

$$p_n(b) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k}\binom{2k}{k} b^k. \tag{2.3}$$

(See [1] for this formula.)

Another expression (formula (34) of [3]) for the Legendre polynomial $\mathbf{P}_n(x)$ is given by

$$\mathbf{P}_n(x) = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}^2 (x-1)^{n-k}(x+1)^k. \tag{2.4}$$

For the rest of this paper, ω will denote $\omega = \sqrt{-1/3}$.

Lemma 3. *Let p be a prime such that $p \geqslant 5$ and $r \geqslant 1$ an integer. Then*

$$\mathbf{P}_{rp^2-1}(\omega) = \Bigl(\frac{\mathrm{i}}{\sqrt{3}}\Bigr)^{rp^2-1} \sum_{k=0}^{\lfloor (rp^2-1)/2\rfloor} \binom{rp^2-1}{2k}\binom{2k}{k}$$

where $\mathrm{i} = \sqrt{-1}$.

Proof. Substitute $n = rp^2-1$ and $x = \sqrt{-1/3}$ in the formulas (2.2) and (2.3), from which we get our result. (Here, $x = \sqrt{-1/3}$ comes from solving $(x^2-1)/4x^2 = 1$ in (2.2).) □

Lemma 4. *With p and ω as above, we have that*

$$\mathbf{P}_{rp^2-1}(\omega) = \Bigl(\frac{1}{\sqrt{3}}\Bigr)^{rp^2-1} \cdot \exp\Bigl((rp^2-1)\cdot\frac{5\pi}{6}\mathrm{i}\Bigr) \sum_{k=0}^{rp^2-1}\binom{rp^2-1}{k}^2 \exp\Bigl(\frac{4k\pi}{3}\mathrm{i}\Bigr).$$

Proof. Substituting $n = rp^2-1$ and $x = \sqrt{-1/3}$ into the formula (2.4), we get

$$\begin{aligned}\mathbf{P}_{rp^2-1}(\omega) &= \frac{1}{2^{rp^2-1}}\sum_{k=0}^{rp^2-1}\binom{rp^2-1}{k}^2(\omega-1)^{rp^2-1-k}(\omega+1)^k \\ &= \Bigl(\frac{\omega-1}{2}\Bigr)^{rp^2-1}\sum_{k=0}^{rp^2-1}\binom{rp^2-1}{k}^2\Bigl(\frac{\omega+1}{\omega-1}\Bigr)^k.\end{aligned} \tag{2.5}$$

Now, a short calculation shows that

$$\frac{\omega-1}{2} = \frac{1}{\sqrt{3}}\exp\Bigl(\frac{5\pi}{6}\mathrm{i}\Bigr)$$

and

$$\frac{\omega+1}{\omega-1} = \exp\Bigl(\frac{4\pi}{3}\mathrm{i}\Bigr).$$

Substituting these back into (2.5), we obtain the result. □

Corollary 5. *Under the same notations and conditions as before, we have*

$$\sum_{k=0}^{\lfloor (rp^2-1)/2 \rfloor} \binom{rp^2-1}{2k}\binom{2k}{k}$$
$$= (-\mathrm{i})^{rp^2-1} \exp\left((rp^2-1)\cdot\frac{5\pi}{6}\,\mathrm{i}\right) \sum_{k=0}^{rp^2-1} \binom{rp^2-1}{k}^2 \exp\left(\frac{4k\pi}{3}\,\mathrm{i}\right).$$

P r o o f. This is obtained by comparing the two expressions for $\mathbf{P}_{rp^2-1}(\omega)$ in Lemmas 3 and 4. □

3. Proof of the main theorem

We keep all the notations and conditions of the previous sections.

Lemma 6. *Let*

$$S = (-\mathrm{i})^{rp^2-1} \exp\left((rp^2-1)\cdot\frac{5\pi}{6}\,\mathrm{i}\right) \sum_{k=0}^{rp^2-1} \exp\left(\frac{4k\pi}{3}\,\mathrm{i}\right).$$

Then

$$S = \begin{cases} 1, & \textit{if } \ r \equiv 1,2 \pmod 6, \\ 0, & \textit{if } \ r \equiv 0 \pmod 3, \\ -1, & \textit{if } \ r \equiv 4,5 \pmod 6. \end{cases}$$

P r o o f. By summing the geometric series, we have

$$\sum_{k=0}^{rp^2-1} \exp\left(\frac{4k\pi}{3}\,\mathrm{i}\right) = \frac{\exp(rp^2\cdot\frac{4}{3}\pi\mathrm{i}) - 1}{\exp(\frac{4}{3}\pi\mathrm{i}) - 1}.$$

Observing that

$$\exp\left(\frac{4\pi}{3}\mathrm{i}\right) - 1 = -\frac{3}{2} - \frac{\sqrt{3}}{2}\,\mathrm{i} = -\sqrt{3}\exp\left(\frac{\pi}{6}\,\mathrm{i}\right)$$

and defining S' by $S = (-\mathrm{i})^{rp^2-1} S'$, we have that

$$\begin{aligned} S' &= \exp\Big((rp^2-1)\cdot\frac{5\pi}{6}\mathrm{i}\Big)\cdot\frac{\exp(rp^2\cdot\frac{4\pi}{3}\mathrm{i})-1}{\exp(\frac{4\pi}{3}\mathrm{i})-1} \\ &= \exp\Big((rp^2-1)\cdot\frac{5\pi}{6}\mathrm{i}\Big)\cdot\frac{\exp(rp^2\cdot\frac{4\pi}{3}\mathrm{i})-1}{-\sqrt{3}\exp(\frac{\pi}{6}\mathrm{i})} \\ &= \frac{1}{\sqrt{3}}\cdot\exp\Big(rp^2\cdot\frac{5\pi}{6}\mathrm{i}\Big)\Big\{\exp\Big(rp^2\cdot\frac{4\pi}{3}\mathrm{i}\Big)-1\Big\} \\ &= \frac{1}{\sqrt{3}}\Big\{\exp\Big(rp^2\cdot\frac{\pi}{6}\mathrm{i}\Big)-\exp\Big(rp^2\cdot\frac{5\pi}{6}\mathrm{i}\Big)\Big\}. \end{aligned}$$

Now, it is completely elementary to check that

$$S = (-\mathrm{i})^{rp^2-1}S' = \begin{cases} 1, & \text{if } rp^2 \equiv 1,2,7,8 \pmod{12}, \\ 0, & \text{if } rp^2 \equiv 0,3,6,9 \pmod{12}, \\ -1, & \text{if } rp^2 \equiv 4,5,10,11 \pmod{12}, \end{cases}$$

equivalently, but more simply

$$S = \begin{cases} 1, & \text{if } rp^2 \equiv 1,2 \pmod{6}, \\ 0, & \text{if } rp^2 \equiv 0 \pmod{3}, \\ -1, & \text{if } rp^2 \equiv 4,5 \pmod{6}. \end{cases}$$

Observe that $x^2 \equiv 1 \pmod{12}$ for any x with $(x,6)=1$. Thus, for $p \geqslant 5$ we have $p^2 \equiv 1 \pmod{6}$, and our result follows as claimed. □

Theorem 7. *Let p be a prime such that $p \geqslant 5$. For $n = rp^2-1$ where $r \geqslant 1$ is an integer, we have the following:*

$$\sum_{k=0}^{\lfloor n/2\rfloor}\binom{2k}{k} \equiv \begin{cases} 1 \pmod{p^2}, & \text{if } r \equiv 1 \text{ or } 2 \pmod{6}; \\ 0 \pmod{p^2}, & \text{if } r \equiv 0 \pmod{3}; \\ -1 \pmod{p^2}, & \text{if } r \equiv 4 \text{ or } 5 \pmod{6}. \end{cases}$$

P r o o f. From Corollary 5, we have that

$$\begin{aligned} &\sum_{k=0}^{\lfloor (rp^2-1)/2\rfloor}\binom{rp^2-1}{2k}\binom{2k}{k} \\ &\qquad = (-\mathrm{i})^{rp^2-1}\exp\Big((rp^2-1)\cdot\frac{5\pi}{6}\mathrm{i}\Big)\sum_{k=0}^{rp^2-1}\binom{rp^2-1}{k}^2\exp\Big(\frac{4k\pi}{3}\mathrm{i}\Big). \end{aligned}$$

According to Lemma 2, we can replace

$$\binom{rp^2-1}{k} = (-1)^k + p^2 f(k) \tag{3.1}$$

where $f\colon \mathbb{N} \to \mathbb{N}$ is a function defined on the set of natural numbers $\mathbb{N}$ (including 0) and takes values also in $\mathbb{N}$, for given r and p.

We will put (3.1) into the equation given in Corollary 5. Then on the left hand side of the equation, we have

$$\sum_{k=0}^{\lfloor (rp^2-1)/2 \rfloor} (1+p^2 f(2k))\binom{2k}{k} = \sum_{k=0}^{\lfloor (rp^2-1)/2 \rfloor} \binom{2k}{k} + p^2 \sum_{k=0}^{\lfloor (rp^2-1)/2 \rfloor} f(2k)\binom{2k}{k}. \tag{3.2}$$

On the right hand side of the equation, we have

$$\begin{aligned}(-\mathrm{i})^{rp^2-1} \exp\Big((rp^2-1)\cdot\frac{5\pi}{6}\,\mathrm{i}\Big) &\sum_{k=0}^{rp^2-1} ((-1)^k + p^2 f(k))^2 \exp\Big(\frac{4k\pi}{3}\,\mathrm{i}\Big) \\ &= S + p^2\cdot(-\mathrm{i})^{rp^2-1}\exp\Big((rp^2-1)\cdot\frac{5\pi}{6}\,\mathrm{i}\Big) \\ &\quad\times \sum_{k=0}^{rp^2-1} (2(-1)^k f(k) + p^2 f(k)^2)\exp\Big(\frac{4k\pi}{3}\,\mathrm{i}\Big)\end{aligned} \tag{3.3}$$

where S is as given in the previous lemma.

Comparing (3.2) and (3.3), as they are equal by Lemma 5, we get

$$\begin{aligned}&\sum_{k=0}^{\lfloor (rp^2-1)/2 \rfloor} \binom{2k}{k} - S \\ &= p^2(-\mathrm{i})^{rp^2-1}\exp\Big((rp^2-1)\frac{5\pi}{6}\,\mathrm{i}\Big)\sum_{k=0}^{rp^2-1}(2(-1)^k f(k) + p^2 f(k)^2)\exp\Big(\frac{4k\pi}{3}\,\mathrm{i}\Big) \\ &\quad - p^2 \sum_{k=0}^{\lfloor (rp^2-1)/2 \rfloor} f(2k)\binom{2k}{k} \\ &= p^2\cdot g(rp^2-1)\end{aligned} \tag{3.4}$$

for some function $g\colon \mathbb{N} \to \mathbb{C}$. Explicitly, let g be defined by

$$\begin{aligned}g(n) := (-\mathrm{i})^n \exp\Big(n\cdot\frac{5\pi}{6}\,\mathrm{i}\Big)\sum_{k=0}^{n}(2(-1)^k f(k) + p^2 f(k)^2)\exp\Big(\frac{4k\pi}{3}\,\mathrm{i}\Big) \\ - \sum_{k=0}^{\lfloor n/2 \rfloor} f(2k)\binom{2k}{k}\end{aligned}$$

for any natural number n.

By Lemma 6, we know that $S = \pm 1$ or 0, so that the left hand side of the equation (3.4) is an integer, and so is the right hand side. Thus, $g(rp^2 - 1)$ is a rational number, whose denominator can possibly (but not necessarily) be divisible by p, but no other primes when it is written in its lowest terms.

Now, $\exp((rp^2-1)\cdot\frac{5\pi}{6}\mathrm{i})$ and $\exp(\frac{4k\pi}{3}\mathrm{i})$ can take only a finite number of values. More precisely, their values are one of $\pm\frac{1}{2}\sqrt{3}\pm\frac{\mathrm{i}}{2}$, $\pm\frac{1}{2}\pm\frac{1}{2}\sqrt{3}\mathrm{i}$, ± 1 or $\pm\mathrm{i}$ depending on the choices of r, p and k. Moreover, since $f(k)$ and $f(2k)$ are integers for any k, the denominator of $g(rp^2-1)$ cannot be divisible by p for any prime $p > 2$, which implies that $g(rp^2-1)$ is an integer.

Therefore the equation (3.4) implies that

$$\sum_{k=0}^{\lfloor (rp^2-1)/2 \rfloor} \binom{2k}{k} \equiv S \pmod{p^2},$$

which combined with Lemma 6, gives the proof of our theorem. □

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Author's address: Jung-Jo Lee, Yonsei University, Seoul 120-749, South Korea, e-mail: jungjolee@gmail.com.