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ON SPECIAL TYPES OF SEMIHOLONOMIC 3-JETS

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Abstract. First we summarize some properties of the nonholonomic r-jets from the functorial point of view. In particular, we describe the basic properties of our original concept of nonholonomic r-jet category. Then we deduce certain properties of the Weil algebras associated with nonholonomic r-jets. Next we describe an algorithm for finding the nonholonomic r-jet categories. Finally we classify all special types of semiholonomic 3-jets.

Keywords: special type of nonholonomic r-jet, nonholonomic r-jet category, classification of semiholonomic 3-jet

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All manifolds and maps are assumed to be infinitely differentiable. Unless otherwise specified, we use the terminology and notation from [8]. The author acknowledges Josef Šilhan for advice concerning representation theory.

1. Introduction

Let $\mathcal{M}f$ be the category of all manifolds and all smooth maps and $\mathcal{M}f_m$ be the category of m-dimensional manifolds and their local diffeomorphisms. Every two manifolds M and N determine the bundle $J^r(M,N) \to M \times N$ of all r-jets of M into N. In [8] we pointed out that J^r is a bundle functor on the product category $\mathcal{M}f_m \times \mathcal{M}f$, $m = \dim M$. Indeed, every local diffeomorphism $f\colon M \to M'$ and every map $g\colon N \to N'$ induce a map

$$J^r(f,g)\colon J^r(M,N) \to J^r(M',N')$$

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by the jet composition

$$J^r(f,g)(X) = (j^r_y g) \circ X \circ (j^r_x f)^{-1}, \quad X \in J^r_x(M,N)_y. \tag{1}$$

Clearly, $J^r(M, N_1 \times N_2) = J^r(M,N_1) \times_M J^r(M,N_2)$.

In [1], C. Ehresmann introduced the bundle $\widetilde{J}^r(M,N) \to M \times N$ of nonholonomic r-jets of M into N, $J^r(M,N) \subset \widetilde{J}^r(M,N)$, see also [5]. He defined a composition

$$X_2 \circ X_1 \in \widetilde{J}^r_x(M,Q)_z \tag{2}$$

for every $X_1 \in \widetilde{J}^r_x(M,N)_y$ and $X_2 \in \widetilde{J}^r_y(N,Q)_z$, that is associative and generalizes the composition of the classical holonomic r-jets. Hence $\widetilde{J}^r$ can be interpreted as a bundle functor on $\mathcal{M}f_m \times \mathcal{M}f$, if we set

$$\widetilde{J}^r(f,g) = (j^r_y g) \circ X \circ (j^r_x f)^{-1}, \quad X \in \widetilde{J}^r_x(M,N)_y, \tag{3}$$

with the composition of nonholonomic r-jets. Even in this case we have $\widetilde{J}^r(M, N_1 \times N_2) = \widetilde{J}^r(M,N_1) \times_M \widetilde{J}^r(M,N_2)$.

The best known example of special type of nonholonomic r-jets are the bundles $\overline{J}^r(M,N)$ of semiholonomic r-jets

$$J^r(M,N) \subset \overline{J}^r(M,N) \subset \widetilde{J}^r(M,N),$$

[1], [5], [9]. There is a simple description of $\overline{J}^r(V,W)$ in the case of two vector spaces V, W, [1]. Analogously to the classical formula

$$J^r(V,W) = V \oplus W \otimes \left(\sum_{i=0}^{r} S^i V^* \right) \tag{4}$$

with symmetric tensor powers of V^*, we have

$$\overline{J}^r(V,W) = V \oplus W \otimes \left(\sum_{i=0}^{r} \overset{i}{\otimes} V^* \right) \tag{5}$$

with arbitrary tensor powers of V^*. The composition of two semiholonomic r-jets is semiholonomic as well. Further, $\overline{J}^r(M, N_1 \times N_2) = \overline{J}^r(M,N_1) \times_M \overline{J}^r(M,N_2)$. We denote by $\pi^r_s \colon \overline{J}^r(M,N) \to \overline{J}^s(M,N)$, $s < r$, the canonical projection, [1].

We have been interested in the general concept of special type of nonholonomic r-jets. In our first attempt, [3], we started from the description of all bundle functors on the category $\mathcal{M}f_m \times \mathcal{M}f$ preserving product in the second factor, [7], [5]. In general, a bundle functor F on $\mathcal{M}f_m \times \mathcal{M}f$ is said to preserve products in the second factor, if

$$F(M, N_1 \times N_2) = F(M, N_1) \times_M F(M, N_2).$$

Further, F is said to be of order r in the first factor, if for every two local diffeomorphisms $f_1,\ f_2\colon\ M_1 \to M_2$ and every $g\colon\ N_1 \to N_2$, $j^r_x f_1 = j^r_x f_2$ implies

$$F(f_1, g) \mid F_x(M_1, N_1) = F(f_2, g) \mid F_x(M_1, N_1),$$

where $F_x(M_1, N_1)$ means the fiber of $F(M_1, N_1)$ over $x \in M_1$. Such functors are identified with pairs (A, H), where A is a Weil algebra and $H\colon\ G^r_m \to \operatorname{Aut} A$ is a group homomorphism of the r-th jet group G^r_m in dimension m into the group $\operatorname{Aut} A$ of all algebra automorphisms of A. Then $F(M, N)$ is the associated bundle $P^r M[T^A N, H_N]$, where $P^r M$ is the r-th order frame bundle of M and H_N is the induced action of G^r_m on $T^A N$. We have $F(f, g) = P^r f[T^A g]$.

In the special case $F = J^r$, the Weil algebra is $\mathbb{D}^r_m = J^r_0(\mathbb{R}^m, \mathbb{R})$, we have $\operatorname{Aut} \mathbb{D}^r_m \approx G^r_m$ and $H = \mathrm{id}_{G^r_m}$. This yields a classical formula $J^r(M, N) = P^r M[T^r_m N]$. In the case $F = \widetilde{J}^r$, the Weil algebra is $\widetilde{\mathbb{D}}^r_m = \widetilde{J}^r_0(\mathbb{R}^m, \mathbb{R})$, $T^{\widetilde{\mathbb{D}}^r_m} N = \widetilde{T}^r_m N = \widetilde{J}^r_0(\mathbb{R}^m, N)$ is the bundle of nonholonomic (m, r)-velocities over N, the jet composition defines an action of G^r_m on $\widetilde{\mathbb{D}}^r_m$ and $\widetilde{J}^r(M, N) = P^r M[\widetilde{T}^r_m N]$.

In our first approach, [3], we considered a G^r_m-invariant Weil algebra Φ, $\mathbb{D}^r_m \subset \Phi \subset \widetilde{\mathbb{D}}^r_m$, and we defined an r-th order jet functor on $\mathcal{M}f_m \times \mathcal{M}f$ by

$$F(M, N) = P^r M[T^\Phi N, i^\Phi_N], \quad F(f, g) = P^r f[T^\Phi g], \tag{6}$$

where i^Φ is the action of G^r_m on Φ. Clearly,

$$J^r(M, N) \subset F(M, N) \subset \widetilde{J}^r(M, N). \tag{7}$$

Conversely, if F is a bundle functor on $\mathcal{M}f_m \times \mathcal{M}f$ satisfying (7) and preserving products in the second factor, then F is determined by a Weil algebra Φ of the above type, [3].

Using the Weil algebra technique, [4], we deduced that the only nonholonomic 2-jet functors on $\mathcal{M}f_m \times \mathcal{M}f$ are J^2, $\overline{J}^2$ and $\widetilde{J}^2$.

However, this model does not includes the composition of jets. That is why we have recently introduced the general concept of nonholonomic r-jet category C, [6]. In Section 2 of the present paper, we describe C in terms of its skeleton. Then we deduce some algebraic properties of the algebra $\widetilde{\mathbb{D}}^r_m$ and we characterize C in terms of the induced sequence $\mathbb{D}^C_m \subset \widetilde{\mathbb{D}}^r_m$ of Weil algebras. Our above mentioned result from [4] implies directly that the only nonholonomic 2-jet categories are J^2, $\overline{J}^2$ and $\widetilde{J}^2$, see Example 2 below. However, there are so many nonholonomic 3-jet categories that we do not find it reasonable to classify all of them without further reasons. So we restrict ourselves to the semiholonomic 3-jet categories and we classify them in Section 4.

2. Nonholonomic r-jet categories

We recall that $X \in \widetilde{J}^r_x(M,N)_y$ is said to be regular if there exists $Z \in \widetilde{J}^r_y(N,M)_x$ such that $Z \circ X = j^r_x \operatorname{id}_M$, [6].

In [6], we introduced a nonholonomic r-jet category C as a rule transforming every pair (M,N) of manifolds into a fibered submanifold $C(M,N) \subset \widetilde{J}^r(M,N)$ such that

(i) $J^r(M,N) \subset C(M,N)$ is a fibered submanifold,

(ii) if $X \in C_x(M,N)_y$ and $Z \in C_y(N,Q)_z$, then $Z \circ X \in C_x(M,Q)_z$,

(iii) if $X \in C_x(M,N)_y$ is regular in $\widetilde{J}^r(M,N)$, then there exists $Z \in C_y(N,M)_x$ such that $Z \circ X = j^r_x \operatorname{id}_M$,

(iv) $C(M, N \times Q) = C(M,N) \times_M C(M,Q)$.

Analogously to the case of J^r, [8], we define $L^C_{m,n} = C_0(\mathbb{R}^m, \mathbb{R}^n)_0$ and

$$L^C = \bigcup_{m,n\in\mathbb{N}} L^C_{m,n}$$

is called the skeleton of C. Clearly, we can reconstruct C from L^C in the same way as in the case of J^r, [8]. We have a left action of $G^r_m \times G^r_n$ on $L^C_{m,n}$

$$(g_1, g_2)(X) = g_2 \circ X \circ g_1^{-1}, \quad g_1 \in G^r_m,\ g_2 \in G^r_n,\ X \in L^C_{m,n} \tag{8}$$

and $C(M,N)$ coincides with the associated bundle

$$C(M,N) = (P^rM \times P^rN)[L^C_{m,n}]. \tag{9}$$

We define $T^C_m N = C_0(\mathbb{R}^m, N)$. This gives rise to a product preserving bundle functor on $\mathcal{M}f$, so a Weil functor $T^{\mathbb{D}^C_m}$, $\mathbb{D}^r_m \subset \mathbb{D}^C_m$. Clearly, each $\mathbb{D}^C_m$ is a G^r_m-invariant Weil subalgebra of $\widetilde{\mathbb{D}}^r_m$. We are going to clarify how C can be determined by such a sequence.

3. Some algebraic properties of $\widetilde{\mathbb{D}}^r_m$

By the iteration theorem for Weil bundles, [5], we have

$$\widetilde{\mathbb{D}}^r_m \approx \underbrace{\mathbb{D}^1_m \otimes \ldots \otimes \mathbb{D}^1_m}_{r\text{-times}}, \quad \mathbb{D}^1_m = \mathbb{R} \times \mathbb{R}^{m*}. \tag{10}$$

Write e^i_s, $i = 1, \ldots, m$, $s = 1, \ldots, r$ for the canonical basis of $\mathbb{R}^{m*}$ and $e^0_s = 1_s$ for the unit in the s-th component of (10). For a sequence $k_1, \ldots, k_r$ of $0, 1, \ldots, m$, we define

$$e^{k_1 \ldots k_r} = e^{k_1}_1 \otimes \ldots \otimes e^{k_r}_r. \tag{11}$$

This is a basis of the vector space $\widetilde{\mathbb{D}}^r_m$, so that every $X \in \widetilde{\mathbb{D}}^r_m$ is of the form $X = x_{k_1 \ldots k_r} e^{k_1 \ldots k_r}$. The multiplication in $\widetilde{\mathbb{D}}^r_m$ is determined by

$$e^{k_1 \ldots k_r} e^{l_1 \ldots l_r} = e^{h_1 \ldots h_r}, \tag{12}$$

where $e^{h_1 \ldots h_r} = 0$ if $k_s \neq 0 \neq l_s$ for at least one s and $h_s = k_s + l_s$ otherwise.

Write $\langle k_1 \ldots k_r \rangle = (i_1 \ldots i_s)$, $s \leqslant r$, for the subsequence of all nonzero indices and $|k_1 \ldots k_r|$ for the set $\{i_1, \ldots, i_s\}$. The semiholonomic subalgebra $\overline{\mathbb{D}}^r_m = \overline{J}^r_0(\mathbb{R}^m, \mathbb{R})$ is characterized by

$$x_{k_1 \ldots k_r} = x_{l_1 \ldots l_r} \quad \text{whenever } \langle k_1 \ldots k_r \rangle = \langle l_1 \ldots l_r \rangle \tag{13}$$

and the holonomic subalgebra $\mathbb{D}^r_m$ satisfies

$$x_{k_1 \ldots k_r} = x_{l_1 \ldots l_r} \quad \text{whenever } |k_1 \ldots k_r| = |l_1 \ldots l_r|. \tag{14}$$

In the holonomic case, a simple assertion is that the set of all Weil algebra homomorphisms $\mathrm{Hom}(\mathbb{D}^r_m, \mathbb{D}^r_n)$ coincides with $L^r_{n,m}$, [5]. This identification is a special case of the following construction.

Proposition 1. *For every $Z \in \widetilde{L}^r_{n,m}$ the rule*

$$Z^h(X) = X \circ Z, \quad X \in \widetilde{\mathbb{D}}^r_m \tag{15}$$

defines a Weil algebra homomorphism $Z^h \colon \widetilde{\mathbb{D}}^r_m \to \widetilde{\mathbb{D}}^r_n$.

Proof. A quick proof is based on a general result concerning Weil bundles, [5], [8]. Consider the bundle functors $\widetilde{T}^r_m$ and $\widetilde{T}^r_n$ on $\mathcal{M}f$. For $f \colon Q \to Q'$ and $X \in (\widetilde{T}^r_m Q)_x$, we have $\widetilde{T}^r_m f(X) = j^r_x f \circ X$. Since the composition of nonholonomic jets in associative, we have $(\widetilde{T}^r_m f(X)) \circ Z = (j^r_x f) \circ X \circ Z = \widetilde{T}^r_n f(X \circ Z)$, so that Z induces a natural transformation $\widetilde{T}^r_m \to \widetilde{T}^r_n$. These are determined by the algebra homomorphisms $\widetilde{\mathbb{D}}^r_m \to \widetilde{\mathbb{D}}^r_n$. $\square$

Write $\widetilde{\mathbb{D}}^r_m = \mathbb{R} \times \widetilde{N}^r_m$, so that $\widetilde{N}^r_m = \widetilde{J}^r_0(\mathbb{R}^m, \mathbb{R})_0$. Since $\widetilde{J}^r$ preserves products in the second factor, we have $\widetilde{L}^r_{m,n} = \widetilde{J}^r_0(\mathbb{R}^m, \mathbb{R}^n)_0 = (\widetilde{N}^r_m)^n$. Analogously, $\overline{L}^r_{m,n} := \overline{J}^r_0(\mathbb{R}^m, \mathbb{R})_0 = (\overline{N}^r_m)^n$ and $L^r_m = J^r_0(\mathbb{R}^m, \mathbb{R}^n)_0 = (N^r_m)^n$ with $\overline{\mathbb{D}}^r_m = \mathbb{R} \times \overline{N}^r_m$ and $\mathbb{D}^r_m = \mathbb{R} \times N^r_m$.

Proposition 2. *We have* $\mathrm{Hom}(\overline{\mathbb{D}}^r_m, \overline{\mathbb{D}}^r_n) = \overline{L}^r_{n,m}$.

Proof. Consider an algebra homomorphism $\varphi\colon \overline{\mathbb{D}}^r_m \to \overline{\mathbb{D}}^r_n$. The algebraic generators of $\overline{N}^r_m$ are $e^i := e^{i0\ldots0} + \ldots + e^{0\ldots0i}$. Write $\varphi^i = \varphi(e^i) \in \overline{N}^r_n$, so that $\Phi := (\varphi^1, \ldots \varphi^m) \in \overline{L}^r_{n,m}$. Then the algebra homomorphism Φ^h coincides with φ on the algebraic generators, so that $\varphi = \Phi^h$. □

Example 1. Direct evaluation in the case $r = 2$ shows that $\widetilde{L}^2_{m,n}$ is a proper subset of $\mathrm{Hom}(\widetilde{\mathbb{D}}^2_n, \widetilde{\mathbb{D}}^2_m)$ only. Indeed, if we consider the standard coordinate expressions $a = (a^p_{i0}, a^p_{0i}, a^p_{ij}) \in \widetilde{L}^2_{m,n}$ and $b = (b^v_{p0}, b^v_{0p}, b^v_{pq}) \in \widetilde{L}^2_{n,p}$ the composition $c = b \circ a = (c^v_{i0}, c^v_{0i}, c^v_{ij}) \in \widetilde{L}^2_{m,p}$, i, $j = 1, \ldots, m$, p, $q = 1, \ldots, n$, $v = 1, \ldots, p$, is of the form

$$
\begin{aligned}
c^v_{i0} &= b^v_{p0} a^p_{i0}, \quad c^v_{0i} = b^v_{0p} a^p_{0i}, \\
c^v_{ij} &= b^v_{pq} a^p_{i0} a^q_{0j} + b^v_{p0} a^p_{ij}.
\end{aligned}
\tag{16}
$$

Thus, for $x = (x_{i0}, x_{0i}, x_{ij}) \in \widetilde{N}^2_m$ and $a \in \widetilde{L}^2_{n,m}$, we have

$$
a^h(x) = x \circ a = (x_{i0} a^i_{p0}, x_{0i} a^i_{0p}, x_{ij} a^i_{p0} a^j_{0q} + x_{i0} a^i_{pq}).
\tag{17}
$$

On the other hand, an algebra homomorphism $f\colon \widetilde{\mathbb{D}}^2_m \to \widetilde{\mathbb{D}}^2_n$ is determined by

$$
\begin{aligned}
f(e^{i0}) &= d^{i0}_{p0} e^{p0} + d^{i0}_{0p} e^{0p} + d^{i0}_{pq} e^{pq}, \\
f(e^{0i}) &= d^{0i}_{p0} e^{p0} + d^{0i}_{0p} e^{0p} + d^{0i}_{pq} e^{pq}.
\end{aligned}
\tag{18}
$$

Then

$$
f(e^{ij}) = f(e^{i0} e^{0j}) = (d^{i0}_{p0} d^{0j}_{0q} + d^{i0}_{0q} d^{0j}_{p0}) e^{pq}.
\tag{19}
$$

By direct evaluation, we find $f(x)$ in the form

$$
\begin{aligned}
&(x_{i0} d^{i0}_{p0} + x_{0i} d^{0i}_{p0}) e^{p0} + (x_{i0} d^{i0}_{0p} + x_{0i} d^{0i}_{0p}) e^{0p} \\
&\qquad + [x_{i0} d^{i0}_{pq} + x_{0i} d^{0i}_{pq} + x_{ij} (d^{i0}_{p0} d^{0j}_{0q} + d^{i0}_{0q} d^{0j}_{p0})] e^{pq}.
\end{aligned}
\tag{20}
$$

Clearly, (20) reduces to (17) iff $d^{0i}_{p0} = 0$, $d^{i0}_{0p} = 0$, $d^{0i}_{pq} = 0$.

Consider the immersion $i_{m,n}\colon \mathbb{R}^m \to \mathbb{R}^{m+n}$, $x \mapsto (x,0)$, and the submersion $s_{m,n}\colon \mathbb{R}^{m,n} \to \mathbb{R}^m$, $(x_1,x_2) \mapsto x_1$, and write $I^r_{m,n} = j^r_0 i_{m,n}$. Since $s_{m,n}\circ i_{m,n} = \mathrm{id}_{\mathbb{R}^m}$, the induced algebra homomorphism $(I^r_{m,n})^h\colon \widetilde{\mathbb{D}}^r_{m+n} \to \widetilde{\mathbb{D}}^r_m$ is surjective. One verifies directly that its coordinate expression is

$$\bar{x}_{k_1\dots k_r} = x_{k_1\dots k_r} \tag{21}$$

with no appearance of $x_{q_1\dots q_r}$ with at least one q_s greater than m, $q_s = 0,1,\dots,m+n$, on the right hand side.

Let $\mathbb{D}^C_m$ be the sequence of Weil algebras determined by a nonholonomic r-jet category C. Then $I^r_{m,n}$ induces a restricted and corestricted algebra homomorphism

$$I^C_{m,n}\colon \mathbb{D}^C_{m+n} \to \mathbb{D}^C_m, \quad I^C_{m,n}(\mathbb{D}^C_{m+n}) = \mathbb{D}^C_m, \tag{22}$$

whose coordinate expression is of the form (21).

Consider now an arbitrary sequence $\mathbb{D}^S_m$ of Weil algebras, $\mathbb{D}^r_m \subset \mathbb{D}^S_m \subset \widetilde{\mathbb{D}}^r_m$, $\mathbb{D}^S_m = \mathbb{R}\times N^S_m$, and write

$$L^S_{m,n} = (N^S_m)^n, \quad L^S = \bigcup_{m,n} L^S_{m,n}. \tag{23}$$

Hence $L^S_{m,n} \subset \widetilde{L}^r_{m,n}$.

Definition 1. The sequence $\mathbb{D}^S_m$ is called admissible, if L^S is a subcategory of $\widetilde{L}^r$.

Proposition 3. *A sequence $\mathbb{D}^S_m$ is determined by a nonholonomic r-jet category C, if and only if it is admissible.*

Proof. For an admissible sequence $\mathbb{D}^S_m$, we define

$$C(M,N) = (P^rM \times P^rN)[L^S_{m,n}].$$

For $X_1 \in C_x(M,N)_y$ and $X_2 \in C_y(N,Q)_z$, $X_1 = \{u,v,\xi_1\}$, $X_2 = \{v,w,\xi_2\}$, $u \in P^r_xM$, $v \in P^r_yN$, $w \in P^r_zQ$, $\xi_1 \in L^S_{m,n}$, $\xi_2 \in L^S_{n,p}$, we set

$$X_2 \circ X_1 = \{u, w, \xi_2 \circ \xi_1\}$$

with composition in L^S on the right hand side. One verifies directly that C has all required properties. □

In particular, if $\mathbb{D}^S_m$ is an admissible sequence, then $I^r_{m,n}$ maps $\mathbb{D}^S_{m+n}$ onto $\mathbb{D}^S_m$. Further, since G^r_m acts on $(N^S_m)^n$ fiberwise, every algebra $\mathbb{D}^S_m$ is G^r_m-invariant.

Thus, in order to find all nonholonomic r-jet categories, we can proceed in the following way.

(i) We determine all G^r_m-invariant Weil algebras $\mathbb{D}^r_m \subset \mathbb{D}^S_m \subset \widetilde{\mathbb{D}}^r_m$ for every m.
(ii) We restrict ourselves to the sequences satisfying (22).
(iii) We analyze under what conditions (23) is a subcategory of $\widetilde{L}^r$.

Example 2. In [4], we deduced that all G^2_m-invariant subalgebras of $\widetilde{\mathbb{D}}^2_m$ are $\mathbb{D}^2_m$, $\overline{\mathbb{D}}^2_m$, and $\widetilde{\mathbb{D}}^2_m$. The sequences satisfying (22) are $\mathbb{D}^2_m$, $\overline{\mathbb{D}}^2_m$ and $\widetilde{\mathbb{D}}^2_m$, $m \in \mathbb{N}$. They determine the categories J^2, $\overline{J}^2$ and $\widetilde{J}^2$.

4. Semiholonomic 3-jet categories

A nonholonomic r-jet category C is called semiholonomic, if $C(M,N) \subset \overline{J}^r(M,N)$ for all M and N. We are going to describe the semiholonomic 3-jet categories. In the course of direct evaluations, we use the coordinate formula for the composition of semiholonomic 3-jets. In the coordinates determined by (5), if $a = (a^p_i, a^p_{ij}, a^p_{ijk}) \in \overline{L}^3_{m,n}$ and $b = (b^v_p, b^v_{pq}, b^v_{pqr}) \in \overline{L}^3_{n,p}$, then $c = b \circ a = (c^v_i, c^v_{ij}, c^v_{ijk}) \in \overline{L}^3_{m,p}$ is of the form

$$
\begin{aligned}
&c^v_i = b^v_p a^p_i, \quad c^v_{ij} = b^v_{pq} a^p_i a^q_j + b^v_p a^p_{ij},\\
&c^v_{ijk} = b^v_{pqr} a^p_i a^q_j a^r_k + b^v_{pq} a^p_{ik} a^q_j + b^v_{pq} a^p_i a^q_{jk} + b^v_{pq} a^p_{ij} a^q_k + b^v_p a^p_{ijk}.
\end{aligned} \tag{24}
$$

Lemma 1. *The only subalgebra $A \subset \overline{\mathbb{D}}^3_m$ satisfying $\pi^3_2(A) = \overline{\mathbb{D}}^2_m$ is $\overline{\mathbb{D}}^3_m$.*

P r o o f. We prove that the kernel of the induced map $\overline{N}^3_m \to \overline{N}^2_m$ is $\overset{3}{\otimes}\mathbb{R}^{m*}$. Indeed, we deduce directly by (24) that the coordinate expression of the product in $\overline{\mathbb{D}}^3_m$ of x, $y \in \overline{N}^3_m$, $z = xy$, is

$$
\begin{aligned}
&z_i = 0, \quad z_{ij} = x_i y_j + x_j y_i,\\
&z_{ijk} = x_{ij} y_k + x_{ik} y_j + x_i y_{jk} + x_{jk} y_i + x_j y_{ik} + x_k y_{ij}.
\end{aligned} \tag{25}
$$

Hence the tensor Z_{ijk} with $z_{ijk} = 1$ and all other coordinates equal to zero is obtained by multiplying $X_{ij} \in \overline{N}^3_m$ and $Y_k \in \overline{N}^3_m$, where the first and second order components of X_{ij} are $x_{ij} = 1$ and zero otherwise and the first and second order components of Y_k are $y_k = 1$ and zero otherwise. □

In [4] we studied the bundles

$$\overline{J}^{r,r-1}(M,N) = \{X \in \overline{J}^r(M,N), \pi^r_{r-1}(X) \in J^{r-1}(M,N)\}$$

of semiholonomic r-jets that are holonomic up to the order $r-1$. Already in [2] we deduced that for every $X \in \overline{J}^{r,r-1}_x(M,N)_y$ there exists a unique $\sigma(X) \in J^r_x(M,N)_y$ satisfying

$$\sigma(X) \circ U = X \circ U \in (T^r_1 N)_y \quad \text{for all} \quad U \in (T^r_1 M)_x.$$

The difference $X - \sigma(X)$ is a well defined element of $T_y N \otimes \overset{r}{\otimes} T^*_x M$. This identifies $\overline{J}^{r,r-1}(M,N)$ with the fiber product over $M \times N$

$$J^r(M,N) \times_{M\times N} TN \otimes (\overset{r}{\otimes} T^*M / S^r T^*M).$$

In the case $\overline{\mathbb{D}}^{r,r-1}_m = \overline{J}^{r,r-1}_0(\mathbb{R}^m, \mathbb{R})$, we obtain

$$\overline{\mathbb{D}}^{r,r-1}_m = \mathbb{D}^r_m \times V, \quad V := \overset{r}{\otimes}\mathbb{R}^{m*}/S^r\mathbb{R}^{m*}. \tag{26}$$

The action of G^r_m on $\overline{\mathbb{D}}^{r,r-1}_m$ is

$$X \circ g = (\sigma(X) \circ g, l(g_1)(X - \sigma(X))), \tag{27}$$

where $l(g_1)$ denotes the standard action of $g_1 = \pi^r_1(g) \in GL(m,\mathbb{R})$ on V. This implies easily the following assertion from [4].

Lemma 2. *The G^r_m-invariant Weil algebras $\mathbb{D}^r_m \subset A \subset \overline{\mathbb{D}}^{r,r-1}_m$ are of the form $A = \mathbb{D}^r_m \times L$, where L is a $GL(m,\mathbb{R})$-invariant linear subspace of $\overset{r}{\otimes}\mathbb{R}^{m*}$ containing $S^r\mathbb{R}^{m*}$.*

Further, using the formulae from [4], one deduces directly the following assertion.

Lemma 3. *Let $A' = \mathbb{D}^r_m \times L'$ be another such algebra. Then the G^r_m-invariant algebra homomorphisms $A \to A'$ are in bijection with the $GL(m,\mathbb{R})$-invariant linear maps $L \to L'$.*

Going back to the case $r = 3$, Lemma 1 implies that we can restrict ourselves to the bundles $\overline{J}^{3,2}(M,N)$. In [10], G. Vosmanská deduced that all natural transformations $\overline{J}^{3,2} \to \overline{J}^{3,2}$ over the identity of J^2 form a 5-parameter family Ψ. Its coordinate expression is

$$\begin{aligned} &\bar a^p_i = a^p_i, \quad \bar a^p_{ij} = a^p_{ij} \quad \text{with } a^p_{ij} = a^p_{ji}, \\ &\bar a^p_{ijk} = a^p_{ijk} + c_1(a^p_{ikj} - a^p_{ijk}) + c_2(a^p_{jik} - a^p_{ijk}) \\ &\qquad + c_3(a^p_{jki} - a^p_{ijk}) + c_4(a^p_{kij} - a^p_{ijk}) + c_5(a^p_{kji} - a^p_{ijk}). \end{aligned} \tag{28}$$

We introduce $\overline{J}_h^{2,3}Y = J_h^1(J_h^2 Y) \cap \overline{J}_h^3 Y$ and $\overline{J}^{2,3}(M,N) = \overline{J}_h^{2,3}(M \times N \to M)$. In coordinates, $\overline{J}^{2,3}(M,N)$ is characterized by

$$a^p_{ij} = a^p_{ji}, \quad a^p_{ijk} = a^p_{jik}, \tag{29}$$

so that $\overline{J}^{2,3}(M,N) \subset \overline{J}^{3,2}(M,N)$. By (24), $\overline{J}^{2,3}$ is a semiholonomic 3-jet category. Further, for every $\psi \in \Psi$, $(\psi \circ \overline{J}^{2,3})(M,N) \subset \overline{J}^{3,2}(M,N)$ is a fibered submanifold and (24) implies that every $\psi \circ \overline{J}^{2,3}$ is a semiholonomic 3-jet category.

If we consider an invariant tensor of degree 3 interpreted as a linear map $\iota\colon \overset{3}{\otimes}\mathbb{R}^{m*} \to \overset{3}{\otimes}\mathbb{R}^{m*}$ and assume it vanishes on $S^3\mathbb{R}^{m*}$, then the kernel of ι determines an invariant subspace of $V = \overset{3}{\otimes}\mathbb{R}^{m*}/S^3\mathbb{R}^{m*}$. By the Invariant tensor theorem, [8], all invariant tensors of degree 3 form a 6-parameter family

$$d_1 x_{ijk} + d_2 x_{ikj} + d_3 x_{jik} + d_4 x_{jki} + d_5 x_{kij} + d_6 x_{kji} \tag{30}$$

and vanishing on $S^3\mathbb{R}^{m*}$ means

$$d_1 + d_2 + d_3 + d_4 + d_5 + d_6 = 0.$$

Hence (30) and (31) determine a 5-parameter family of invariant subspaces of V. According to the representation theory, every invariant subspace L satisfying $S^3\mathbb{R}^* \subset L \subset \overset{3}{\otimes}\mathbb{R}^{m*}$ is one of this family.

Hence we can formulate our classification result as follows.

Proposition 4. *All semiholonomic* 3*-jet categories are* $\overline{J}^3$, $\overline{J}^{3,2}$, J^3 *and* $\psi \circ \overline{J}^{2,3}$ *for all* $\psi \in \Psi$.

Example 3. There is an interesting problem to geometrize the semiholonomic 3-jet categories of the form $\psi \circ \overline{J}^{2,3}$, $\psi \in \Psi$. The simpliest case is $x_{ijk} = x_{ikj}$. This corresponds to the functor $J_h^2(J_h^1 Y) \cap \overline{J}_h^3 Y$ restricted to the product fibered manifolds $M \times N \to M$.

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