

Belmesnaoui Aqzzouz; Jawad H'michane
AM-Compactness of some classes of operators

Commentationes Mathematicae Universitatis Carolinae, Vol. 53 (2012), No. 4, 509--518

Persistent URL: <http://dml.cz/dmlcz/143185>

Terms of use:

© Charles University in Prague, Faculty of Mathematics and Physics, 2012

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.



This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* <http://project.dml.cz>

AM-Compactness of some classes of operators

BELMESNAOUI AQZZOUZ, JAWAD H'MICHANE

Abstract. We characterize Banach lattices on which each regular order weakly compact (resp. b-weakly compact, almost Dunford-Pettis, Dunford-Pettis) operator is AM-compact.

Keywords: AM-compact operator, order weakly compact operator, b-weakly compact operator, almost Dunford-Pettis operator, b-AM-compact operator, order continuous norm, discrete Banach lattice

Classification: 46A40, 46B40, 46B42

1. Introduction and notation

The class of AM-compact operators is introduced and studied by Dodds-Fremlin [14] and its domination problem is characterized in [5]. Recall that a regular operator T from a Banach lattice E into a Banach space F is said to be AM-compact if it carries each order bounded subset of E onto a relatively compact subset of F .

On the other hand, each regular compact operator is AM-compact, but an AM-compact operator is not necessarily compact. In fact, the identity operator of the Banach lattice ℓ^1 is AM-compact (because ℓ^1 is discrete with order continuous norm) but it is not compact. However, if E is an AM-space with unit, the class of regular compact operators coincides with that of AM-compact operators. For a more detailed study of this class of operators we refer the reader to the book of Zaanen [21].

In this paper we are interested in three classes of operators. The first one is bigger than that of AM-compact operators. It is the class of order weakly compact operators introduced by Dodds [13]. Recall that an operator T from a Banach lattice E into a Banach space F is said to be order weakly compact if for each $x \in E_+$, the set $T([0, x])$ is relatively weakly compact in F . Note that an order weakly compact operator is not necessarily AM-compact. In fact, the identity operator $Id_{L^1[0,1]} : L^1[0,1] \rightarrow L^1[0,1]$ is order weakly compact (because the norm of $L^1[0,1]$ is order continuous), but it is not AM-compact (because $L^1[0,1]$ is not discrete).

The second class is that of b-weakly compact operators introduced by Alpay-Altin-Tonyali [3]. An operator T from a Banach lattice E into a Banach space F is said to be b-weakly compact if for each b-order bounded subset A of E (i.e. order bounded in the topological bidual E''), $T(A)$ is relatively weakly compact in F . Note that there is an AM-compact operator which is not b-weakly compact

and conversely there is a b-weakly compact operator which is not AM-compact. In fact, the identity operator $Id_{L^1[0,1]} : L^1[0,1] \rightarrow L^1[0,1]$ is b-weakly compact (because $L^1[0,1]$ is KB-space), but it is not AM-compact (because $L^1[0,1]$ is not discrete), and conversely the identity operator $Id_{c_0} : c_0 \rightarrow c_0$ is AM-compact (because c_0 is discrete with order continuous norm), but is not b-weakly compact (because c_0 is not KB-space).

The third class is that of almost Dunford-Pettis operators introduced by Sanchez in [18]. Recall from [20] that an operator T from a Banach lattice E into a Banach space F is called almost Dunford-Pettis if the sequence $(\|T(x_n)\|)$ converges to 0 for every weakly null sequence (x_n) consisting of pairwise disjoint elements in E . Note that there is an AM-compact operator which is not almost Dunford-Pettis, and conversely there is an almost Dunford-Pettis operator which is not AM-compact. In fact, the identity operator $Id_{L^1[0,1]} : L^1[0,1] \rightarrow L^1[0,1]$ is almost Dunford-Pettis (because $L^1[0,1]$ has the positive Schur property) but it is not AM-compact, and conversely the identity operator $Id_{c_0} : c_0 \rightarrow c_0$ is AM-compact but is not almost Dunford-Pettis (because c_0 does not have the positive Schur property).

In [6], we studied the AM-compactness of positive Dunford-Pettis operators. The aim of this paper is to extend this study to other classes of operators, by characterizing Banach lattices for which each regular order weakly compact (resp. b-weakly compact, almost Dunford-Pettis, Dunford-Pettis) operator is AM-compact. Also, we will give some interesting consequences.

To state our results, we need to fix some notation and recall some definitions. A vector lattice is said to be Dedekind σ -complete if every nonempty countable subset that is bounded from above has a supremum. A nonzero element x of a vector lattice E is discrete if the order ideal generated by x equals the lattice subspace generated by x . The vector lattice E is discrete, if it admits a complete disjoint system of discrete elements. A Banach lattice is a Banach space $(E, \|\cdot\|)$ such that E is a vector lattice and its norm satisfies the following property: for each $x, y \in E$ such that $|x| \leq |y|$, we have $\|x\| \leq \|y\|$. Note that the topological dual E' , endowed with the dual norm and the dual order, is also a Banach lattice. A norm $\|\cdot\|$ of a Banach lattice E is order continuous if for each generalized sequence (x_α) such that $x_\alpha \downarrow 0$ in E , the sequence (x_α) converges to 0 for the norm $\|\cdot\|$ where the notation $x_\alpha \downarrow 0$ means that the sequence (x_α) is decreasing, its infimum exists and $\inf(x_\alpha) = 0$. A Banach lattice E is said to be a KB-space whenever every increasing norm bounded sequence of E^+ is norm convergent. As an example, each reflexive Banach lattice is a KB-space. A Banach lattice E is said to be an AM-space if for each $x, y \in E$ such that $\inf(x, y) = 0$, we have $\|x + y\| = \max\{\|x\|, \|y\|\}$. A Banach lattice E is said to have weakly sequentially continuous lattice operations whenever $x_n \rightarrow 0$ in $\sigma(E, E')$ implies $\|x_n\| \rightarrow 0$ in $\sigma(E, E')$. Note that every AM-space has this property ([2, Theorem 4.31]). Also, any discrete Banach lattice with an order continuous norm has weakly sequentially continuous lattice operations ([17, Proposition 2.5.23]).

For a bounded linear mapping $T : E \rightarrow F$ between two Banach lattices, we will use the term operator. It is positive if $T(x) \geq 0$ in F whenever $x \geq 0$ in E . An operator $T : E \rightarrow F$ is regular if $T = T_1 - T_2$ where T_1 and T_2 are positive operators from E into F . It is well known that each positive linear mapping on a Banach lattice is continuous. If an operator $T : E \rightarrow F$ between two Banach lattices is positive, then its dual operator $T' : F' \rightarrow E'$ is likewise positive, where T' is defined by $T'(f)(x) = f(T(x))$ for each $f \in F'$ and for each $x \in E$.

For terminology concerning Banach lattice theory and positive operators, we refer the reader to the excellent book of Aliprantis-Burkinshaw [2].

2. Preliminaries

Recall that an operator T from a Banach space E into another F is said to be Dunford-Pettis if it carries weakly compact subsets of E onto compact subsets of F . A Banach space E has the Dunford-Pettis property if every weakly compact operator defined on E (and taking their values in a Banach space F) is Dunford-Pettis.

Note that if E is a Banach lattice and X, Y are two Banach spaces, and if $T : E \rightarrow X$ and $S : X \rightarrow Y$ are two operators such that T is order weakly compact and S is Dunford-Pettis, then the composed operator $S \circ T$ is AM-compact.

To give a characterization of AM-compact operators, we need the following lemma.

Lemma 2.1. *Let E be a Banach lattice. Then the following assertions are equivalent.*

- (1) *Every positive operator from E into E is AM-compact.*
- (2) *The identity operator of the Banach lattice E is AM-compact.*
- (3) *E is discrete and its norm is order continuous.*

PROOF: (1) \implies (2) Obvious.

(2) \implies (3) Since the identity operator of E is AM-compact, then for each $x \in E_+$, the order interval $[0, x]$ is norm relatively compact, and since $[0, x]$ is norm closed, then $[0, x]$ is norm compact. Finally, Corollary 21.13 of [1] implies that E is discrete with order continuous norm.

(3) \implies (1) Let T be a positive operator from E into E . Since E is discrete and its norm is order continuous, it follows from Corollary 21.13 of [1] that for each $x \in E_+$, the order interval $[0, x]$ is norm compact and hence $T[0, x]$ is norm compact. \square

Let E be a Banach lattice. For each $u \in E_+$, we denote E_u the principal ideal generated by u , that we endow with the norm $\|\cdot\|_\infty$ defined by $\|x\|_\infty = \inf\{\lambda > 0 : \|x\| \leq \lambda u\}$. It is an AM-space having u as the unit and $[-u, u]$ as the closed unit ball (see Theorem 4.21 of [2]), and the natural embedding $i_u : (E_u, \|\cdot\|_\infty) \rightarrow E$ is continuous.

Moreover, for every $f \in E'$ we have $f \circ i_u \in (E_u)'$ and $\|f \circ i_u\|_{(E_u)'} = \sup\{|(f \circ i_u)(y)| : y \in [-u, u]\} = \sup\{|f(y)| : |y| \leq u\} = |f|(u)$.

Note that an operator $T : E \rightarrow X$ is AM-compact if and only if for every $u \in E_+$ the composed map $T \circ i_u : E_u \rightarrow E \rightarrow X$ is compact. Thus $T : E \rightarrow X$ is AM-compact if and only if for every order bounded sequence (x_n) of E , the sequence $(T(x_n))$ has a norm convergent subsequence in X .

Now we are in position to give this characterization.

Proposition 2.2. *Let E be a Banach lattice, X a Banach space and T an operator from E into X . Then T is AM-compact if and only if for every order bounded sequence (x_n) in E such that $(T(x_n))$ converges weakly to x in X , we have $\lim_n \|T(x_n) - x\| = 0$.*

PROOF: Let $T : E \rightarrow X$ be an AM-compact operator and A an order bounded subset of E and let (x_n) be a sequence in A such that the sequence $(T(x_n))$ converges weakly to x in X . Since $T(A)$ is norm relatively compact and $(T(x_n))$ converges weakly to x in X , we obtain $\lim_n \|T(x_n) - x\| = 0$.

Conversely, consider the operator $T : E \rightarrow X$ and let A be an order bounded subset of E . Choose $x \in E_+$ with $A \subset [-x, x]$. Let E_x be the principal ideal generated by x in E and endowed with the norm $\|\cdot\|_\infty$ and (x_n) be a weakly null sequence in E_x . Since the identity mapping $i_x : (E_x, \|\cdot\|_\infty) \rightarrow (E, \|\cdot\|)$ is continuous, (x_n) converges weakly to 0 in E . Hence (Tx_n) converges weakly to zero in X , and thus $\|Tx_n\| \rightarrow 0$ by the assumption. Thus we have verified that $T \circ i_x : E_x \rightarrow X$ is a Dunford-Pettis operator. Since $(E_x, \|\cdot\|_\infty)$ is an AM-space with unit, then by Theorem 2.1.3 of [2], $(E_x, \|\cdot\|_\infty)$ can be identified with a suitable $C(K)$ -space. It follows from Theorem 4 of [15], that $T \circ i_x$ is weakly compact. Thus $T(A)$ is a relatively weakly compact subset of X .

Now we claim that $T(A)$ is relatively norm compact. Indeed, otherwise there would exist a sequence (Tx_n) in $T(A)$ without a norm convergent subsequence. By the relative weak compactness of $T(A)$ we may assume that (Tx_n) converges weakly to a point $x \in X$. But then we have a contradiction with the assumption. Therefore, $T(A)$ is a norm relatively compact subset of X , and hence $T : E \rightarrow X$ is AM-compact. \square

As a consequence of Proposition 2.2, we obtain the following characterization of a discrete Banach lattice with order continuous norm.

Corollary 2.3. *Let E be a Banach lattice. Then E is discrete and its norm is order continuous if and only if every order bounded weakly convergent sequence (x_n) in E is norm convergent.*

PROOF: Let (x_n) be an order bounded and weakly convergent sequence in E . Since E is discrete with order continuous norm, it follows from Lemma 2.1 that its identity operator is AM-compact. And hence Proposition 2.2 implies that (x_n) is norm convergent.

Conversely, let (x_n) be an order bounded and weakly convergent sequence in E . Then (x_n) is norm convergent and it follows from Proposition 2.2 that the identity operator of E is AM-compact. Finally, Lemma 2.1 implies that E is discrete and its norm is order continuous. \square

3. Major results

Note that each b-weakly compact operator is order weakly compact, but the converse is false in general. However, if the Banach lattice E has the (b)-property (i.e. each subset $A \subset E$ is order bounded in E whenever it is order bounded in its topological bidual E''), then the class of b-weakly compact operators on E coincides with that of order weakly compact operators on E .

On the other hand, each almost Dunford-Pettis operator is b-weakly compact. (In fact, let (x_n) be a disjoint b-order bounded sequence of E . Then (x_n) is an order bounded disjoint sequence of the topological bidual E'' . So, $x_n \rightarrow 0$ for the topology $\sigma(E'', E''')$, and hence $x_n \rightarrow 0$ for the topology $\sigma(E, E')$. If $T : E \rightarrow X$ is almost Dunford-Pettis, then $T(x_n)$ converges in norm to 0 and hence it follows from Proposition 2.8 of [3] that T is b-weakly compact). However, a b-weakly compact operator is not necessarily almost Dunford-Pettis. In fact, the identity operator $Id_{\ell^2} : \ell^2 \rightarrow \ell^2$ is b-weakly compact, but it is not almost Dunford-Pettis.

Now, we are in position to give necessary and sufficient conditions under which each regular order weakly compact (resp. b-weakly compact, almost Dunford-Pettis, Dunford-Pettis) operator $T : E \rightarrow F$ is AM-compact.

Theorem 3.1. *Let E and F be two Banach lattices such that the lattice operations of F are weakly sequentially continuous. Then the following statements are equivalent.*

- (1) Every regular order weakly compact operator $T : E \rightarrow F$ is AM-compact.
- (2) Every regular b-weakly compact operator $T : E \rightarrow F$ is AM-compact.
- (3) Every regular almost Dunford-Pettis operator $T : E \rightarrow F$ is AM-compact.
- (4) One of the following conditions is valid:
 - (i) E' is discrete,
 - (ii) F is discrete with order continuous norm.

PROOF: (1) \implies (2) Since every regular b-weakly compact operator is order weakly compact, it is evident that every regular b-weakly compact operator is AM-compact.

(2) \implies (3) Since every regular almost Dunford-Pettis operator is b-weakly compact, then every regular almost Dunford-Pettis operator is AM-compact.

(3) \implies (4) Suppose that E' is not discrete. So, we have to show that F is discrete and its norm is order continuous.

Suppose that F is not discrete or its norm is not order continuous. It follows from Corollary 2.4 the existence of an order bounded sequence $(y_n) \subset F$ which converges weakly to some y and $\lim_n \|y_n - y\| > \varepsilon$. Consider the sequence $(v_n) = (|y_n - y|)$. Since the lattice operations of F are weakly sequentially continuous, then (v_n) converges weakly to 0 and we have $\lim_n \|v_n\| > \varepsilon$. Now, by Corollary 2.3.5 of [17], there exist a subsequence $(k_n) \subset \mathbf{N}$ and a disjoint sequence $(z_n) \subset F_+$ such that $z_n \leq v_{k_n}$ and $\|z_n\| \geq \frac{1}{2}$ for all $n \in \mathbf{N}$. Since (v_n) is order bounded then (z_n) is order bounded and hence there exists $z \in F_+$ such that $(z_n) \subset [0, z]$. By Lemma 3.4 of [7] there exists a positive disjoint sequence (g_n) of

F' with $\|g_n\| \leq 1$ such that

$$g_n(z_n) = 1 \text{ for all } n \text{ and } g_n(z_m) = 0 \text{ for } n \neq m.$$

On the other hand, as E' is not discrete, it follows from Theorem 3.1 of Chen-Wickstead [11] the existence of a sequence $(f_n) \subset E'$ such that $f_n \rightarrow 0$ in $\sigma(E', E)$ as $n \rightarrow \infty$ and $\|f_n\| = f > 0$ for all n and some $f \in E'$.

Now, we consider the operators $S, T : E \rightarrow F$ defined by

$$S(x) = \left(\sum_{n=1}^{\infty} f_n(x) \cdot z_n \right) \quad \text{and} \quad T(x) = f(x) \cdot z \quad \text{for all } x \in E.$$

Since $\sum_{n=1}^{\infty} \|f_n(x) \cdot z_n\| \leq \sum_{n=1}^{\infty} f(|x|) \cdot \|z_n\| \leq f(|x|) \cdot \|z\|$, the series defining S converges in norm for each $x \in E$. So, the operator S is well defined and is positive. Note that S and T are the same operators considered in Theorem 2 of [19].

Clearly, $0 \leq S \leq T$ holds. (In fact, for each $x \in E^+$ and each $n \geq 1$, we have $|\sum_{k=1}^n f_k(x) \cdot z_k| \leq \sum_{k=1}^n f(x) \cdot z_k \leq f(x) \cdot z$. Then $|\sum_{n=1}^{\infty} f_n(x) \cdot z_n| \leq f(x) \cdot z$ for each $x \in E^+$. Hence $0 \leq S(x) \leq T(x)$ for each $x \in E^+$.)

The operator T is compact and hence almost Dunford-Pettis. After that, it follows from the Corollary 2.3 of [9] that the operator S is almost Dunford-Pettis.

It remains to show that S is not AM-compact. Choose $u \in E_+$ such that $f(u) > 0$, and note that $(f_n \circ i_u)_{n=1}^{\infty}$ has no norm convergent subsequence in $(E_u)'$. In fact, for each $y \in E_u$ we have $f_n \circ i_u(y) = f_n(y) \rightarrow 0$ as $n \rightarrow \infty$. Then $f_n \circ i_u \rightarrow 0$ in $\sigma((E_u)', E_u)$. As $\|f_n \circ i_u\|_{(E_u)'} = \|f_n\|(u) = f(u) > 0$ for all n , we conclude that $(f_n \circ i_u)_{n=1}^{\infty}$ has no norm convergent subsequence in $(E_u)'$.

If S is AM-compact, then $S \circ i_u : E_u \rightarrow E \rightarrow F$ is compact and so is $(S \circ i_u)'$. As we have $(S \circ i_u)'(g) = (\sum_{n=1}^{\infty} g(z_n) \cdot (f_n \circ i_u))$ for all $g \in F'$, then $(S \circ i_u)'(g_k) = (f_k \circ i_u)$ for all k . Hence $((S \circ i_u)'(g_k))$ has a norm convergent subsequence in $(E_u)'$. We conclude that $(f_k \circ i_u)_k$ has a convergent subsequence in $(E_u)'$. This is a contradiction and then S is not AM-compact.

(4)(i) \implies (1) Follows from Proposition 7 of [4].

(4)(ii) \implies (1) Since $T : E \rightarrow F$ is a regular operator, then the image by T , of each order interval of E , is an order bounded subset of F . Finally, the result follows from Corollary 21.13 of [1]. □

Remark 3.2. The assumption “the lattice operations of F are weakly sequentially continuous” is essential in Theorem 3.1. For instance, every regular operator $T : L^1[0, 1] \rightarrow L^2[0, 1]$ is AM-compact. But neither $(L^1[0, 1])'$ is discrete nor $L^2[0, 1]$ is discrete with order continuous norm.

As consequences of Theorem 3.1, we obtain the following results:

Corollary 3.3. *Let F be a Banach lattice with weakly sequentially continuous lattice operations. Then the following statements are equivalent.*

- (1) *Every regular order weakly compact operator $T : \ell^\infty \rightarrow F$ is AM-compact.*

- (2) Every regular b -weakly compact operator $T : \ell^\infty \rightarrow F$ is AM-compact.
- (3) Every regular almost Dunford-Pettis operator $T : \ell^\infty \rightarrow F$ is AM-compact.
- (4) F is discrete with order continuous norm.

Corollary 3.4. *Let E be a Banach lattice, then the following statements are equivalent.*

- (1) Every regular order weakly compact operator $T : E \rightarrow c$ is AM-compact.
- (2) Every regular b -weakly compact operator $T : E \rightarrow c$ is AM-compact.
- (3) Every regular almost Dunford-Pettis operator $T : E \rightarrow c$ is AM-compact.
- (4) E' is discrete.

To give another consequence of Theorem 3.1, we need to recall from [8] that an operator T from a Banach lattice E into a Banach space X is said to be b -AM-compact if it carries each b -order bounded subset of E into a relatively compact subset of X .

Note that a regular order weakly compact (resp. b -weakly compact, almost Dunford-Pettis) operator is not necessarily b -AM-compact. In fact, the identity operator $Id_{L^1[0,1]} : L^1[0,1] \rightarrow L^1[0,1]$ is order weakly compact (resp. b -weakly compact, almost Dunford-Pettis) but it is not b -AM-compact (because $L^1[0,1]$ is not a discrete KB-space).

Theorem 3.5. *Let E and F be two Banach lattices such that the norm of E is order continuous and the lattice operations of E and F are weakly sequentially continuous. Then the following statements are equivalent.*

- (1) Every regular operator $T : E \rightarrow F$ is b -AM-compact.
- (2) Every regular order weakly compact operator $T : E \rightarrow F$ is b -AM-compact.
- (3) Every regular AM-compact operator $T : E \rightarrow F$ is b -AM-compact.
- (4) One of the following conditions is valid:
 - (a) E is a discrete KB-space,
 - (b) F is a discrete KB-space.

PROOF: (1) \implies (2) Obvious.

(2) \implies (3) Since every regular AM-compact is order weakly compact, then every regular AM-compact operator is b -AM-compact.

(3) \implies (4) Since the norm of E is order continuous and the lattice operations of E are weakly sequentially continuous, it follows from Corollary 2.3 of [12] that E is discrete.

Suppose that E is not a KB-space and that F is not a discrete KB-space. Since the norm of E is order continuous, then it follows from [10] that E contains a complemented copy of c_0 . Hence, there exists a positive projection $P : E \rightarrow c_0$ and let $i : c_0 \rightarrow E$ be the injection of c_0 in E . And as F is not a discrete KB-space, it follows from Corollary 3.9 of [8] that there exists a regular operator $S : c_0 \rightarrow F$ which is not b -AM-compact.

Consider the operator $T = S \circ P : E \rightarrow c_0 \rightarrow F$, since S and P are two regular operators and the identity operator Id_{c_0} is AM-compact, then $T = S \circ Id_{c_0} \circ P$

is AM-compact. But T is not b-AM-compact. Otherwise, the operator $T \circ i = S$ would be b-AM-compact, which is a contradiction.

(4) \implies (1) Follows from [8, Corollary 2.4]. \square

Remarks 3.6. (1) The assumption “the norm of E is order continuous” is essential in Theorem 3.5. For instance, every positive operator $T : l^\infty \rightarrow c_0$ is b-AM-compact. But neither l^∞ nor c_0 is a discrete KB-space.

(2) The assumption “the lattice operations of E are weakly sequentially continuous” is essential in Theorem 3.5. For instance, from [16, Theorem] it follows that each regular operator $T : L^1[0, 1] \rightarrow c_0$ is Dunford-Pettis. Since $T = T \circ Id_{L^1[0,1]}$ and $Id_{L^1[0,1]}$ is b-weakly compact, it follows from Proposition 3.4 of [8] that the operator $T : L^1[0, 1] \rightarrow c_0$ is b-AM-compact. But neither $L^1[0, 1]$ nor c_0 is a discrete KB-space.

(3) The assumption “the lattice operations of F are weakly sequentially continuous” is essential in Theorem 3.5. For instance, from Theorem 6.8 of Wnuk [20] every regular operator $T : c_0 \rightarrow (l^\infty)'$ is compact. But neither c_0 nor $(l^\infty)'$ is a discrete KB-space.

Let us recall that a Banach space X has the Dunford-Pettis property if $\lim_n x'_n(x_n) = 0$ whenever (x_n) converges weakly to zero in X and (x'_n) converges weakly to zero in X' .

It follows from Theorem 5.82 of [2] that a Banach space X has the Dunford-Pettis property if and only if every weakly compact operator from X to an arbitrary Banach space is Dunford-Pettis.

We end this paper by establishing a result on the AM-compactness of Dunford-Pettis operators.

Theorem 3.7. *Let E and F be two Banach lattices such that E is Dedekind σ -complete and the lattice operations of F are weakly sequentially continuous. Then the following statements are equivalent.*

- (1) *Every regular Dunford-Pettis operator $T : E \rightarrow F$ is AM-compact.*
- (2) *One of the following conditions is valid:*
 - (a) *the norm of E is order continuous,*
 - (b) *F is discrete with order continuous norm.*

PROOF: (2)(a) \implies (1). Let $T : E \rightarrow F$ be a regular Dunford-Pettis operator and let A be an order bounded subset of E . Since E has an order continuous norm, then it follows from Theorem 4.9 of [2] that A is weakly relatively compact. On the other hand, since the operator T is Dunford-Pettis, then $T(A)$ is norm relatively compact and hence T is AM-compact.

(2)(b) \implies (1). In this case it follows from Corollary 21.13 of [1] that every regular operator $T : E \rightarrow F$ is AM-compact.

(1) \implies (2). Assume that the norm of E is not order continuous and that F is not discrete with order continuous norm. Since E is Dedekind σ -complete, it follows from Corollary 2.4.3 of [17] that E contains a sublattice which is isomorphic to l^∞ and there exists a positive projection P from E onto l^∞ . As the lattice

operations of F are weakly sequentially continuous and F is not discrete with order continuous norm, it follows from Corollary 3.3 that there exists a regular almost Dunford-Pettis operator $S : l^\infty \rightarrow F$ which is not AM-compact. Since $S : l^\infty \rightarrow F$ is almost Dunford-Pettis, it is order weakly compact and as l^∞ is an AM-space with unit, $S : l^\infty \rightarrow F$ is weakly compact. As l^∞ has the Dunford-Pettis property, then $S : l^\infty \rightarrow F$ is Dunford-Pettis. We consider the operator product $T = S \circ P : E \rightarrow F$. Note that T is Dunford-Pettis because the operator S is Dunford-Pettis and the class of Dunford-Pettis operators is a two-sided ideal. But it is not AM-compact. If not, the operator $T \circ i = S$ would be AM-compact and this is a contradiction. \square

Remark 3.8. The assumption “ E is Dedekind σ -complete” is essential in Theorem 3.7. In fact, every regular Dunford-Pettis operator $T : c \rightarrow c$ is AM-compact (In fact, since T is Dunford-Pettis then T is almost Dunford-Pettis. As c' is discrete and the lattice operations of c are weakly sequentially continuous, then it follows from Theorem 3.1 that the operator T is AM-compact), but the norm of the Banach lattice c is not order continuous.

REFERENCES

- [1] Aliprantis C.D., Burkinshaw O., *Locally Solid Riesz Spaces*, Academic Press, New York-London, 1978.
- [2] Aliprantis C.D., Burkinshaw O., *Positive Operators*, reprint of the 1985 original, Springer, Dordrecht, 2006.
- [3] Alpay S., Altin B., Tonyali C., *On property (b) of vector lattices*, Positivity **7** (2003), no. 1–2, 135–139.
- [4] Alpay S., Altin B., *On Riesz spaces with b-property and b-weakly compact operators*, Vladikavkaz. Mat. Zh. **11** (2009), no. 2, 19–26.
- [5] Aqzzouz B., Nouira R., Zraoula L., *Compactness properties of operators dominated by AM-compact operators*, Proc. Amer. Math. Soc. **135** (2007), no. 4, 1151–1157.
- [6] Aqzzouz B., Zraoula L., *AM-compactness of positive Dunford-Pettis operators on Banach lattices*, Rend. Circ. Mat. Palermo (2) **56** (2007), no. 3, 305–316.
- [7] Aqzzouz B., Elbour A., Hmichane J., *The duality problem for the class of b-weakly compact operators*, Positivity **13** (2009) no. 4, 683–692.
- [8] Aqzzouz B., Hmichane J., *The class of b-AM-compact operators*, Quaestiones Mathematicae, to appear.
- [9] Aqzzouz B., Elbour A., *Some characterizations of almost Dunford-Pettis operators and applications*, Positivity **15** (2011), 369–380.
- [10] Aqzzouz B., Hmichane J., *The b-weak compactness of order weakly compact operators*, Complex Anal. Oper. Theory, DOI 10. 1007/s 11785-011-0138-1.
- [11] Chen Z.L., Wickstead A.W., *Some applications of Rademacher sequences in Banach lattices*, Positivity **2** (1998), no. 2, 171–191.
- [12] Chen Z.L., Wickstead A.W., *Relative weak compactness of solid hulls in Banach lattices*, Indag. Math. (N.S.) **9** (1998), no. 2, 187–196.
- [13] Dodds P.G., *o-Weakly compact mappings of vector lattices*, Trans. Amer. Math. Soc. **214** (1975), 389–402.
- [14] Dodds P.G., Fremlin D.H., *Compact operators on Banach lattices*, Israel J. Math. **34** (1979) 287–320.

- [15] Grothendieck A., *Sur les applications linéaires faiblement compactes d'espaces du type $C(K)$* , Canad. J. Math. **5** (1953), 129–173.
- [16] Holub J.R., *A note on Dunford-Pettis operators*, Glasgow Math. J. **29** (1987), no. 2, 271–273.
- [17] Meyer-Nieberg P., *Banach lattices*, Universitext, Springer, Berlin, 1991.
- [18] Sanchez J.A., *Operators on Banach lattices* (Spanish), Ph.D. Thesis, Complutense University, Madrid, 1985.
- [19] Wickstead A.W., *Converses for the Dodds-Fremlin and Kalton-Saab Theorems*, Math. Proc. Camb. Philos. Soc. **120** (1996), 175–179.
- [20] Wnuk W., *Banach lattices with the weak Dunford-Pettis property*, Atti Sem. Mat. Fis. Univ. Modena **42** (1994), no. 1, 227–236.
- [21] Zaanen A.C., *Riesz Spaces II*, North Holland Publishing Co., Amsterdam, 1983.

UNIVERSITÉ MOHAMMED V-SOUISSI, FACULTÉ DES SCIENCES ECONOMIQUES, JURIDIQUES ET SOCIALES, DÉPARTEMENT D'ÉCONOMIE, B.P. 5295, SALAALJADIDA, MOROCCO

E-mail: baqzzouz@hotmail.com

UNIVERSITÉ IBN TOFAIL, FACULTÉ DES SCIENCES, DÉPARTEMENT DE MATHÉMATIQUES, B.P. 133, KÉNITRA, MOROCCO

(Received April 4, 2011, revised May 23, 2012)