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TWO IDENTITIES RELATED TO DIRICHLET CHARACTER OF POLYNOMIALS

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Abstract. Let q be a positive integer, χ denote any Dirichlet character $\bmod q$. For any integer m with $(m,q)=1$, we define a sum $C(\chi,k,m;q)$ analogous to high-dimensional Kloosterman sums as follows:

$$C(\chi,k,m;q)=\sum_{a_1=1}^{q}{}' \sum_{a_2=1}^{q}{}' \dots \sum_{a_k=1}^{q}{}' \chi(a_1+a_2+\dots+a_k+m\overline{a_1a_2\dots a_k}),$$

where $a\cdot\overline{a}\equiv 1 \bmod q$. The main purpose of this paper is to use elementary methods and properties of Gauss sums to study the computational problem of the absolute value $|C(\chi,k,m;q)|$, and give two interesting identities for it.

Keywords: Dirichlet character of polynomials, sum analogous to Kloosterman sum, identity, Gauss sum

MSC 2010: 11L05

1. Introduction

Let q be a positive integer, χ denote any Dirichlet character $\bmod q$. For any integer m with $(m,q)=1$, we define a sum analogous to Kloosterman sums $C(\chi,k,m;q)$ as follows:

$$C(\chi,k,m;q)=\sum_{a_1=1}^{q}{}' \sum_{a_2=1}^{q}{}' \dots \sum_{a_k=1}^{q}{}' \chi(a_1+a_2+\dots+a_k+m\overline{a_1a_2\dots a_k}),$$

where $a\cdot\overline{a}\equiv 1 \bmod q$.

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It is clear that this sum very similar to a hyper-Kloosterman sum, it is also a special case of the general character sums of the polynomials

$$\sum_{x_1=N_1+1}^{N_1+M_1}\sum_{x_2=N_2+1}^{N_2+M_2}\cdots\sum_{x_s=N_s+1}^{N_s+M_s}\chi(f(x_1,x_2,\ldots,x_s)), \tag{1}$$

where M_i and N_i are any positive integers, and $f(x_1,x_2,\ldots,x_s)$ is a polynomial.

It is a very important and difficult problem in analytic number theory to give a sharper upper bound estimate for (1). If $q=p$ is an odd prime and $s=1$, then Weil (see [1]) obtained the following important conclusion:

Let χ be a qth-order character $\bmod p$. If $f(x)$ is not a perfect qth power $\bmod p$, then we have the estimate

$$\sum_{x=N+1}^{N+M}\chi(f(x))\ll p^{\frac{1}{2}}\ln p, \tag{2}$$

where $A\ll B$ denotes $|A|<cB$ for some constant c, which in this case depends only on the degree of $f(x)$. Some related results can also be found in [2], [5] and [6]. The main term $p^{\frac{1}{2}}$ in (2) is the best possible. In fact, Zhang Wenpeng and Yao Weili [6] found some polynomials $f(x)=(x-r)^m(x-s)^n$ such that

$$\sum_{a=1}^{q}\chi((a-r)^m(a-s)^n)=\sqrt{q}\,\chi((sm-rm)^m(rn-sn)^n))\overline{\chi}((m+n)^{m+n}),$$

where $(r-s,q)=1$, m, n and χ also satisfy some special conditions.

In this paper, we shall use elementary methods and properties of Gauss sums to study the computational problem of $C(\chi,k,m;q)$, and give two interesting identities. That is, we shall prove the following conclusions:

Theorem 1. *Let q be an odd number, k and m be positive integers such that $(km,q)=1$ and $(k+1,\varphi(q))=1$. Then for any primitive character $\chi \bmod q$, we have the identity*

$$|C(\chi,k,m;q)|=\left|\sum_{a_1=1}^{q}{}'\sum_{a_2=1}^{q}{}'\cdots\sum_{a_k=1}^{q}{}'\chi(a_1+a_2+\ldots+a_k+m\overline{a_1a_2\ldots a_k})\right|=\sqrt{q^k},$$

where $\overline{a}$ denotes the solution of the congruent equation $ax\equiv 1 \bmod q$.

Theorem 2. *Let q be an odd perfect square, k and m be positive integers such that $(km,q)=1$ and $(k+1,\varphi(q))=1$. Then for any primitive character $\chi \bmod q$, we have the identity*

$$\Bigl|\sum_{a=1}^{q}{}'\chi(a+m\overline{a}^k)\Bigr| = \sqrt{q}.$$

For the general integer $q \geqslant 3$ (or the general positive integer k), whether there exists an identity for

$$\Bigl|\sum_{a=1}^{q}{}'\chi(a+m\overline{a}^k)\Bigr|,$$

is an open problem.

2. Several Lemmas

To complete the proof of our Theorems, we need the following several lemmas.

Lemma 1. *Let p be an odd prime, α be a positive integer, χ be any primitive character $\bmod\, p^\alpha$. Then for any integer m and positive integer k with $(m(k+1), p(p-1))=1$, we have the identity*

$$\Bigl|\sum_{a_1=1}^{p^\alpha}{}'\sum_{a_2=1}^{p^\alpha}{}'\dots\sum_{a_k=1}^{p^\alpha}{}'\chi(a_1+a_2+\dots+a_k+m\overline{a_1a_2\dots a_k})\Bigr| = p^{k\alpha/2},$$

where $\sum\limits_{a=1}^{p^\alpha}{}'$ denotes the summation over all $1\leqslant a\leqslant p^\alpha$ such that $(a,p)=1$.

P r o o f. From the properties of the classical Gauss sums and the reduced residue system $\bmod\, p^\alpha$ we have (in what follows we use $e(t)=\mathrm{e}^{2\pi \mathrm{i}t}$)

$$\begin{aligned}(3)\qquad &\sum_{a_1=1}^{p^\alpha}{}'\sum_{a_2=1}^{p^\alpha}{}'\dots\sum_{a_k=1}^{p^\alpha}{}'\chi(a_1+a_2+\dots+a_k+m\overline{a_1a_2\dots a_k})\\
&=\frac{1}{\tau(\overline{\chi})}\sum_{a_1=1}^{p^\alpha}{}'\dots\sum_{a_k=1}^{p^\alpha}{}'\sum_{b=1}^{p^\alpha}\overline{\chi}(b)\, e\Bigl(\frac{b(a_1+a_2+\dots+a_k+m\overline{a_1a_2\dots a_k})}{p^\alpha}\Bigr)\\
&=\frac{1}{\tau(\overline{\chi})}\sum_{a_1=1}^{p^\alpha}{}'\dots\sum_{a_k=1}^{p^\alpha}{}'\sum_{b=1}^{p^\alpha}\overline{\chi}(b)\, e\Bigl(\frac{a_1+a_2+\dots+a_k+mb^{k+1}\overline{a_1a_2\dots a_k}}{p^\alpha}\Bigr)\\
&=\frac{1}{\tau(\overline{\chi})}\sum_{a_1=1}^{p^\alpha}{}'\dots\sum_{a_k=1}^{p^\alpha}{}'\sum_{b=1}^{p^\alpha}\overline{\chi}(b)\, e\Bigl(\frac{a_1b^{k+1}+a_2+\dots+a_k+m\overline{a_1a_2\dots a_k}}{p^\alpha}\Bigr).\end{aligned}$$

Since $(k+1, p(p-1)) = 1$, there exist two integers r and s such that $r\varphi(p^\alpha) + s(k+1) = 1$. If b runs through a reduced residue system $\bmod p^\alpha$, then b^{k+1} also runs through a reduced residue system $\bmod p^\alpha$. So from (3) we have

$$
\begin{aligned}
(4)\qquad & \sideset{}{'}\sum_{a_1=1}^{p^\alpha}\sideset{}{'}\sum_{a_2=1}^{p^\alpha}\dots\sideset{}{'}\sum_{a_k=1}^{p^\alpha}\chi(a_1+a_2+\ldots+a_k+m\overline{a_1a_2\ldots a_k})\\
&= \frac{1}{\tau(\overline{\chi})}\sideset{}{'}\sum_{a_1=1}^{p^\alpha}\dots\sideset{}{'}\sum_{a_k=1}^{p^\alpha}\sum_{b=1}^{p^\alpha}\overline{\chi}(b^{s(k+1)})\ e\Big(\frac{a_1b^{k+1}+a_2+\ldots+a_k+m\overline{a_1a_2\ldots a_k}}{p^\alpha}\Big)\\
&= \frac{1}{\tau(\overline{\chi})}\sideset{}{'}\sum_{a_1=1}^{p^\alpha}\dots\sideset{}{'}\sum_{a_k=1}^{p^\alpha}\sum_{b=1}^{p^\alpha}\overline{\chi}^s(b)\ e\Big(\frac{a_1b+a_2+\ldots+a_k+m\overline{a_1a_2\ldots a_k}}{p^\alpha}\Big)\\
&= \frac{\tau(\overline{\chi}^s)}{\tau(\overline{\chi})}\sideset{}{'}\sum_{a_1=1}^{p^\alpha}\dots\sideset{}{'}\sum_{a_k=1}^{p^\alpha}\chi^s(a_1)\ e\Big(\frac{a_2+a_3+\ldots+a_k+m\overline{a_1a_2\ldots a_k}}{p^\alpha}\Big)\\
&= \frac{\tau^2(\overline{\chi}^s)}{\tau(\overline{\chi})}\sideset{}{'}\sum_{a_2=1}^{p^\alpha}\dots\sideset{}{'}\sum_{a_k=1}^{p^\alpha}\chi^s(m\overline{a_2a_3\ldots a_k})\ e\Big(\frac{a_2+a_3\ldots+a_k}{p^\alpha}\Big)\\
&= \frac{\tau^{k+1}(\overline{\chi}^s)}{\tau(\overline{\chi})}\chi^s(m).
\end{aligned}
$$

Note that $|\tau(\overline{\chi})| = |\tau(\overline{\chi}^s)| = p^{\alpha/2}$, $|\chi^s(m)| = 1$, from (4) we may immediately deduce

$$\Big|\sideset{}{'}\sum_{a_1=1}^{p^\alpha}\sideset{}{'}\sum_{a_2=1}^{p^\alpha}\dots\sideset{}{'}\sum_{a_k=1}^{p^\alpha}\chi(a_1+a_2+\ldots+a_k+m\overline{a_1a_2\ldots a_k})\Big| = p^{k\alpha/2}.$$

This proves Lemma 1. □

Now we introduce a hyper-Kloosterman sum by

$$K(q, m+1, z) = \sum_{\substack{x_1, x_2, \ldots, x_m \bmod q\\ (x_1,p)=\ldots=(x_m,\ p)=1}} e\Big(\frac{x_1+\ldots+x_m+z\overline{x}_1\ldots\overline{x}_m}{q}\Big)$$

for $q = p^\alpha$, $m \geqslant 1$, and z not divisible by p. Define an exponential sum by

$$I(q, m, z) = \sum_{\substack{x \bmod q\\ (x,p)=1}} e\Big(\frac{mx+z\overline{x}^m}{q}\Big).$$

Under the above notations, we have the following:

Lemma 2. *Let p be an odd prime, $q = p^{2\alpha}$. Then for any integers $m \geqslant 1$ and z with $(z, p) = 1$, we have the identity*

$$K(q, m+1, z) = q^{(m-1)/2} I(q, m, z).$$

P r o o f. This identity follows from R. A. Smith [3] or Yangbo Ye [4]. □

Lemma 3. *Let p be an odd prime, α be any positive integer, χ be any primitive character* $\bmod\, p^{2\alpha}$. *Then for any integer m and positive integer k with $(k+1, p(p-1)) = (km, p) = 1$, we have the identity*

$$\Bigl|\sideset{}{'}\sum_{a_1=1}^{p^{2\alpha}} \sideset{}{'}\sum_{a_2=1}^{p^{2\alpha}} \cdots \sideset{}{'}\sum_{a_k=1}^{p^{2\alpha}} \chi(a_1 + a_2 + \ldots + a_k + m\overline{a_1 a_2 \ldots a_k})\Bigr|$$
$$= p^{\alpha(k-1)} \Bigl|\sideset{}{'}\sum_{a=1}^{p^{2\alpha}} \chi(ka + m\overline{a}^k)\Bigr|.$$

P r o o f. From Lemma 2 and the method of proving (4) we have

$$(5) \qquad \sideset{}{'}\sum_{a_1=1}^{p^{2\alpha}} \sideset{}{'}\sum_{a_2=1}^{p^{2\alpha}} \cdots \sideset{}{'}\sum_{a_k=1}^{p^{2\alpha}} \chi(a_1 + a_2 + \ldots + a_k + m\overline{a_1 a_2 \ldots a_k})$$
$$= \frac{1}{\tau(\overline{\chi})} \sum_{b=1}^{p^{2\alpha}} \overline{\chi}(b) \sideset{}{'}\sum_{a_1=1}^{p^{2\alpha}} \cdots \sideset{}{'}\sum_{a_k=1}^{p^{2\alpha}} e\Bigl(\frac{a_1 + a_2 + \ldots + a_k + mb^{k+1}\overline{a_1 a_2 \ldots a_k}}{p^{2\alpha}}\Bigr)$$
$$= \frac{p^{\alpha(k-1)}}{\tau(\overline{\chi})} \sum_{b=1}^{p^{2\alpha}} \overline{\chi}(b) \sideset{}{'}\sum_{a=1}^{p^{2\alpha}} e\Bigl(\frac{ka + mb^{k+1}\overline{a}^k}{p^{2\alpha}}\Bigr)$$
$$= \frac{p^{\alpha(k-1)}}{\tau(\overline{\chi})} \sum_{b=1}^{p^{2\alpha}} \overline{\chi}(b) \sideset{}{'}\sum_{a=1}^{p^{2\alpha}} e\Bigl(\frac{kab + mb\overline{a}^k}{p^{2\alpha}}\Bigr)$$
$$= \frac{p^{\alpha(k-1)}}{\tau(\overline{\chi})} \sideset{}{'}\sum_{a=1}^{p^{2\alpha}} \sum_{b=1}^{p^{2\alpha}} \overline{\chi}(b)\ e\Bigl(\frac{b(ka + m\overline{a}^k)}{p^{2\alpha}}\Bigr)$$
$$= p^{\alpha(k-1)} \cdot \sideset{}{'}\sum_{a=1}^{p^{2\alpha}} \chi(ka + m\overline{a}^k),$$

from which Lemma 3 follows. □

Lemma 4. *Let q_1 and q_2 be two positive integers with $(q_1, q_2) = 1$, $\chi_1 \bmod q_1$ and $\chi_2 \bmod q_2$. Then for any integers $m \geqslant 1$ and n with $(n, q_1q_2) = 1$, we have*

$$\begin{aligned}&\sideset{}{'}\sum_{a_1=1}^{q_1q_2}\sideset{}{'}\sum_{a_2=1}^{q_1q_2}\cdots\sideset{}{'}\sum_{a_m=1}^{q_1q_2}\chi_1\chi_2(a_1+a_2+\ldots+a_m+n\overline{a}_1\overline{a}_2\ldots\overline{a}_m)\\&\quad=\sideset{}{'}\sum_{a_1=1}^{q_1}\sideset{}{'}\sum_{a_2=1}^{q_1}\cdots\sideset{}{'}\sum_{a_m=1}^{q_1}\chi_1(a_1+a_2+\ldots+a_m+n\overline{a}_1\overline{a}_2\ldots\overline{a}_m)\\&\quad\quad\times\sideset{}{'}\sum_{b_1=1}^{q_2}\sideset{}{'}\sum_{b_2=1}^{q_2}\cdots\sideset{}{'}\sum_{b_m=1}^{q_2}\chi_2(b_1+b_2+\ldots+b_m+n\overline{b}_1\overline{b}_2\ldots\overline{b}_m)\end{aligned}$$

P r o o f. Since $(q_1, q_2) = 1$, from the properties of the reduced residue system $\bmod q_1q_2$ we have

$$\begin{aligned}&\sideset{}{'}\sum_{a_1=1}^{q_1q_2}\sideset{}{'}\sum_{a_2=1}^{q_1q_2}\cdots\sideset{}{'}\sum_{a_m=1}^{q_1q_2}\chi_1\chi_2(a_1+a_2+\ldots+a_m+n\overline{a}_1\overline{a}_2\ldots\overline{a}_m)\\&=\sideset{}{'}\sum_{a_1=1}^{q_1}\sideset{}{'}\sum_{b_1=1}^{q_2}\sideset{}{'}\sum_{a_2=1}^{q_1}\sideset{}{'}\sum_{b_2=1}^{q_2}\cdots\sideset{}{'}\sum_{a_m=1}^{q_1}\sideset{}{'}\sum_{b_m=1}^{q_2}\chi_1\chi_2\bigg(\sum_{i=1}^{m}(a_iq_2+b_iq_1)+n\prod_{i=1}^{m}\overline{a_iq_2+b_iq_1}\bigg)\\&=\sideset{}{'}\sum_{a_1=1}^{q_1}\sideset{}{'}\sum_{a_2=1}^{q_1}\cdots\sideset{}{'}\sum_{a_m=1}^{q_1}\chi_1\bigg(\sum_{i=1}^{m}a_iq_2+n\prod_{i=1}^{m}\overline{a_iq_2}\bigg)\\&\quad\times\sideset{}{'}\sum_{b_1=1}^{q_2}\sideset{}{'}\sum_{b_2=1}^{q_2}\cdots\sideset{}{'}\sum_{b_m=1}^{q_2}\chi_2\bigg(\sum_{i=1}^{m}b_iq_1+n\prod_{i=1}^{m}\overline{b_iq_1}\bigg)\\&=\sideset{}{'}\sum_{a_1=1}^{q_1}\sideset{}{'}\sum_{a_2=1}^{q_1}\cdots\sideset{}{'}\sum_{a_m=1}^{q_1}\chi_1(a_1+a_2+\ldots+a_m+n\overline{a}_1\overline{a}_2\ldots\overline{a}_m)\\&\quad\times\sideset{}{'}\sum_{b_1=1}^{q_2}\sideset{}{'}\sum_{b_2=1}^{q_2}\cdots\sideset{}{'}\sum_{b_m=1}^{q_2}\chi_2(b_1+b_2+\ldots+b_m+n\overline{b}_1\overline{b}_2\ldots\overline{b}_m).\end{aligned}$$

This proves Lemma 4. □

3. Proof of the Theorems

In this section, we shall complete the proof of our Theorems. First we prove Theorem 1. Let $q = p_1^{\alpha_1}p_2^{\alpha_2}\ldots p_r^{\alpha_r}$ denote the factorization of q into prime powers. Then for any primitive character $\chi \bmod q$, we have $\chi = \chi_1\chi_2\ldots\chi_r$, where χ_i is a primitive character $\bmod p_i^{\alpha_i}$, $i = 1, 2, \ldots, r$. From Lemma 1 and Lemma 4 with

$(km, q) = 1$, $(k+1, \varphi(q)) = 1$ we have

$$\begin{aligned}
&\Big|\sideset{}{'}\sum_{a_1=1}^{q}\sideset{}{'}\sum_{a_2=1}^{q}\dots\sideset{}{'}\sum_{a_k=1}^{q}\chi(a_1+a_2+\ldots+a_k+m\overline{a_1a_2\ldots a_k})\Big| \\
&\quad = \prod_{i=1}^{r}\Big|\sideset{}{'}\sum_{a_1=1}^{p_i^{\alpha_i}}\sideset{}{'}\sum_{a_2=1}^{p_i^{\alpha_i}}\dots\sideset{}{'}\sum_{a_k=1}^{p_i^{\alpha_i}}\chi(a_1+a_2+\ldots+a_k+m\overline{a_1a_2\ldots a_k})\Big| \\
&\quad = \prod_{i=1}^{r}\sqrt{p_i^{\alpha_i}} = \sqrt{q}.
\end{aligned}$$

This proves Theorem 1. □

Now we prove Theorem 2. Applying Lemma 1 and Lemma 3 we have the identity

$$\begin{aligned}
p^{\alpha k} &= \Big|\sideset{}{'}\sum_{a_1=1}^{p^{2\alpha}}\sideset{}{'}\sum_{a_2=1}^{p^{2\alpha}}\dots\sideset{}{'}\sum_{a_k=1}^{p^{2\alpha}}\chi(a_1+a_2+\ldots+a_k+mk\overline{a_1a_2\ldots a_k})\Big| \\
&= p^{\alpha(k-1)}\cdot\Big|\sideset{}{'}\sum_{a=1}^{p^{2\alpha}}\chi(ka+mk\overline{a}^k)\Big|
\end{aligned}$$

or

$$\Big|\sideset{}{'}\sum_{a=1}^{p^{2\alpha}}\chi(a+m\overline{a}^k)\Big| = p^{\alpha}. \tag{6}$$

From (6), Lemma 4 and the properties of the primitive character $\chi \bmod q$ we may immediately deduce the identity

$$\Big|\sideset{}{'}\sum_{a=1}^{q}\chi(a+m\overline{a}^k)\Big| = \sqrt{q}.$$

This completes the proof of Theorem 2. □

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