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THE DIRICHLET BOUNDARY VALUE PROBLEMS FOR STRONGLY SINGULAR HIGHER-ORDER NONLINEAR FUNCTIONAL-DIFFERENTIAL EQUATIONS

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Abstract. The a priori boundedness principle is proved for the Dirichlet boundary value problems for strongly singular higher-order nonlinear functional-differential equations. Several sufficient conditions of solvability of the Dirichlet problem under consideration are derived from the a priori boundedness principle. The proof of the a priori boundedness principle is based on the Agarwal-Kiguradze type theorems, which guarantee the existence of the Fredholm property for strongly singular higher-order linear differential equations with argument deviations under the two-point conjugate and right-focal boundary conditions.

Keywords: higher order functional-differential equation, Dirichlet boundary value problem, strong singularity, Fredholm property, a priori boundedness principle

MSC 2010: 34K06, 34K10

1. Statement of the main results

1.1. Statement of the problem and a survey of the literature. Consider the functional differential equation

$$u^{(n)}(t) = F(u)(t) \tag{1.1}$$

with the two-point boundary conditions

$$u^{(i-1)}(a) = 0 \ \ (i = 1, \ldots, m), \quad u^{(i-1)}(b) = 0 \ \ (i = 1, \ldots, n - m). \tag{1.2}$$

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Here $n \geqslant 2$, m is the integer part of $n/2$, $-\infty < a < b < +\infty$, and the operator F is acting from the set of $(m-1)$-th time continuously differentiable on $]a,b[$ functions to the set $L_{\mathrm{loc}}(]a,b[)$. By $u^{(j-1)}(a)$ $(u^{(j-1)}(b))$ we denote the right (the left) limit of the function $u^{(j-1)}$ at the point $a(b)$.

The problem is singular in the sense that for an arbitrary x the right-hand side of equation (1.41) may have nonintegrable singularities at the points a and b.

Throughout the paper we use the following notation:

- ▷ $\mathbb{R}^+ = [0, +\infty[$;
- ▷ $[x]_+$ the positive part of a number x, that is $[x]_+ = \frac{1}{2}(x + |x|)$;
- ▷ $L_{\mathrm{loc}}(]a,b[)(L_{\mathrm{loc}}(]a,b]))$ is the space of functions $y\colon\ ]a,b[\ \to \mathbb{R}$, which are integrable on $[a+\varepsilon, b-\varepsilon]$ for arbitrarily small $\varepsilon > 0$;
- ▷ $L_{\alpha,\beta}(]a,b[)(L^2_{\alpha,\beta}(]a,b[))$ is the space of integrable (square integrable) with the weight $(t-a)^\alpha(b-t)^\beta$ functions $y\colon\ ]a,b[\ \to \mathbb{R}$, with the norm

$$\|y\|_{L_{\alpha,\beta}} = \int_a^b (s-a)^\alpha (b-s)^\beta |y(s)|\,\mathrm{d}s$$
$$\left(\|y\|_{L^2_{\alpha,\beta}} = \left(\int_a^b (s-a)^\alpha (b-s)^\beta y^2(s)\,\mathrm{d}s\right)^{1/2}\right);$$

- ▷ $L([a,b]) = L_{0,0}(]a,b[), L^2([a,b]) = L^2_{0,0}(]a,b[)$;
- ▷ $M(]a,b[)$ is the set of measurable functions $\tau\colon\ ]a,b[\ \to\ ]a,b[$;
- ▷ $\widetilde{L}^2_{\alpha,\beta}(]a,b[)(\widetilde{L}^2_{\alpha}(]a,b]))$ is the Banach space of $y \in L_{\mathrm{loc}}(]a,b[)(L_{\mathrm{loc}}(]a,b]))$ functions, with the norm

$$\|y\|_{\widetilde{L}^2_{\alpha,\beta}} \equiv \max\left\{\left[\int_a^t (s-a)^\alpha \left(\int_s^t y(\xi)\,\mathrm{d}\xi\right)^2 \mathrm{d}s\right]^{1/2} \colon\ a \leqslant t \leqslant \frac{a+b}{2}\right\}$$
$$+ \max\left\{\left[\int_t^b (b-s)^\beta \left(\int_t^s y(\xi)\,\mathrm{d}\xi\right)^2 \mathrm{d}s\right]^{1/2} \colon\ \frac{a+b}{2} \leqslant t \leqslant b\right\} < +\infty.$$

- ▷ $L_n(]a,b[)$ is the Banach space of $y \in L_{\mathrm{loc}}(]a,b[)$ functions, with the norm

$$\|y\|_{\widetilde{L}^2_{\alpha,\beta}} = \sup\left\{[(s-a)(b-t)]^{m-1/2}\int_s^t (\xi-a)^{n-2m}|y(\xi)|\,\mathrm{d}\xi\colon\ a < s \leqslant t < b\right\} < +\infty.$$

- ▷ $C^{n-1}_{\mathrm{loc}}(]a,b[), (\widetilde{C}^{n-1}_{\mathrm{loc}}(]a,b[))$ is the space of functions $y\colon\ ]a,b[\ \to \mathbb{R}$ which are continuous (absolutely continuous) together with $y', y'', \ldots, y^{(n-1)}$ on $[a+\varepsilon, b-\varepsilon]$ for arbitrarily small $\varepsilon > 0$.
- ▷ $\widetilde{C}^{n-1,m}(]a,b[)$ is the space of functions $y \in \widetilde{C}^{n-1}_{\mathrm{loc}}(]a,b[)$, such that

$$\int_a^b |x^{(m)}(s)|^2\,\mathrm{d}s < +\infty. \tag{1.3}$$

▷ $C_1^{m-1}(]a,b[)$ is the Banach space of functions $y \in C_{\rm loc}^{m-1}(]a,b[)$, such that

$$\limsup_{t\to a} \frac{|x^{(i-1)}(t)|}{(t-a)^{m-i+1/2}} < +\infty \quad (i=1,\dots,m), \tag{1.4}$$
$$\limsup_{t\to b} \frac{|x^{(i-1)}(t)|}{(b-t)^{m-i+1/2}} < +\infty \quad (i=1,\dots,n-m),$$

with the norm:

$$\|x\|_{C_1^{m-1}} = \sum_{i=1}^{m} \sup\Big\{\frac{|x^{(i-1)}(t)|}{\alpha_i(t)} \colon\ a<t<b\Big\},$$

where $\alpha_i(t) = (t-a)^{m-i+1/2}(b-t)^{m-i+1/2}$.

▷ $\widetilde{C}_1^{m-1}(]a,b[)$ is the Banach space of functions $y \in \widetilde{C}_{\rm loc}^{m-1}(]a,b[)$, such that conditions (1.3) and (1.4) hold, with the norm:

$$\|x\|_{\widetilde{C}_1^{m-1}} = \sum_{i=1}^{m} \sup\Big\{\frac{|x^{(i-1)}(t)|}{\alpha_i(t)} \colon\ a<t<b\Big\} + \Big(\int_a^b |x^{(m)}(s)|^2\,{\rm d}s\Big)^{1/2}.$$

▷ $D_n(]a,b[\times\mathbb{R}^+)$ is the set of such functions $\delta\colon\ ]a,b[\times\mathbb{R}^+ \to L_n(]a,b[)$ that $\delta(t,\cdot)\colon\ \mathbb{R}^+\to\mathbb{R}^+$ is nondecreasing for every $t\in\ ]a,b[$, and $\delta(\cdot,\varrho)\in L_n(]a,b[)$ for any $\varrho\in\mathbb{R}^+$.

▷ $D_{2n-2m-2,2m-2}(]a,b[\times\mathbb{R}^+)$ is the set of such functions $\delta\colon\ ]a,b[\times\mathbb{R}^+ \to \widetilde{L}^2_{2n-2m-2,2m-2}(]a,b[)$ that $\delta(t,\cdot)\colon\ \mathbb{R}^+\to\mathbb{R}^+$ is nondecreasing for every $t\in\ ]a,b[$, and $\delta(\cdot,\varrho)\in \widetilde{L}^2_{2n-2m-2,2m-2}(]a,b[)$ for any $\varrho\in\mathbb{R}^+$.

▷ A solution of problem (1.1), (1.2) is sought in the space $\widetilde{C}^{n-1,m}(]a,b[)$.

The singular ordinary differential and functional-differential equations have been studied with sufficient completeness under different boundary conditions, see for example [1], [3], [4], [5], [6], [7], [8], [9], [11], [12], [13], [14], [16], [21], [22], [23], [24], [25] and the references cited therein. But the equation (1.1), even under the boundary condition (1.2), have not been studied in the case when the operator F has the form

$$F(x)(t) = \sum_{j=1}^{m} p_j(t)x^{(j-1)}(\tau_j(t)) + f(x)(t), \tag{1.5}$$

where the singularity of the functions $p_j\colon\ L_{\rm loc}([a,b])$ is such that the inequalities

$$\int_a^b (s-a)^{n-1}(b-s)^{2m-1}[(-1)^{n-m}p_1(s)]_+\,{\rm d}s < +\infty, \tag{1.6}$$
$$\int_a^b (s-a)^{n-j}(b-s)^{2m-j}|p_j(s)|\,{\rm d}s < +\infty \quad (j=2,\dots,m),$$

are not fulfilled (in this case we say that the linear part of the operator F is strongly singular), the operator f is continuously acting from $C_1^{m-1}(]a,b[)$ to $L_{\widetilde{L}^2_{2n-2m-2,2m-2}}(]a,b[)$, and the inclusion

$$\sup\{f(x)(t)\colon \|x\|_{C_1^{m-1}} \leqslant \varrho\} \in \widetilde{L}^2_{2n-2m-2,2m-2}(]a,b[) \tag{1.7}$$

holds. The first step in studying the differential equations with strong singularities was made by R. P. Agarwal and I. Kiguradze in the article [2], where the linear ordinary differential equations under conditions (1.2), in the case when the functions p_j have strong singularities at the points a and b, are studied. Also the ordinary differential equations with strong singularities under two-point boundary conditions are studied in the articles of I. Kiguradze [10], [19], and N. Partsvania [20]. In the papers [18], [15] these results are generalized to linear differential equations with deviating arguments, i.e., the Agarwal-Kiguradze type theorems, which guarantee Fredholm's property for linear differential equations with deviating arguments are proved.

In this paper, on the bases of articles [2] and [17] we prove the a priori boundedness principle for the problem (1.1), (1.2) in the case when the operator has the form (1.5).

Now we introduce some results from the articles [18], [15], which we need for this work. Consider the equation

$$u^{(n)}(t) = \sum_{j=1}^{m} p_j(t)u^{(j-1)}(\tau_j(t)) + q(t) \quad \text{for } a < t < b. \tag{1.8}$$

For problem (1.8), (1.2) we assume, that when $n = 2m$, then the conditions

$$p_j \in L_{\mathrm{loc}}(]a,b[) \quad (j = 1,\ldots,m) \tag{1.9}$$

are fulfilled and when $n = 2m+1$, along with (1.9), the condition

$$\limsup_{t\to b} \left|(b-t)^{2m-1}\int_{t_1}^{t} p_1(s)\,\mathrm{d}s\right| < +\infty \quad \Bigl(t_1 = \frac{a+b}{2}\Bigr) \tag{1.10}$$

holds.

By $h_j\colon\ ]a,b[\times]a,b[\ \to \mathbb{R}_+$ and $f_j\colon [a,b]\times M(]a,b[) \to C_{\mathrm{loc}}(]a,b[\times]a,b[)$ $(j = 1,\ldots,m)$ we denote the functions and operators, respectively, defined by the equalities

$$\begin{aligned} h_1(t,s) &= \left|\int_s^t (\xi-a)^{n-2m}[(-1)^{n-m}p_1(\xi)]_+\,\mathrm{d}\xi\right|, \\ h_j(t,s) &= \left|\int_s^t (\xi-a)^{n-2m}p_j(\xi)\,\mathrm{d}\xi\right| \quad (j = 2,\ldots,m), \end{aligned} \tag{1.11}$$

and

$$(1.12)\qquad f_j(c,\tau_j)(t,s) = \left| \int_s^t (\xi-a)^{n-2m} |p_j(\xi)| \left| \int_\xi^{\tau_j(\xi)} (\xi_1 - c)^{2(m-j)} \,\mathrm{d}\xi_1 \right|^{1/2} \mathrm{d}\xi \right|.$$

Let $k = 2k_1 + 1$ ($k_1 \in \mathbb{N}$), then we denote

$$k!! = \begin{cases} 1 & \text{for } k \leqslant 0, \\ 1 \cdot 3 \cdot 5 \cdot \ldots \cdot k & \text{for } k \geqslant 1. \end{cases}$$

Now we can introduce the main theorem of the paper [18].

Theorem 1.1. *Let there exist numbers $t^* \in \,]a,b[$, $l_{kj} > 0$, $\bar{l}_{kj} \geqslant 0$, and $\gamma_{kj} > 0$ ($k = 0,1$; $j = 1,\ldots,m$) such that along with*

$$(1.13)\quad B_0 \equiv \sum_{j=1}^{m} \Big(\frac{(2m-j)2^{2m-j+1} l_{0j}}{(2m-1)!!(2m-2j+1)!!} + \frac{2^{2m-j-1}(t^*-a)^{\gamma_{0j}} \bar{l}_{0j}}{(2m-2j-1)!!(2m-3)!!\sqrt{2\gamma_{0j}}} \Big) < \frac{1}{2},$$

$$(1.14)\quad B_1 \equiv \sum_{j=1}^{m} \Big(\frac{(2m-j)2^{2m-j+1} l_{1j}}{(2m-1)!!(2m-2j+1)!!} + \frac{2^{2m-j-1}(b-t^*)^{\gamma_{0j}} \bar{l}_{1j}}{(2m-2j-1)!!(2m-3)!!\sqrt{2\gamma_{1j}}} \Big) < \frac{1}{2},$$

the conditions

$$(1.15)\qquad (t-a)^{2m-j} h_j(t,s) \leqslant l_{0j}, \quad (t-a)^{m-\gamma_{0j}-1/2} f_j(a,\tau_j)(t,s) \leqslant \bar{l}_{0j}$$

for $a < t \leqslant s \leqslant t^*$, and

$$(1.16)\qquad (b-t)^{2m-j} h_j(t,s) \leqslant l_{1j}, \quad (b-t)^{m-\gamma_{1j}-1/2} f_j(b,\tau_j)(t,s) \leqslant \bar{l}_{1j}$$

for $t^* \leqslant s \leqslant t < b$ hold. Then problem (1.8), (1.2) *is uniquely solvable in the space $\widetilde{C}^{n-1,m}(]a,b[)$.*

Also, in [15] the following theorem is proved:

Theorem 1.2. *Let all the conditions of Theorem* 1.1 *be satisfied. Then the unique solution u of problem* (1.8), (1.2) *for every $q \in \widetilde{L}^2_{2n-2m-2,2m-2}(]a,b[)$ admits the estimate*

$$\|u^{(m)}\|_{L^2} \leqslant r\|q\|_{\widetilde{L}^2_{2n-2m-2,2m-2}}, \tag{1.17}$$

with

$$r = \frac{2^m(1+b-a)(2n-2m-1)}{(\nu_n - 2\max\{B_0,B_1\})(2m-1)!!}, \quad \nu_{2m}=1,\ \nu_{2m+1}=\frac{2m+1}{2},$$

and thus the constant $r>0$ depends only on the numbers l_{kj}, $\bar{l}_{kj}$, γ_{kj} ($k=1,2$; $j=1,\dots,m$), and a, b, t^*, n.

Remark 1.1. Under the conditions of Theorem 1.2, for every

$$q \in \widetilde{L}^2_{2n-2m-2,2m-2}(]a,b[)$$

the unique solution u of problem (1.8), (1.2) admits the estimate

$$\|u^{(m)}\|_{\widetilde{C}_1^{m-1}} \leqslant r_n\|q\|_{\widetilde{L}^2_{2n-2m-2,2m-2}}, \tag{1.18}$$

with

$$r_n = \left(1+\sum_{j=1}^m \frac{2^{m-j+1/2}}{(m-j)!(2m-2j+1)^{1/2}(b-a)^{m-j+1/2}}\right) \times \frac{2^m(1+b-a)(2n-2m-1)}{(\nu_n-2\max\{B_0,B_1\})(2m-1)!!}.$$

1.2. Theorems on solvability of problem (1.1), (1.2)**.**

Define an operator $P\colon C_1^{m-1}(]a,b[)\times C_1^{m-1}(]a,b[)\to L_{\mathrm{loc}}(]a,b[)$ by the equality

$$P(x,y)(t) = \sum_{j=1}^m p_j(x)(t)y^{(j-1)}(\tau_j(t)) \quad \text{for } a<t<b \tag{1.19}$$

where $p_j\colon C_1^{m-1}(]a,b[)\to L_{\mathrm{loc}}(]a,b[)$, and $\tau_j\in M(]a,b[)$. Also, for any $\gamma>0$ define a set A_γ by the relation

$$A_\gamma = \{x\in \widetilde{C}_1^{m-1}(]a,b[)\colon\ \|x\|_{\widetilde{C}_1^{m-1}}\leqslant\gamma\}. \tag{1.20}$$

For formulating the a priori boundedness principle we have to introduce

Definition 1.1. Let γ_0 and γ be positive numbers. We say that the continuous operator $P\colon C_1^{m-1}(]a,b[) \times C_1^{m-1}(]a,b[) \to L_n(]a,b[)$ is γ_0,γ consistent with boundary condition (1.2) if:

(i) For any $x \in A_{\gamma_0}$ and almost all $t \in]a,b[$ the inequality

$$\sum_{j=1}^{m} |p_j(x)(t)x^{(j-1)}(\tau_j(t))| \leqslant \delta(t, \|x\|_{\widetilde{C}_1^{m-1}})\|x\|_{\widetilde{C}_1^{m-1}} \tag{1.21}$$

holds, where $\delta \in D_n(]a,b[\times \mathbb{R}^+)$.

(ii) For any $x \in A_{\gamma_0}$ and $q \in \widetilde{L}^2_{2n-2m-2,2m-2}(]a,b[)$ the equation

$$y^{(n)}(t) = \sum_{j=1}^{m} p_j(x)(t)y^{(j-1)}(\tau_j(t)) + q(t) \tag{1.22}$$

under boundary conditions (1.2) has a unique solution y in the space $\widetilde{C}^{n-1,m}(]a,b[)$ and

$$\|y\|_{\widetilde{C}_1^{m-1}} \leqslant \gamma\|q\|_{\widetilde{L}^2_{2n-2m-2,2m-2}}. \tag{1.23}$$

Definition 1.2. We say that the operator P is γ consistent with boundary condition (1.2), if the operator P is γ_0,γ consistent with boundary condition (1.2) for any $\gamma_0 > 0$.

In the sequel it will always be assumed that the operator F_p defined by equality

$$F_p(x)(t) = \left| F(x)(t) - \sum_{j=1}^{m} p_j(x)(t)x^{(j-1)}(\tau_j(t))(t) \right|$$

is continuously acting from $C_1^{m-1}(]a,b[)$ to $L_{\widetilde{L}^2_{2n-2m-2,2m-2}}(]a,b[)$, and

$$\widetilde{F}_p(t,\varrho) \equiv \sup\{F_p(x)(t)\colon \|x\|_{C_1^{m-1}} \leqslant \varrho\} \in \widetilde{L}^2_{2n-2m-2,2m-2}(]a,b[) \tag{1.24}$$

for each $\varrho \in [0,+\infty[$.

Then the following theorem is valid

Theorem 1.3. *Let the operator P be γ_0, γ consistent with boundary condition* (1.2)*, and let there exist a positive number $\varrho_0 \leqslant \gamma_0$, such that*

$$\|\widetilde{F}_p(\cdot, \min\{2\varrho_0, \gamma_0\})\|_{\widetilde{L}^2_{2n-2m-2,2m-2}} \leqslant \frac{\gamma_0}{\gamma}. \tag{1.25}$$

Let, moreover, for any $\lambda \in]0,1[$ an arbitrary solution $x \in A_{\gamma_0}$ of the equation

$$x^{(n)}(t) = (1-\lambda)P(x,x)(t) + \lambda F(x)(t) \tag{1.26}$$

under the conditions (1.2) *admit the estimate*

$$\|x\|_{\widetilde{C}_1^{m-1}} \leqslant \varrho_0. \tag{1.27}$$

Then problem (1.1), (1.2) *is solvable in the space $\widetilde{C}^{n-1,m}(]a,b[)$.*

Theorem 1.3 with $\varrho_0 = \gamma_0$ immediately yields

Corollary 1.1. *Let the operator P be γ_0, γ consistent with boundary condition* (1.2)*, and*

$$|F(x)(t) - \sum_{j=1}^{m} p_j(x)(t)x^{(j-1)}(\tau_j(t))(t)| \leqslant \eta(t, \|x\|_{\widetilde{C}_1^{m-1}}) \tag{1.28}$$

for $x \in A_{\gamma_0}$ and almost all $t \in]a,b[$, and

$$\|\eta(\cdot, \gamma_0)\|_{\widetilde{L}^2_{2n-2m-2,2m-2}} \leqslant \frac{\gamma_0}{\gamma}, \tag{1.29}$$

where $\eta \in D_{2n-2m-2,2m-2}(]a,b[\times \mathbb{R}^+)$. Then problem (1.1), (1.2) *is solvable in the space $\widetilde{C}^{n-1,m}(]a,b[)$.*

Corollary 1.2. *Let the operator P be γ consistent with boundary condition* (1.2)*, let inequality* (1.28) *hold for $x \in \widetilde{C}_1^{m-1}(]a,b[)$ and almost all $t \in]a,b[$, where $\eta(\cdot, \varrho) \in \widetilde{L}^2_{2n-2m-2,2m-2}(]a,b[)$ for any $\varrho \in \mathbb{R}^+$, and*

$$\limsup_{\varrho \to +\infty} \frac{1}{\varrho} \|\eta(\cdot, \varrho)\|_{\widetilde{L}^2_{2n-2m-2,2m-2}} < \frac{1}{\gamma}. \tag{1.30}$$

Then problem (1.1), (1.2) *is solvable in the space $\widetilde{C}^{n-1,m}(]a,b[)$.*

When we discuss problem (1.41), (1.2), and $n = 2m+1$, we assume that the continuous operator $p_1\colon \widetilde{C}_1^{m-1}(]a,b[) \to L_{\mathrm{loc}}(]a,b[)$ is such that

$$\limsup_{t\to b}\left|(b-t)^{2m-1}\int_{t_1}^{t} p_1(x)(s)\,\mathrm{d}s\right| < +\infty \quad \Bigl(t_1 = \frac{a+b}{2}\Bigr) \tag{1.31}$$

for any $x \in \widetilde{C}_1^{m-1}(]a,b[)$.

Now define operators $h_j\colon C_1^{m-1}(]a,b[) \times]a,b[\times]a,b[\to \mathbb{R}_+$, $f_j\colon C_1^{m-1}(]a,b[) \times [a,b] \times M(]a,b[) \to L_{\mathrm{loc}}(]a,b[\times]a,b[)$ $(j = 1,\dots,m)$ by the equalities

$$\begin{aligned} h_1(x,t,s) &= \left|\int_s^t (\xi-a)^{n-2m}[(-1)^{n-m}p_1(x)(\xi)]_+\,\mathrm{d}\xi\right|, \\ h_j(x,t,s) &= \left|\int_s^t (\xi-a)^{n-2m}p_j(x)(\xi)\,\mathrm{d}\xi\right| \quad (j=2,\dots,m), \end{aligned} \tag{1.32}$$

and

$$f_j(x,c,\tau_j)(t,s) = \left|\int_s^t (\xi-a)^{n-2m}|p_j(x)(\xi)|\left|\int_\xi^{\tau_j(\xi)}(\xi_1-c)^{2(m-j)}\,\mathrm{d}\xi_1\right|^{1/2}\mathrm{d}\xi\right|. \tag{1.33}$$

Theorem 1.4. *Let the continuous operator $P\colon C_1^{m-1}(]a,b[) \times C_1^{m-1}(]a,b[) \to L_n(]a,b[)$ fulfil the condition* (1.21) *where $\delta \in D_n(]a,b[\times\mathbb{R}^+)$, $\tau_j \in M(]a,b[)$ and the numbers γ_0, $t^* \in]a,b[$, $l_{kj} > 0$, $\bar{l}_{kj} > 0$, $\gamma_{kj} > 0$ $(k = 0,1;\ j = 1,\dots,m)$, are such that the inequalities*

$$(t-a)^{2m-j}h_j(x,t,s) \leqslant l_{0j}, \quad \limsup_{t\to a}(t-a)^{m-\frac12-\gamma_{0j}}f_j(x,a,\tau_j)(t,s) \leqslant \bar{l}_{0j} \tag{1.34}$$

for $a < t \leqslant s \leqslant t^*$, $\|x\|_{\widetilde{C}_1^{m-1}} \leqslant \gamma_0$,

$$(b-t)^{2m-j}h_j(x,t,s) \leqslant l_{1j}, \quad \limsup_{t\to b}(b-t)^{m-\frac12-\gamma_{1j}}f_j(x,b,\tau_j)(t,s) \leqslant \bar{l}_{1j} \tag{1.35}$$

for $t^* \leqslant s \leqslant t < b$, $\|x\|_{\widetilde{C}_1^{m-1}} \leqslant \gamma_0$, and conditions (1.13), (1.14) *hold. Let moreover the operator F and a function $\eta \in D_{2n-2m-2,2m-2}(]a,b[\times\mathbb{R}^+)$ be such that condition* (1.28) *and inequality*

$$\|\eta(\cdot,\gamma_0)\|_{\widetilde{L}^2_{2n-2m-2,2m-2}} < \frac{\gamma_0}{r_n}, \tag{1.36}$$

are fulfilled, where

$$\begin{aligned} r_n = \left(1 + \sum_{j=1}^m \frac{2^{m-j+1/2}}{(m-j)!(2m-2j+1)^{1/2}(b-a)^{m-j+1/2}}\right) \\ \times \frac{2^m(1+b-a)(2n-2m-1)}{(\nu_n - 2\max\{B_0,B_1\})(2m-1)!!}. \end{aligned}$$

Then problem (1.1), (1.2) *is solvable in the space $\widetilde{C}^{n-1,m}(]a,b[)$.*

Theorem 1.5. *Let the operator F and the function η be such that conditions* (1.28), (1.30) *hold and the continuous operator $P\colon C_1^{m-1}(]a,b[)\times C_1^{m-1}(]a,b[)\to L_n(]a,b[)$ fulfils condition* (1.21) *where $\delta\in D_n(]a,b[\times\mathbb{R}^+)$. Let moreover measurable functions $\tau_j\in M(]a,b[)$ and numbers $t^*\in\,]a,b[$, $l_{kj}>0$, $\bar l_{kj}>0$, $\gamma_{kj}>0$ ($k=0,1$; $j=1,\dots,m$) be such that the inequalities*

$$(1.37)\qquad (t-a)^{2m-j}h_j(x,t,s)\leqslant l_{0j},\quad \limsup_{t\to a}(t-a)^{m-\frac12-\gamma_{0j}}f_j(x,a,\tau_j)(t,s)\leqslant \bar l_{0j}$$

for $a<t\leqslant s\leqslant t^*$, $x\in \widetilde C_1^{m-1}(]a,b[)$,

$$(1.38)\qquad (b-t)^{2m-j}h_j(x,t,s)\leqslant l_{1j},\quad \limsup_{t\to b}(b-t)^{m-\frac12-\gamma_{1j}}f_j(x,b,\tau_j)(t,s)\leqslant \bar l_{1j}$$

for $t^*\leqslant s\leqslant t<b$, $x\in\widetilde C_1^{m-1}(]a,b[)$, and conditions (1.13), (1.14) *hold. Then problem* (1.1), (1.2) *is solvable in the space $\widetilde C^{n-1,m}(]a,b[)$.*

Remark 1.2. Let $\gamma_0>0$, let the operators $\alpha_j(t)p_j(x)(t)$ ($j=1,\dots,m$) be continuously acting from the space $C_1^{m-1}(]a,b[)$ to the space $L_n(]a,b[)$, let there exist functions $\delta_j\in D_n(]a,b[)$ such that for any $x\in A_{\gamma_0}$

$$(1.39)\qquad |p_j(x)(t)|\alpha_j(t)\leqslant\delta_j(t,\|x\|_{\widetilde C_1^{m-1}})\quad\text{for } a<t<b,$$

and constants $\kappa>0$, $\varepsilon>0$ such that

$$(1.40)\qquad \begin{aligned}&|\tau_j(t)-t|\leqslant\kappa(t-a)\quad (j=1,\dots,m)\quad\text{for } a<t<a+\varepsilon,\\ &|\tau_j(t)-t|\leqslant\kappa(b-t)\quad (j=1,\dots,m)\quad\text{for } b-\varepsilon<t<b.\end{aligned}$$

Then the operator P defined by equality (1.19), continuously acting from A_{γ_0} to the space $L_n(]a,b[)$, and there exists a function $\delta\in D_n(]a,b[)$ such that item (*ii*) of Definition 1.1 holds.

Now consider the equation with deviating arguments

$$(1.41)\qquad u^{(n)}(t)=f(t,u(\tau_1(t)),u'(\tau_2(t)),\dots,u^{(m-1)}(\tau_m(t)))\quad\text{for } a<t<b,$$

where $-\infty<a<b<+\infty$, $f\colon\,]a,b[\,\times\mathbb{R}^m\to\mathbb{R}$ is a function satisfying the local Caratheodory conditions and $\tau_j\in M(]a,b[)$ ($j=0,\dots,n-1$) are measurable functions.

Corollary 1.3. *Let the functions* $\tau_j \in M(]a,b[)$ *and the numbers* $t^* \in]a,b[$, $\kappa \geqslant 0$, $\varepsilon > 0$, $l_{kj} > 0$, $\bar{l}_{kj} > 0$, $\gamma_{kj} > 0$ $(k = 1,2;\ j = 1,\dots,m)$ *be such that conditions* (1.13), (1.14), (1.15), (1.16), (1.40) *and the inclusions*

$$\alpha_j p_j \in L_n(]a,b[) \quad (j = 1,\dots,m) \tag{1.42}$$

are fulfilled. Let moreover

$$\Big| f(t, x(\tau_1(t)), x'(\tau_2(t)), \dots, x^{(m-1)}(\tau_m(t))) - \sum_{j=1}^{m} p_j(t) x^{(j-1)}(\tau_j(t))(t) \Big| \leqslant \eta(t, \|x\|_{\widetilde{C}_1^{m-1}}) \tag{1.43}$$

for $x \in \widetilde{C}_1^{m-1}(]a,b[)$ *and almost all* $t \in]a,b[$, *where* $\eta(\cdot,\varrho) \in \widetilde{L}^2_{2n-2m-2,2m-2}(]a,b[)$ *for any* $\varrho \in \mathbb{R}^+$, *and let condition* (1.30) *hold. Then problem* (1.41), (1.2) *is solvable in the space* $\widetilde{C}^{n-1,m}(]a,b[)$.

Remark 1.3. Conditions (1.42) do not imply conditions (1.6).

Now for illustration of our results consider on $]a,b[$ the second order functional-differential equations

$$u''(t) = -\frac{\lambda |u(t)|^k}{[(t-a)(b-t)]^{2+k/2}} u(\tau(t)) + q(x)(t), \tag{1.44}$$

$$u''(t) = -\frac{\lambda |\sin u^k(t)|}{[(t-a)(b-t)]^{2}} u(\tau(t)) + q(x)(t), \tag{1.45}$$

where $\lambda, k \in \mathbb{R}^+$, the function $\tau \in M(]a,b[)$, the operator $q\colon C_1^{m-1}(]a,b[) \to \widetilde{L}^2_{0,0}(]a,b[)$ is continuous and

$$\eta(t,\varrho) \equiv \sup\{|q(x)(t)|\colon \|x\|_{\widetilde{C}_1^{m-1}} \leqslant \varrho\} \in \widetilde{L}^2_{0,0}(]a,b[).$$

Than Theorems 1.4 and 1.5 yield

Corollary 1.4. *Let a continuous operator* $q\colon C_1^{m-1}(]a,b[) \to \widetilde{L}^2_{0,0}(]a,b[)$, *a function* $\tau \in M(]a,b[)$, *and numbers* $\gamma_0 > 0$, $\lambda \geqslant 0$, $k > 0$, *be such that*

$$|\tau(t) - t| \leqslant \begin{cases} (t-a)^{3/2} & \text{for } a < t \leqslant (a+b)/2, \\ (b-t)^{3/2} & \text{for } (a+b)/2 \leqslant t < b, \end{cases} \tag{1.46}$$

$$\|\eta(t,\gamma_0)\|_{\widetilde{L}^2_{0,0}} \leqslant \left(1 + \sqrt{\frac{2}{b-a}}\right)^{-1} \frac{(b-a)^2 - 16\lambda\gamma_0^k(1 + [2(b-a)]^{1/4})}{2(1+b-a)(b-a)^2}, \tag{1.47}$$

and

$$\lambda < \frac{(b-a)^2}{32\gamma_0^k(1+[2(b-a)]^{1/4})}. \tag{1.48}$$

Then problem (1.44), (1.2) *is solvable.*

Corollary 1.5. *Let a continuous operator* $q\colon C_1^{m-1}(]a,b[) \to \widetilde{L}_{0,0}^2(]a,b[)$, *a function* $\tau \in M(]a,b[)$, *and a number* $\lambda \geqslant 0$ *be such that inequalities* (1.30) *with* $n=2$, (1.46) *and*

$$\lambda < \frac{(b-a)^2}{32(1+[2(b-a)]^{1/4})}, \tag{1.49}$$

hold. Then problem (1.45), (1.2) *is solvable.*

2. Auxiliary propositions

2.1. Lemmas on some properties of the equation $x^{(n)}(t) = \lambda(t)$**.**
First, we introduce two lemmas without proofs. The first lemma is proved in [2].

Lemma 2.1. *Let* $i \in 1,2$, $x \in \widetilde{C}_{\mathrm{loc}}^{m-1}(]t_0,t_1[)$ *and*

$$x^{(j-1)}(t_i) = 0 \ (j=1,\dots,m), \qquad \int_{t_0}^{t_1} |x^{(m)}(s)|^2 \,\mathrm{d}s < +\infty. \tag{2.1}$$

Then

$$\left|\int_{t_i}^{t} \frac{(x^{(j-1)}(s))^2}{(s-t_i)^{2m-2j+2}} \,\mathrm{d}s\right|^{1/2} \leqslant \frac{2^{m-j+1}}{(2m-2j+1)!!}\left|\int_{t_i}^{t} |x^{(m)}(s)|^2 \,\mathrm{d}s\right|^{1/2} \tag{2.2}$$

for $t_0 \leqslant t \leqslant t_1$.

This second lemma is a particular case of Lemma 4.1 in [11].

Lemma 2.2. *If $x \in C^{n-1}_{\text{loc}}(]a, a_1])$, then for any $s, t \in]a, a_1]$ the equality*

$$(-1)^{n-m} \int_s^t (\xi - a)^{n-2m} x^{(n)}(\xi) x(\xi)\, \mathrm{d}\xi = w_n(x)(t) - w_n(x)(s) + \nu_n \int_s^t |x^{(m)}(\xi)|^2\, \mathrm{d}\xi$$

is valid, where $\nu_{2m} = 1$, $\nu_{2m+1} = \frac{1}{2}(2m+1)$,

$$\begin{aligned} w_{2m}(x)(t) &= \sum_{j=1}^{m} (-1)^{m+j-1} x^{(2m-j)}(t) x(t), \\ w_{2m+1}(x)(t) &= \sum_{j=1}^{m} (-1)^{m+j} [(t-a) x^{(2m+1-j)}(t) - j x^{(2m-j)}(t)] x^{(j-1)}(t) \\ &\quad - \frac{t-a}{2} |x^{(m)}(t)|^2. \end{aligned}$$

Lemma 2.3. *Let numbers $a_1 \in]a, b[$, $t_{0k} \in]a, a_1[$, and ε_{ik}, ε_i, β_k, $\beta \in \mathbb{R}^+$, $k \in \mathbb{N}$, $i = 1, \ldots, n-m$ be such that*

$$\lim_{k \to +\infty} t_{0k} = a, \quad \lim_{k \to +\infty} \beta_k = \beta, \quad \lim_{k \to +\infty} \varepsilon_{i,k} = \varepsilon_i. \tag{2.3}$$

Let, moreover,

$$\lambda \in \widetilde{L}^2_{2n-2m-2,0}(]a, a_1]) \tag{2.4}$$

be a nonnegative function, $x_k \in \widetilde{C}^{n-1,m}(]a, b[)$ a solution of the problem

$$x^{(n)}(t) = \beta_k \lambda(t), \tag{2.5}$$

$$x^{(i-1)}(t_{0k}) = 0 \ (i = 1, \ldots, m), \quad x^{(i-1)}(a_1) = \varepsilon_{i,k} \ (i = 1, \ldots, n-m), \tag{2.6}$$

and $x \in \widetilde{C}^{n-1,m}(]a, b[)$ a solution of the problem

$$x^{(n)}(t) = \beta \lambda(t), \tag{2.7}$$

$$x^{(i-1)}(a) = 0 \ \ (i = 1, \ldots, m), \quad x^{(i-1)}(a_1) = \varepsilon_i \ \ (i = 1, \ldots, n-m). \tag{2.8}$$

Then

$$\lim_{k \to +\infty} x_k^{(j-1)}(t) = x^{(j-1)}(t) \quad (j = 1, \ldots, n) \quad \text{uniformly in }]a, a_1]. \tag{2.9}$$

P r o o f. First, let us prove our lemma under the assumption that there exists a number $r_1 > 0$ such that the estimates

$$\int_{t_{0k}}^{a_1} |x_k^{(m)}(s)|^2\, \mathrm{d}s \leqslant r_1, \quad k \in \mathbb{N} \tag{2.10}$$

hold. Now, suppose that $t_1, \dots, t_n$ are such numbers that $t_{0k} < t_1 < \dots < t_n < a_1$ $(k \in \mathbb{N})$, and $g_i(t)$ are polynomials of $(n-1)$-th degree, satisfying the conditions $g_j(t_j) = 1$, $g_j(t_i) = 0$ $(i \neq j;\ i, j = 1, \dots, n)$. Then if x_k is a solution of problem (2.5), (2.6), and x is a solution of problem (2.7), (2.8). For the solution $x - x_k$ of the equation $\mathrm{d}^n(x(t) - x_k(t))/\mathrm{d}t^n = (\beta - \beta_k)\lambda(t)$, the representation

(2.11)

$$x(t) - x_k(t) = \sum_{j=1}^{n} \left((x(t_j) - x_k(t_j)) - \frac{\beta - \beta_k}{(n-1)!} \int_{t_1}^{t_j} (t_j - s)^{n-1} \lambda(s)\, \mathrm{d}s \right) g_j(t) + \frac{\beta - \beta_k}{(n-1)!} \int_{t_1}^{t} (t - s)^{n-1} \lambda(s)\, \mathrm{d}s \ \ k \in \mathbb{N} \quad \text{for} \quad t_{0k} \leqslant t \leqslant a_1$$

is valid. On the other hand, in view of inequality (2.10), the identities

$$x_k^{(i-1)}(t) = \frac{1}{(m-i)!} \int_{t_{0k}}^{t} (t - s)^{m-i} x_k^{(m)}(s)\, \mathrm{d}s \quad (i = 1, 2,\ k \in \mathbb{N})$$

by Schwartz inequality yield

$$|x_k^{(i-1)}(t)| \leqslant r_2 (t - a)^{m-i-1/2} \quad \text{for } t_{0k} \leqslant t \leqslant a_1 \ (i = 1, 2,\ k \in \mathbb{N}), \tag{2.12}$$

where $r_2 = r_1/(m-i)!\sqrt{2m - 2i + 1}$. By virtue of the Arzela-Ascoli lemma and (2.3), (2.12) the sequence $\{x_k\}_{k=1}^{+\infty}$ contains a subsequence $\{x_{k_l}\}_{l=1}^{+\infty}$ which is uniformly convergent in $]a, a_1]$. Suppose $\lim_{l \to +\infty} x_{k_l}(t) = x_0(t)$. Thus (2.11) by (2.3) yields the existence of such $r_3 > 0$ that

$$|x_{k_l}^{(j-1)}(t)| \leqslant r_3 + |x^{(j-1)}(t)| \quad (j = 1, \dots, n) \quad \text{for } t_{0k_l} \leqslant t \leqslant a_1,$$

and then without loss of generality we can assume that

$$\lim_{l \to +\infty} x_{k_l}^{(j-1)}(t) = x_0^{(j-1)}(t) \quad (j = 1, \dots, n) \quad \text{uniformly in }]a, a_1]. \tag{2.13}$$

Then in virtue of (2.3), (2.11), and (2.13) we have

$$x(t) - x_0(t) = \sum_{j=1}^{n} ((x(t_j) - x_0(t_j))) g_j(t) \quad \text{for } a \leqslant t \leqslant a_1.$$

From the last two relations by (2.10) it is clear that $x^{(n)} = x_0^{(n)}$ and $x_0 \in \widetilde{C}^{n-1,m}(]a, b[)$. So, the function $x_0 \in \widetilde{C}^{n-1,m}(]a, b[)$ is a solution of problem (2.7), (2.8). In view of (2.4) all the conditions of Theorem 1.1 are fulfilled, thus problem

(2.7), (2.8) is uniquely solvable in the space $\widetilde{C}^{n-1,m}(]a,b[)$ and $x = x_0$. Therefore (2.13) implies

$$\lim_{l\to+\infty} x_{k_l}^{(j-1)}(t) = x^{(j-1)}(t) \quad (j=1,\dots,n) \quad \text{uniformly in }]a,a_1]. \tag{2.14}$$

Now suppose that relations (2.9) are not fulfilled. Then there exist $\delta \in]0,\frac{1}{2}(a_1-a)[$, $\varepsilon > 0$, and an increasing sequence of natural numbers $\{k_l\}_{l=1}^{+\infty}$ such that

$$\max\Big\{\sum_{j=1}^{n} |x_{k_l}^{(j-1)}(t) - x^{(j-1)}(t)|\colon\ a+\delta \leqslant t \leqslant a_1\Big\} > \varepsilon \quad (l \in \mathbb{N}). \tag{2.15}$$

By virtue of the Arczela-Ascoli lemma and condition (2.10) the sequence $\{x_{k_l}^{(j-1)}\}_{l=1}^{+\infty}$ $(j=1,\dots,m)$, without loss of generality, can be assumed to be uniformly converging in $]a+\delta,a_1]$. Then, in view of what we have shown above, equality (2.14) holds. But this contradicts condition (2.15). Thus (2.9) holds if the conditions (2.10) are fulfilled.

Let now the conditions (2.10) be not fulfilled. Then there exists a subsequence $\{t_{0k_l}\}_{l=1}^{+\infty}$ of the sequence $\{t_{0k}\}_{k=1}^{+\infty}$, such that

$$\int_{t_{0k}}^{a_1} |x_{k_l}^{(m)}(s)|^2 \,\mathrm{d}s \geqslant l \quad (l \in \mathbb{N}). \tag{2.16}$$

Suppose that $\beta_l = \big(\int_{t_{0k}}^{a_1} |x_{k_l}^{(m)}(s)|^2 \,\mathrm{d}s\big)^{-1}$ and $v_l(t) = u_{k_l}(t)\beta_l$. Thus in view of (2.16) and our notation

$$\int_{t_{0k_l}}^{a_1} |v_{k_l}^{(m)}(s)|^2 \,\mathrm{d}s = 1 \quad (l \in \mathbb{N}), \quad \lim_{l\to+\infty} \beta_l = 0, \tag{2.17}$$

$$v_l^{(n)}(t) = \beta_l \lambda(t), \tag{2.18}$$

$$\begin{gathered} v_l^{(i-1)}(t_{0k_l}) = 0 \quad (i=1,\dots,m), \\ v_l^{(i-1)}(a_1) = \varepsilon_{i,k_l}\beta_l \quad (i=1,\dots,n-m,\ l \in \mathbb{N}). \end{gathered} \tag{2.19}$$

From the first part of our lemma it follows by (2.17) that the limit $\lim\limits_{l\to+\infty} v_l(t) \equiv v_0(t)$ exists, and v_0 is a solution of the homogeneous problem corresponding to (2.18), (2.19). Thus $v_0 \equiv 0$. On the other hand, from (2.17) it is clear that $\int_{t_{0k_l}}^{a_1} |v_0^{(m)}(s)|^2 \,\mathrm{d}s = 1$, which contradicts $v_0 \equiv 0$. Thus our assumption is invalid and (2.10) holds. $\square$

Analogously one can prove

Lemma 2.4. *Let numbers $b_1 \in]a,b[$, $t_{0k} \in]b_1,b[$, and ε_{ik}, ε_i, β_k, $\beta \in \mathbb{R}^+$, $k \in \mathbb{N}$, $i = 1,\dots,n-m$ be such that*

$$\lim_{k\to+\infty} t_{0k} = b, \qquad \lim_{k\to+\infty} \beta_k = \beta, \qquad \lim_{k\to+\infty} \varepsilon_{i,k} = \varepsilon_i.$$

Let, moreover, $\lambda \in \widetilde{L}^2_{0,2m-2}(]b_1,b])$ be a nonnegative function, $x_k \in \widetilde{C}^{n-1,m}(]a,b[)$ a solution of the problem (2.5) under the conditions

$$x^{(i-1)}(b_1) = \varepsilon_{i,k} \ \ (i=1,\dots,m), \quad x^{(i-1)}(t_{0k}) = 0 \ \ (i=1,\dots,n-m),$$

and $x \in \widetilde{C}^{n-1,m}(]a,b[)$ a solution of the equation (2.7) under the conditions

$$x^{(i-1)}(b_1) = \varepsilon_i \ \ (i=1,\dots,m), \quad x^{(i-1)}(b) = 0 \ \ (i=1,\dots,n-m). \tag{2.20}$$

Then the equalities (2.9) hold.

Lemma 2.5. *Let $a < a_1 < b_1 < b$, $\varepsilon_i \in \mathbb{R}^+$ and let*

$$\lambda \in \widetilde{L}^2_{2n-2m-2,0}(]a,a_1]) \quad (\lambda \in \widetilde{L}^2_{0,2m-2}(]b_1,b]))$$

be a nonnegative function. Then for the solution $x \in \widetilde{C}^{n-1,m}(]a,b[)$ of the problem (2.7), (2.8) ((2.7), (2.20)) with $\beta = 1$, the estimate

$$\int_a^{a_1} |x^{(m)}(s)|^2 \,\mathrm{d}s \leqslant \Theta_1(x,a_1,\lambda) \tag{2.21}$$

$$\Bigl(\int_{b_1}^{b} |x^{(m)}(s)|^2 \,\mathrm{d}s \leqslant \Theta_2(x,b_1,\lambda)\Bigr) \quad (k \in \mathbb{N})$$

is valid, where

$$\Theta_1(x,a_1,\lambda) = 2|w_n(x)(a_1)| + \gamma_1 \|\lambda\|^2_{\widetilde{L}^2_{2n-2m-2,0}(]a,a_1])}, \tag{2.22}$$

$$(\Theta_2(x,b_1,\lambda) = 2|w_n(x)(b_1)| + \gamma_2 \|\lambda\|^2_{\widetilde{L}^2_{0,2m-2}(]b_1,b])}),$$

and

$$\gamma_1 = \Bigl(\frac{2^{m-1}(2m+1)}{(2m-1)!!}\Bigr)^2, \quad \gamma_2 = \Bigl(\frac{2^{m-1}(2m+1)(b-a+1)}{(2m-1)!!}\Bigr)^2.$$

P r o o f. Suppose that x_k is a solution of problem (2.5), (2.6) with $\beta_k = 1$, $\varepsilon_{ik} = \varepsilon_i$. Then in view of Lemma 2.3, relations (2.9) hold. On the other hand, by Lemma 2.2 we get

$$\nu_n \int_{t_{0k}}^{a_1} |x_k^{(m)}(s)|^2 \,\mathrm{d}s \leqslant -w_n(x_k)(a_1) + \int_{t_{0k}}^{a_1} (s-a)^{n-2m} \lambda(s) |x_k(s)| \,\mathrm{d}s. \tag{2.23}$$

Now, on the basis of Lemma 2.1, Schwartz's and Young's inequalities we get

$$\begin{aligned}
&\Big|\int_{t_{0k}}^{a_1}(s-a)^{n-2m}\lambda(s)x_k(s)\,\mathrm{d}s\Big|\\
&\qquad=\Big|\int_{t_{0k}}^{a_1}[(n-2m)x_k(s)+(s-a)^{n-2m}x_k'(s)]\Big(\int_s^{a_1}\lambda(\xi)\,\mathrm{d}\xi\Big)\,\mathrm{d}s\Big|\\
&\qquad\leqslant\Big[(n-2m)\Big(\int_{t_{0k}}^{a_1}\frac{x_k^2(s)}{(s-a)^{2m}}\,\mathrm{d}s\Big)^{1/2}\\
&\qquad\quad+\Big(\int_{t_{0k}}^{a_1}\frac{x_k'^2(s)}{(s-a)^{2m-2}}\,\mathrm{d}s\Big)^{1/2}\Big]\|\lambda\|_{\widetilde{L}_{2n-2m-2,0}(]a,a_1])}\\
&\qquad\leqslant\frac{2^{m-1}(2m+1)}{(2m-1)!!}\Big(\int_{t_{0k}}^{a_1}|x_k^{(m)}(s)|^2\,\mathrm{d}s\Big)^{1/2}\|\lambda\|_{\widetilde{L}_{2n-2m-2,0}(]a,a_1])}\\
&\qquad\leqslant\frac{1}{2}\int_{t_{0k}}^{a_1}|x_k^{(m)}(s)|^2\,\mathrm{d}s+\frac{1}{2}\Big(\frac{2^{m-1}(2m+1)}{(2m-1)!!}\Big)^2\|\lambda\|^2_{\widetilde{L}_{2n-2m-2,0}(]a,a_1])}.
\end{aligned}$$

Thus from (2.23) by the definition of the numbers ν_n we immediately obtain the estimate

$$\int_{t_{0k}}^{a_1}|x_k^{(m)}(s)|\,\mathrm{d}s\leqslant 2|w_n(x_k)(a_1)|+\Big(\frac{2^{m-1}(2m+1)}{(2m-1)!!}\Big)^2\|\lambda\|^2_{\widetilde{L}_{2n-2m-2,0}(]a,a_1])}\quad(k\in\mathbb{N}).$$

By (2.9) from the last inequality (2.21) and (2.22) follow. Thus the lemma is proved for problem (2.7), (2.8).

Analogously, by using Lemma 2.4 one can prove the case of problem (2.7), (2.20). □

2.2. Lemmas on Banach space $\widetilde{C}_1^{m-1}(]a,b[)$.

Definition 2.1. Let $\varrho\in\mathbb{R}^+$ and let the function $\eta\in L_{\mathrm{loc}}(]a,b[)$ be nonnegative. Then $S(\varrho,\eta)$ is a set of such $y\in C_{\mathrm{loc}}^{n-1}(]a,b[)$ that

$$\Big|y^{(i-1)}\Big(\frac{a+b}{2}\Big)\Big|\leqslant\varrho\quad(i=1,\dots,n),\tag{2.24}$$

$$|y^{(n-1)}(t)-y^{(n-1)}(s)|\leqslant\int_s^t\eta(\xi)\,\mathrm{d}\xi\quad\text{for } a<s\leqslant t<b,\tag{2.25}$$

and

$$y^{(i-1)}(a)=0\ \ (i=1,\dots,m),\quad y^{(i-1)}(b)=0\ \ (i=1,\dots,n-m).\tag{2.26}$$

Lemma 2.6. *Let for a function* $y \in \widetilde{C}^{n-1,m}(]a,b[)$, *conditions* (2.26) *be satisfied. Then* $y \in \widetilde{C}_1^{m-1}(]a,b[)$ *and the estimates*

$$(2.27)\quad |y^{(i-1)}(t)| \leqslant \frac{|t-c_k|^{m-i+1/2}}{(m-i)!(2m-2i+1)^{1/2}} \left| \int_{c_k}^{t} |y^{(m)}(s)|^2 \,\mathrm{d}s \right|^{1/2} \quad \text{for } a<t<b,$$

$i=1,\ldots,m$, *hold for* $k=1,2$, *where* $c_1=a$, $c_2=b$.

P r o o f. First note that in view of inclusion $y \in \widetilde{C}^{n-1,m}(]a,b[)$, the equality

$$(2.28)\qquad y^{(i-1)}(t) = \sum_{j=i}^{l} \frac{(t-c)^{j-i}}{(j-i)!} y^{(j-1)}(c) + \frac{1}{(l-i)!} \int_{c}^{t} (t-s)^{l-i} y^{(l)}(s)\,\mathrm{d}s \quad \text{for } a<t<b$$

for $i=1,\ldots,l$, $l=1,\ldots,n$, holds, where

(1) $c \in [a,b]$ if $l \leqslant m$;
(2) $c \in]a,b]$ if $l=m+1$ and $n=2m+1$;
(3) $c \in]a,b[$ if $l>m$,

and there exists $r>0$ such that

$$(2.29)\qquad \int_a^b |y^{(m)}(s)|^2\,\mathrm{d}s \leqslant r.$$

Equality (2.28) with $l=m$, $c=a$ and with $l=m$, $c=b$ by conditions (2.26), (2.29) and the Schwartz inequality yields (2.27). From (2.27) and (2.29) it is clear that $y \in \widetilde{C}_1^{m}(]a,b[)$. □

Lemma 2.7. *Let* $\varrho \in \mathbb{R}^+$, *and let* $\eta \in \widetilde{L}^2_{2n-2m-2,2m-2}(]a,b[)$ *be a nonnegative function. Then* $S(\varrho,\eta)$ *is a compact subset of the space* $\widetilde{C}_1^{m-1}(]a,b[)$.

P r o o f. Condition (2.25) yields the inequality $|y^{(n)}(t)| \leqslant \eta(t)$. Thus there exists such a function $\eta_1 \in \widetilde{L}^2_{2n-2m-2,2m-2}(]a,b[)$ that

$$(2.30)\qquad y^{(n)}(t) = \eta_1(t), \quad \text{for } a<t<b,$$

$$(2.31)\qquad |\eta_1(t)| \leqslant \eta(t) \quad \text{for } a<t<b.$$

From Theorem 1.1 it follows that problem (2.30), (2.26) has a unique solution $y \in C^{n-1,m}(]a,b[)$, i.e. there exists $r>0$ such that the inequality (2.29) holds.

For any $y \in S(\varrho,\eta)$, from equality (2.28) with $l=n$, by (2.24), (2.30) and (2.31) we get

$$(2.32)\qquad |y^{(i-1)}(t)| \leqslant \gamma_i(t) \quad \text{for } a<t<b \quad (i=1,\ldots,n),$$

where

$$\gamma_i(t) = \varrho_i + \frac{1}{(n-i)!}\left| \int_c^t (t-s)^{n-i}\eta(s)\,\mathrm{d}s \right| \quad (i = 1, \dots, n).$$

Let now $y_k \in S(\varrho, \eta)$ $(k \in \mathbb{N})$. By virtue of the Arzela-Ascoli lemma and conditions (2.25), (2.32) the sequence $\{y_k\}_{k=1}^{+\infty}$ contains a subsequence $\{y_{k_l}\}_{l=1}^{+\infty}$ such that $\{y_{k_l}^{(i-1)}\}_{l=1}^{+\infty}$ $(i = 1, \dots, n)$ are uniformly convergent on $]a, b[$. Thus without loss of generality we can assume that $\{y_k^{(i-1)}\}_{k=1}^{+\infty}$ $(i = 1, \dots, n-1)$ are uniformly convergent on $]a, b[$. Let $\lim\limits_{k\to+\infty} y_k(t) = y_0(t)$, then $y_0 \in \widetilde{C}_{\mathrm{loc}}^{n-1}(]a, b[)$ and

$$\lim_{k\to+\infty} y_k^{(i-1)}(t) = y_0^{(i-1)}(t) \quad (i = 1, \dots, n) \quad \text{uniformly on }]a, b[. \tag{2.33}$$

From (2.33) in view of the inclusions $y_k \in S(\varrho, \eta)$ it immediately follows that

$$\left| y_0^{(i-1)}\Big(\frac{a+b}{2}\Big) \right| \leqslant \varrho \quad (i = 1, \dots, n), \tag{2.34}$$

$$y_0^{(i-1)}(a) = 0 \;\; (j = 1, \dots, m), \quad y_0^{(i-1)}(b) = 0 \;\; (j = 1, \dots, n-m), \tag{2.35}$$

and

$$|y_0^{(n-1)}(t) - y_0^{(n-1)}(s)| \leqslant \int_s^t \eta(\xi)\,\mathrm{d}\xi \quad \text{for } a < s \leqslant t < b. \tag{2.36}$$

From (2.34), (2.35), (2.36) it is clear that $y_0 \in S(\varrho, \eta)$. To complete the proof we must show that

$$\lim_{k\to+\infty} \|y_k(t) - y_0(t)\|_{\widetilde{C}_1^{m-1}} = 0 \tag{2.37}$$

and

$$S(\varrho, \eta) \subset \widetilde{C}_1^{m-1}(]a, b[). \tag{2.38}$$

Let, $x_k = y_0 - y_k$ and $a_1 \in\,]a, b[$, $b_1 \in\,]a_1, b[$. Then it is clear that $x_k \in S(\varrho', \eta')$ where $\varrho' = 2\varrho$, $\eta' = 2\eta$. Thus for any x_k there exists $\eta_k \in \widetilde{L}_{2n-2m-2,2m-2}^2(]a, b[)$ such that

$$x_k^{(n)}(t) = \eta_k(t), \tag{2.39}$$

$$x_k^{(i-1)}(a) = 0 \;\; (i = 1, \dots, n), \quad x_k^{(i-1)}(b) = 0 \;\; (i = 1, \dots, n-m) \tag{2.40}$$

where

$$|\eta_k(t)| \leqslant 2\eta(t) \quad \text{for } a < t < b \;\; (k \in \mathbb{N}). \tag{2.41}$$

On the other hand, from (2.27) with $y = x_k$, in view of (2.40) we get

$$(2.42)\qquad \begin{aligned} |x_k^{(i-1)}(t)| &\leqslant \Big(\int_a^t |x_k^{(m)}(s)|^2\,\mathrm{d}s\Big)^{1/2}(t-a)^{m-i+1/2} \quad \text{for } a<t<a_1,\\ |x_k^{(i-1)}(t)| &\leqslant \Big(\int_t^b |x_k^{(m)}(s)|^2\,\mathrm{d}s\Big)^{1/2}(b-t)^{m-i+1/2} \quad \text{for } b_1<t<b, \end{aligned}$$

for $i = 1, \dots, m$.

Let now w_n be the operator defined in Lemma 2.2 and Θ_1, Θ_2 the functions defined by (2.22) with $\lambda = \eta_k$. Then conditions (2.33) yield

$$(2.43)\qquad \lim_{k\to+\infty} w_n(x_k)(a_1) = 0, \qquad \lim_{k\to+\infty} w_n(x_k)(b_1) = 0 \quad (k \in \mathbb{N}),$$

and from the definition of the norm $\|\cdot\|_{\widetilde{L}^2_{\alpha,\beta}}$, (2.41) and (2.43) it follows that for any $\varepsilon > 0$ we can choose $a_1 \in\,]a, \min\{a+1, b\}[$, $b_1 \in\,]\max\{b-1, b\}, a_1[$ and $k_0 \in \mathbb{N}$ such that

$$(2.44)\qquad \begin{aligned} \Theta_1(x_k, a_1, 2\eta) &\leqslant \frac{\varepsilon}{6}(b-b_1)^{m-1/2}\ (k \geqslant k_0),\\ \Theta_2(x_k, b_1, 2\eta) &\leqslant \frac{\varepsilon}{6}(a_1-a)^{m-1/2}\ (k \geqslant k_0). \end{aligned}$$

By using Lemma 2.5 for x_k, in view of (2.42) and (2.44) we get

$$(2.45)\qquad \int_a^{a_1} |x_k^{(m)}(s)|^2\,\mathrm{d}s \leqslant \frac{\varepsilon}{6}, \qquad \int_{b_1}^b |x_k^{(m)}(s)|^2\,\mathrm{d}s \leqslant \frac{\varepsilon}{6} \quad (k \geqslant k_0),$$

$$(2.46)\qquad \frac{|x_k^{(i-1)}(t)|}{\alpha_i(t)} \leqslant \frac{\varepsilon}{2m} \quad \text{for} \quad t \in\,]a, a_1] \cup [b_1, b[\quad (1 \leqslant i \leqslant m,\ k \geqslant k_0).$$

Also, in view of (2.33) without loss of generality we can assume that

$$(2.47)\qquad \frac{|x_k^{(i-1)}(t)|}{\alpha_i(t)} \leqslant \frac{\varepsilon}{2m} \quad \text{for} \quad a_1 \leqslant t \leqslant b_1 \quad (1 \leqslant i \leqslant m,\ k \geqslant k_0),$$

and

$$(2.48)\qquad \int_{a_1}^{b_1} |x_k^{(m)}(s)|^2\,\mathrm{d}s \leqslant \frac{\varepsilon}{6} \quad (k \geqslant k_0).$$

From (2.45), (2.46), (2.47), (2.48), equality (2.37) immediately follows.

Let now $y \in S(\varrho, \eta)$ and $y_k = \delta_k y$, where $\lim\limits_{k\to+\infty} \delta_k = 0$. Then by (2.33) it is clear that $y_0 \equiv 0$ and then (2.37) implies $y \in \widetilde{C}_1^{m-1}(]a, b[)$, i.e. the inclusion (2.38) holds. $\square$

Lemma 2.8. *Let* $\tau_j \in M(]a,b[)$, $\alpha \geqslant 0$, $\beta \geqslant 0$ *and let there exist* $\delta \in]0, b-a[$ *such that*

$$|\tau_j(t) - t| \leqslant k_1 (t-a)^{\beta} \quad \text{for } a < t \leqslant a+\delta. \tag{2.49}$$

Then

$$\left| \int_t^{\tau(t)} (s-a)^{\alpha} \, \mathrm{d}s \right| \leqslant \begin{cases} k_1 [1 + k_1 \delta^{\beta-1}]^{\alpha} (t-a)^{\alpha+\beta} & \text{for } \beta \geqslant 1, \\ k_1 [\delta^{1-\beta} + k_1]^{\alpha} (t-a)^{\alpha\beta+\beta} & \text{for } 0 \leqslant \beta < 1 \end{cases}$$

for $a < t \leqslant a + \delta$.

Proof. First note that

$$\left| \int_t^{\tau(t)} (s-a)^{\alpha} \, \mathrm{d}s \right| \leqslant (\max\{\tau(t), t\} - a)^{\alpha} |\tau(t) - t| \quad \text{for } a \leqslant t \leqslant a+\delta,$$

and $\max\{\tau(t), t\} \leqslant t + |\tau(t) - t|$ for $a \leqslant t \leqslant a + \delta$. Then in view of condition (2.49) we get

$$\left| \int_t^{\tau(t)} (s-a)^{\alpha} \, \mathrm{d}s \right| \leqslant k_1 [(t-a) + k_1 (t-a)^{\beta}]^{\alpha} (t-a)^{\beta} \quad \text{for } a \leqslant t \leqslant a+\delta.$$

The last inequality yields the validity of our lemma. □

Analogously one can prove

Lemma 2.9. *Let* $\tau_j \in M(]a,b[)$, $\alpha \geqslant 0$, $\beta \geqslant 0$ *and let there exist* $\delta \in]0, b-a[$ *such that*

$$|\tau_j(t) - t| \leqslant k_1 (b-t)^{\beta} \quad \text{for } b-\delta \leqslant t < b. \tag{2.50}$$

Then

$$\left| \int_t^{\tau(t)} (b-t)^{\alpha} \, \mathrm{d}s \right| \leqslant \begin{cases} k_1 [1 + k_1 \delta^{\beta-1}]^{\alpha} (b-t)^{\alpha+\beta} & \text{for } \beta \geqslant 1, \\ k_1 [\delta^{1-\beta} + k_1]^{\alpha} (b-t)^{\alpha\beta+\beta} & \text{for } 0 \leqslant \beta < 1 \end{cases}$$

for $b - \delta \leqslant t < b$.

2.3. Lemmas on the solutions of auxiliary problems.

Throughout this section we assume that the operator

$$P\colon\ C_1^{m-1}(]a,b[) \times C_1^{m-1}(]a,b[) \to L_n(]a,b[)$$

is γ_0, γ consistent with boundary condition (1.2), and the operator $q\colon C_1^{m-1}(]a,b[) \to \widetilde{L}^2_{2n-2m-2,2m-2}(]a,b[)$ is continuous.

Consider for any $x \in \widetilde{C}_1^{m-1}(]a,b[) \subset C_1^{m-1}(]a,b[)$ the nonhomogeneous equation

$$y^{(n)}(t) = \sum_{i=1}^{m} p_i(x)(t) y^{(i-1)}(\tau_i(t)) + q(x)(t), \tag{2.51}$$

and the corresponding homogeneous equation

$$y^{(n)}(t) = \sum_{i=1}^{m} p_i(x)(t) y^{(i-1)}(\tau_i(t)), \tag{2.52}$$

and let E^n be the set of solutions of problem (2.51), (2.26).

From inequality (1.23) of item (ii) of Definition 1.1 it follows that the boundary problem (2.51), (2.26) has a unique solution y in the space $\widetilde{C}^{n-1,m}(]a,b[)$. But in view of Lemma 2.6 it is clear that $y \in \widetilde{C}_1^{m-1}(]a,b[)$. Thus $E^n \cap \widetilde{C}_1^{m-1}(]a,b[) \neq \emptyset$, and there exists the operator $U\colon \widetilde{C}_1^{m-1}(]a,b[) \to E^n \cap \widetilde{C}_1^{m-1}(]a,b[)$ defined by the equality

$$U(x)(t) = y(t).$$

Lemma 2.10. *$U\colon \widetilde{C}_1^{m-1}(]a,b[) \to E^n \cap \widetilde{C}_1^{m-1}(]a,b[)$ is a continuous operator.*

Proof. Let $x_k \in \widetilde{C}_1^{m-1}(]a,b[)$ and $y_k(t) = U(x_k)(t)$ $(k = 1,2)$, $y = y_2 - y_1$, and let the operator P be defined by (1.19). Then

$$y^{(n)}(t) = P(x_2, y)(t) + q_0(x_1, x_2)(t)$$

where $q_0(x_1,x_2)(t) = P(x_2,y_1)(t) - P(x_1,y_1)(t) + q(x_2)(t) - q(x_1)(t)$. Hence, by item (ii) of Definition 1.1 we have

$$\|U(x_2) - U(x_1)\|_{\widetilde{C}_1^{m-1}} \leqslant \gamma \|q_0(x_1,x_2)\|_{\widetilde{L}^2_{2n-2m-2,2m-2}}.$$

Since the operators P and q are continuous, this estimate implies the continuity of the operator U. □

3. Proofs

P r o o f of Remark 1.1. Let x be a solution of problem (1.8), (1.2), then inequalities (2.27) imply the estimate

$$|x^{(i-1)}(t)| \leqslant \frac{[(b-t)(t-a)]^{m-i+1/2}}{(m-i)!(2m-2i+1)^{1/2}}\Big(\frac{2}{b-a}\Big)^{m-i+1/2}\|x^{(m)}\|_{L^2} \tag{3.1}$$

for $a \leqslant t \leqslant b$. This estimate, by the definition of the norm in the space $\widetilde{C}^{m-1}(]a,b[)$ and estimate (1.17) immediately yields (1.18). □

P r o o f of Theorem 1.3. Let δ and λ be the functions and numbers appearing in Definition 1.1. We set

$$\eta(t) = \delta(t,\gamma_0)\gamma_0 + \widetilde{F}_p(t,\min\{2\varrho_0,\gamma_0\}), \tag{3.2}$$

$$\chi(s) = \begin{cases} 1 & \text{for } 0 \leqslant s \leqslant \varrho_0, \\ 2 - s/\varrho_0 & \text{for } \varrho_0 < s < 2\varrho_0, \\ 0 & \text{for } s \geqslant 2\varrho_0, \end{cases} \tag{3.3}$$

$$q(x)(t) = \chi(\|x\|_{\widetilde{C}_1^{m-1}})F_p(x)(t). \tag{3.4}$$

From (1.24) it is clear that the nonnegative functions $\widetilde{F}_p, \eta$, admit the inclusion

$$\widetilde{F}_p(\cdot,\min\{2\varrho_0,\gamma_0\}),\ \eta \in \widetilde{L}^2_{2n-2m-2,2m-2}(]a,b[), \tag{3.5}$$

and for every $x \in A_{\gamma_0} \subset \widetilde{C}_1^{m-1}(]a,b[)$ and almost all $t \in\,]a,b[$ the inequality

$$|q(x)(t)| \leqslant \widetilde{F}_p(t,\min\{2\varrho_0,\gamma_0\}) \quad \text{for } a < t < b \tag{3.6}$$

holds.

Let $U\colon A_{\gamma_0} \to E^n \cap \widetilde{C}_1^{m-1}(]a,b[)$ be the operator appearing in Lemma 2.10, from which it follows that U is a continuous operator. On the other hand, from items (i) and (ii) of Definition 1.1, (1.25) and (3.6) it is clear that for each $x \in A_{\gamma_0}$, the conditions

$$\|y\|_{\widetilde{C}_1^{m-1}} \leqslant \gamma_0, \quad |y^{(n-1)}(t) - y^{(n-1)}(s)| \leqslant \int_s^t \eta(\xi)\,\mathrm{d}\xi \quad \text{for } a < t < b$$

hold. Thus in view of Definition 2.1 the operator U maps the ball A_{γ_0} into its own subset $S(\varrho_1,\eta)$. From Lemma 2.2 it follows that $S(\varrho_1,\eta)$ is a compact subset of the ball $A_{\gamma_0} \subset \widetilde{C}_1^{m-1}(]a,b[)$, i.e. the operator u maps the ball A_{γ_0} into its own compact subset. Therefore, owing to Schauders's principle, there exists $x \in S(\varrho_1, \eta) \subset A_{\gamma_0}$ such that

$$x(t) = U(x)(t) \quad \text{for } a < t < b.$$

Thus by (2.51) and notation (3.4), the function x ($x \in A_{\gamma_0}$) is a solution of problem (1.26), (1.2), where

$$\lambda = \chi(\|x\|_{\widetilde{C}_1^{m-1}}). \tag{3.7}$$

If $\gamma_0 = \varrho_0$ then in view of condition $x \in A_{\gamma_0}$, by (3.3) we have that $\lambda = 1$, and then in view of (2.51) and (3.4) the function x is a solution of problem (1.1), (1.2) which admits the estimate (1.27).

Let us show now that x admits estimate (1.27) in the case when $\varrho_0 < \gamma_0$. Assume the contrary. Then either

$$\varrho_0 < \|x\|_{\widetilde{C}_1^{m-1}} < 2\varrho_0, \tag{3.8}$$

or

$$\|x\|_{\widetilde{C}_1^{m-1}} \geqslant 2\varrho_0. \tag{3.9}$$

If condition (3.8) holds, then by virtue of (3.3) and (3.7) we have that $\lambda \in \,]0, 1[$, which by the conditions of our theorem guarantees the validity of estimate (1.27). But this contradicts (3.8).

Assume now that (3.9) is fulfilled. Then by virtue of (3.3) and (3.7) we have that $\lambda = 0$. Therefore $x \in A_{\gamma_0}$ is a solution of problem (2.52), (1.2). Thus from item (ii) of Definition 1.1 it is obvious that $x \equiv 0$, because problem (2.52), (1.2) has only the trivial solution. But this contradicts condition (3.9), i.e. estimate (1.27) is valid. From estimate (1.27) and (3.3) we have that $\lambda = 1$, and then in view of (2.51) and (3.4) the function x is a solution of problem (1.1), (1.2) which admits the estimate (1.27). □

P r o o f of Corollary 1.2. First note that in view of condition (1.30) there exists such $\gamma_0 > 2\varrho_0$ that condition (1.25) holds, and in view of Definition 1.2 the operator P is γ_0, γ consistent.

On the other hand, (1.30) implies the existence of a number ϱ_0 such that

$$\gamma \|\eta(\cdot, \varrho)\|_{\widetilde{L}^2_{2n-2m-2,2m-2}} < \varrho \quad \text{for } \varrho > \varrho_0. \tag{3.10}$$

Let x be a solution of problem (1.26), (1.2) for some $\lambda \in \,]0, 1[$. Then $y = x$ is also a solution of problem (1.22), (1.2) where $q(t) = \lambda(F(x)(t) - P(x, x)(t))$. Let now

$\varrho = \|x\|_{\widetilde{C}_1^{m-1}}$ and assume that

$$\varrho > \varrho_0 \tag{3.11}$$

holds. Then in view of the γ-consistency of the operator p with boundary conditions (1.2), inequality (1.23) holds and thus by condition (1.28) we have

$$\varrho = \|x\|_{\widetilde{C}_1^{m-1}} \leqslant \gamma \|q(x)\|_{\widetilde{L}^2_{2n-2m-2,2m-2}} \leqslant \gamma \|\eta(\cdot,\varrho)\|_{\widetilde{L}^2_{2n-2m-2,2m-2}}.$$

But the last inequality contradicts (3.10). Thus assumption (3.11) is not valid and $\varrho \leqslant \varrho_0$. Therefore for any $\lambda \in \,]0,1[$ an arbitrary solution of problem (1.26), (1.2) admits the estimate (1.27). Therefore all the conditions of Theorem 1.3 are fulfilled, from which the solvability of problem (1.1), (1.2) follows. □

P r o o f of Theorem 1.4. Let r_n be the constant defined in Remark 1.1. First we prove that the operator P is γ_0, r_n consistent with boundary conditions (1.2). From the conditions of our theorem it is obvious that the item (i) of Definition 1.1 is satisfied. Let now x be an arbitrary fixed function from the set A_{γ_0} and let $p_j(t) \equiv p_j(x)(t)$. Thus in view of (1.34), (1.35) all the assumptions of Theorem 1.1 are satisfied, and then for any $q \in \widetilde{L}^2_{2n-2m-2,2m-2}(]a,b[)$ problem (1.22), (1.2) has a unique solution y. Also in view of Remark 1.1 there exists a constant $r_n > 0$ (which depends only on the numbers $l_{kj}, \overline{l}_{kj}, \gamma_{kj}$ $(k = 0,1;\ j = 1,\dots,m)$ and a, b, t^*, n) such that estimate (1.23) holds with $\gamma = r_n$. So, the operator P is γ_0, r_n consistent with boundary conditions (1.2). Therefore all the assumptions of Corollary 1.1 are fulfilled, from which the solvability of problem (1.1), (1.2) follows. □

P r o o f of Theorem 1.5. Let r_n be the constant defined in Remark 1.1. First we prove that the operator P is r_n consistent with boundary conditions (1.2). From the conditions of our theorem it is obvious that the item (i) of Definition 1.1 is satisfied. Let now γ_0 be an arbitrary nonnegative number, x an arbitrary fixed function from the space A_{γ_0} and let $p_j(t) \equiv p_j(x)(t)$. Then in view of (1.37), (1.38) all the assumptions of Theorem 1.1 are satisfied and then for any $q \in \widetilde{L}^2_{2n-2m-2,2m-2}(]a,b[)$ problem (1.22), (1.2) has a unique solution y. Also in view of Remark 1.1 there exists a constant $r_n > 0$ (which depends only on the numbers l_{kj}, $\overline{l}_{kj}$, γ_{kj} $(k = 0,1;$ $j = 1,\dots,m)$ and a, b, t^*, n,) such that estimate (1.23) holds with $\gamma = r_n$. So, the operator P is γ_0, r_n consistent with boundary conditions (1.2) for arbitrary $\gamma_0 > 0$. Thus by Definition 1.1, the operator P is r_n consistent with boundary conditions (1.2). Therefore all the assumptions of Corollary 1.2 are fulfilled, from which the solvability of problem (1.1), (1.2) follows. □

Proof of Remark 1.2. By Schwartz's inequality, the definition of the norm $\|y\|_{\widetilde{C}_1^{m-1}}$ and inequalities (1.39), (2.2) for any $x, y \in A_{\gamma_0}$ and $z = y - x$ we have

$$
\begin{aligned}
(3.12)\qquad |p_j(y)(t)z^{(j-1)}(\tau_j(t))| &= |p_j(y)(t)z^{(j-1)}(t)| + |p_j(y)(t)|\left|\int_t^{\tau_j(t)} z^{(j)}(\psi)\,\mathrm{d}\psi\right| \\
&\leqslant \|z\|_{\widetilde{C}_1^{m-1}}|p_j(y)(t)|\alpha_j(t)\biggl(1 + \frac{1}{\alpha_j(t)}\biggl(\int_t^{\tau_j(t)} (\psi - a)^{2m-2j}\,\mathrm{d}\psi\biggr)^{1/2}\biggr)
\end{aligned}
$$

for $a < t < b$. On the other hand, from the conditions (1.40) by Lemmas 2.8 and 2.9 it is cleat that

$$
\begin{aligned}
\alpha_j^{-1}(s)\biggl(\int_s^{\tau_j(s)} (\xi - a)^{2m-2j}\,\mathrm{d}\xi\biggr)^{1/2} &\leqslant \frac{\sqrt{\kappa(1+\kappa)}}{\varepsilon^{m-j+1/2}} \quad \text{for } s \in\,]a, a+\varepsilon] \cup [b-\varepsilon, b[, \\
\alpha_j^{-1}(s)\biggl(\int_s^{\tau_j(s)} (\xi - a)^{2m-2j}\,\mathrm{d}\xi\biggr)^{1/2} &\leqslant \varepsilon^{-2m+2j-1}\biggl(\int_a^b (\xi - a)^{2m-2j}\,\mathrm{d}\xi\biggr)^{1/2} \\
&= \frac{(b-a)^{m-j+1/2}}{\sqrt{2m-2j+1}\,\varepsilon^{2m-2j+1}} \quad \text{for } s \in\,]a+\varepsilon, b-\varepsilon[.
\end{aligned}
$$

Then if we put

$$
(3.13)\qquad \kappa_0 = \max_{1 \leqslant j \leqslant m}\left\{\frac{\sqrt{\kappa(1+\kappa)}}{\varepsilon^{m-j+1/2}}, \frac{(b-a)^{m-j+1/2}}{\sqrt{2m-2j+1}\,\varepsilon^{2m-2j+1}}\right\},
$$

from (3.12) by the last estimates we get the inequality

$$
\begin{aligned}
(3.14)\qquad |p_j(y)(t)z^{(j-1)}(\tau_j(t))| &\leqslant \|z\|_{\widetilde{C}_1^{m-1}}(1+\kappa_0)|p_j(y)(t)|\alpha_j(t) \\
&\leqslant \|z\|_{\widetilde{C}_1^{m-1}}(1+\kappa_0)\delta_j(t, \|y\|_{\widetilde{C}_1^{m-1}})
\end{aligned}
$$

for $a < t < b$. Analogously we get that

$$
|(p_j(y)(t) - p_j(x)(t))x^{(j-1)}(\tau_j(t))| \leqslant \|x\|_{\widetilde{C}_1^{m-1}}(1+\kappa_0)|p_j(y)(t) - p_j(x)(t)|\alpha_j(t)
$$

for $a < t < b$. From (3.14) and the last inequality it is obvious that the operator P defined by equality (1.19) is continuously acting from A_{γ_0} to the space $L_n(]a, b[)$, and the item (ii) of Definition 1.1 holds with $\delta(t, \varrho) = (1+\kappa_0)\sum\limits_{j=1}^{m}\delta_j(t, \varrho)$. □

Proof of Corollary 1.3. From conditions (1.42) and (1.40) by Remark 1.2 we obtain that the operator P defined by equality (1.19) with $p_j(x)(t) = p_j(t)$ is continuously acting from A_{γ_0} to the space $L_n(]a, b[)$ for any $\gamma_0 > 0$, i.e., it is continuously acting from $\widetilde{C}_1^{m-1}(]a, b[)$ to the space $L_n(]a, b[)$.

Therefore it is clear that all the conditions of Theorem 1.5 are satisfied with

$$F(x)(t) = f(t, x(\tau_1(t)), x'(\tau_2(t)), \dots, x^{(m-1)}(\tau_m(t))), \quad \delta(t, \varrho) = (1 + \kappa_0) \sum_{j=1}^{m} |p_j(t)|,$$

where the constant κ_0 is defined by equality (3.13). Thus problem (1.41), (1.2) is solvable. □

P r o o f of Corollary 1.4. Let the operators $F, p_1\colon C^{m-1}(]a,b[) \to L_{\rm loc}(]a,b[)$, and the function $\eta\colon\ ]a,b[\times \mathbb{R}^+ \to \mathbb{R}^+$ be defined by equalities

$$F(x)(t) = -\frac{\lambda |x(t)|^k}{[(t-a)(b-t)]^{2+k/2}} x(\tau(t)) + q(x)(t), \;\; p_1(x)(t) = -\frac{\lambda |x(t)|^k}{[(t-a)(b-t)]^{2+k/2}}.$$

Then it is easy to verify that in view of (1.46), (1.47), (1.48), conditions (1.13), (1.14), (1.28), (1.34), (1.35), (1.36), (1.37), (1.38), (1.39), (1.40), (1.41), (1.42), (1.43) are satisfied with

$$\text{(3.15)}\; \delta(t, \varrho) = \frac{\varrho^k \lambda}{[(t-a)(b-t)]^2}, \quad l_{01} = l_{11} = \frac{4\gamma_0^k \lambda}{(b-a)^2}, \quad \bar{l}_{01} = \bar{l}_{11} = \frac{16\gamma_0^k \lambda}{(b-a)^2},$$
$$r_2 = \left(1 + \sqrt{\frac{2}{b-a}}\right) \frac{2(1+b-a)(b-a)^2}{(b-a)^2 - 16\lambda\gamma_0^k (1 + [2(b-a)]^{1/4})},$$
$$B_0 = B_1 = \frac{16\lambda\gamma_0^k}{(b-a)^2} (1 + [2(b-a)]^{1/4}), \quad t^* = \frac{a+b}{2}, \quad \gamma_{01} = \gamma_{11} = \frac{1}{4}.$$

Thus all the condition of Theorem 1.4 are satisfied, from which solvability of problem (1.44), (1.2) follows. □

P r o o f of Corollary 1.5. Let the operators $F, p_1\colon C^{m-1}(]a,b[) \to L_{\rm loc}(]a,b[)$, and the function $\eta\colon\ ]a,b[\times \mathbb{R}^+ \to \mathbb{R}^+$ be defined by equalities

$$F(x)(t) = -\frac{\lambda |\sin x^k(t)|}{[(t-a)(b-t)]^2} x(\tau(t)) + q(x)(t), \quad p_1(x)(t) = -\frac{\lambda |\sin x^k(t)|}{[(t-a)(b-t)]^2}.$$

Then it is easy to verify that in view of (1.30), (1.46), and (1.49), all the conditions of Theorem 1.5 are fulfilled, where δ, l_{01}, $\bar{l}_{01}$, r_2, B_0, B_1, t^*, γ_{01}, γ_{11}, are defined by (3.15) with $\varrho = 1$, $\gamma_0 = 1$, which implies solvability of problem (1.44), (1.2). □

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