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Spaces with large star cardinal number

Commentationes Mathematicae Universitatis Carolinae, Vol. 53 (2012), No. 4, 637--643

Persistent URL: http://dml.cz/dmlcz/143196

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Abstract. In this paper, we prove the following statements:

1. For any cardinal $\kappa$, there exists a Tychonoff star-Lindelöf space $X$ such that $\sigma(X) \geq \kappa$.

2. There is a Tychonoff discretely star-Lindelöf space $X$ such that $aa(X)$ does not exist.

3. For any cardinal $\kappa$, there exists a Tychonoff pseudocompact $\sigma$-starcompact space $X$ such that $st-l(X) \geq \kappa$.

Keywords: star-Lindelöf number, the Aquaro number, the absolute Aquaro number, star-Lindelöf, centered-Lindelöf, discretely star-Lindelöf, absolutely discretely star-Lindelöf, $\sigma$-starcompact, pseudocompact

Classification: 54A25, 54D20

1. Introduction

By a space, we mean a topological space. Recall from [6] that a space $X$ is starcompact if for every open cover $U$ of $X$, there exists a finite subset $F$ of $X$ such that $St(F,U) = X$, where $St(F,U) = \bigcup\{U \in U : U \cap F \neq \emptyset\}$. It is well-known that starcompactness is equivalent to countably compactness for Hausdorff spaces (see [3], [6]).

A space $X$ is discretely absolutely star-Lindelöf (see [12], [13]) if for every open cover $U$ of $X$ and every dense subset $D$ of $X$, there exists a countable subset $F$ of $D$ such that $F$ is discrete and closed in $X$ and $St(F,U) = X$.

A space $X$ is star-Lindelöf (see [1], [2], [3], [4], [6] under different names) (discretely star-Lindelöf) (see [11], [15]) if for every open cover $U$ of $X$, there exists a countable subset (a countable discrete closed subset, respectively) $F$ of $X$ such that $St(F,U) = X$. It is clear that every separable space is star-Lindelöf as well as every space of countable extent (in particular, every countably compact space or every Lindelöf space).

A space $X$ is centered-Lindelöf (see [1], [6]) if every open cover $U$ of $X$ has a $\sigma$-centered subcover. A family of sets is centered if every finite subfamily has non-empty intersection and a family is $\sigma$-centered if it can be represented as the union of countably many centered subfamilies.

The author acknowledges the support from NSFC Project 11271036.
A space $X$ is $\sigma$-starcompact (see [14]) if for every open cover $\mathcal{U}$ of $X$, there exists a $\sigma$-compact subset $F$ of $X$ such that $\text{St}(F, \mathcal{U}) = X$.

From the above definitions, it is not difficult to see that every discretely absolutely star-Lindelöf space is discretely star-Lindelöf, every discretely star-Lindelöf space is star-Lindelöf, every star-Lindelöf space is centered-Lindelöf and every star-Lindelöf space is $\sigma$-starcompact.

As natural generalizations of star-Lindelöfness and discretely star-Lindelöfness, one can consider the following cardinal functions:

**Definition 1.1** ([1], [6], [7]). The *star-Lindelöf number* of the space $X$ is the cardinal number

$$\text{st-l}(X) = \min\{\kappa : \text{for every open cover } \mathcal{U} \text{ of } X, \text{ there exists a subset } F \subseteq X \text{ such that } |F| \leq \kappa \text{ and } \text{St}(F, \mathcal{U}) = X\}.$$

**Definition 1.2** ([7]). The *Aquaro number* of the space $X$ is the cardinal number

$$a(X) = \min\{\kappa : \text{for every open cover } \mathcal{U} \text{ of } X, \text{ there exists a discrete closed subset } F \subseteq X \text{ such that } |F| \leq \kappa \text{ and } \text{St}(F, \mathcal{U}) = X\}.$$

As a natural generalization of discretely absolutely star-Lindelöfness, we can define the following cardinal function:

**Definition 1.3.** The *absolute Aquaro number* of the space $X$ is the cardinal number

$$aa(X) = \min\{\kappa : \text{for every open cover } \mathcal{U} \text{ of } X \text{ and for every dense subset } D \text{ of } X, \text{ there exists a discrete closed subset } F \subseteq D \text{ such that } |F| \leq \kappa \text{ and } \text{St}(F, \mathcal{U}) = X\}.$$

It is easily proved that the following inequalities hold for every space $X$:

$$\text{st-l}(X) \leq a(X) \leq aa(X).$$

Bonanzinga-Matveev [1] and Matveev [6] asked if the $\text{st-l}(X)$ of a Tychonoff centered-Lindelöf space $X$ cannot be greater than $c$. The author [10] answered negatively the question by giving an example to show that for any cardinal $\kappa$ there exists a Tychonoff centered-Lindelöf space $X$ such that $\text{st-l}(X) \geq \kappa$. In [14], the author constructed an example showing that there exists a Tychonoff $\sigma$-starcompact space that is not star-Lindelöf. However, the author’s space is not pseudocompact and its star-Lindelöf number is not greater than $c$. It is natural for us to consider the following questions:

**Question 1.** Is it true that the Aquaro number of a Tychonoff star-Lindelöf space cannot be greater than $c$?

**Question 2.** Is it true that the absolute Aquaro number of a Tychonoff discretely star-Lindelöf space cannot be greater than $c$?
Question 3. Is it true that the star-Lindelöf number of a Tychonoff pseudocompact $\sigma$-starcompact space cannot be greater than $c$?

The purpose of this paper is to answer negatively the above three questions by showing the three statements stated in the abstract.

The cardinality of a set $A$ is denoted by $|A|$. Let $\omega$ denote the first infinite cardinal and $c$ denote the cardinality of the continuum. As usual, a cardinal is the initial ordinal and an ordinal is the set of smaller ordinals. When viewed as a space, every cardinal has the usual order topology. For each ordinal $\alpha$, $\beta$ with $\alpha < \beta$, we write $(\alpha, \beta) = \{\gamma : \alpha < \gamma < \beta\}$ and $(\alpha, \beta] = \{\gamma : \alpha < \gamma \leq \beta\}$. Other terms and symbols that we do not define will be used as in [5].

2. Spaces with large star cardinal number

In this section, we show the three statements stated in the abstract. All examples of this section are of the form

$$(X \times \alpha) \cup (Y \times \{\alpha\})$$

where $X$ is a space, $Y$ is a subspace of $X$ and $\alpha$ is an ordinal. The first two examples use Matveev’s space. We now sketch the construction of Matveev’s space $M$ defined in [8], [9]. Let $\kappa$ be an infinite cardinal and $D = \{0, 1\}$ be the discrete space. For every $\alpha < \kappa$, let $z_\alpha$ be the point of $D^\kappa$ defined by $z_\alpha(\alpha) = 1$ and $z_\alpha(\beta) = 0$ for $\beta \neq \alpha$. Put $Z = \{z_\alpha : \alpha < \kappa\}$. For a given ordinal $\tau$, Matveev’s space $M(\kappa, \tau)$ is the subspace

$$M(\kappa, \tau) = (D^\kappa \times \tau) \cup (Z \times \{\tau\})$$

of the product space $D^\kappa \times (\tau + 1)$. Then $M(\kappa, \tau)$ is Tychonoff and $Z \times \{\tau\}$ is a discrete closed set of $M(\kappa, \tau)$ with $|Z \times \{\tau\}| = \kappa$.

We need the following lemma:

Lemma 2.1 ([9], [10]). Assume that there exists a family $\{V_\alpha : \alpha < \kappa\}$ of open sets in $D^\kappa$ such that $z_\alpha \in V_\alpha$ for each $\alpha < \kappa$. Then there exists a countable set $S \subseteq D^\kappa$ such that $S \cap V_\alpha \neq \emptyset$ for each $\alpha < \kappa$ and $\overline{D^\kappa \setminus S} \cap Z = \emptyset$.

Theorem 2.2. For any cardinal $\kappa$, there exists a Tychonoff star-Lindelöf space $X$ such that $a(X) \geq \kappa$.

Proof: Since for any cardinal $\kappa$ there is a larger regular uncountable cardinal, we can assume that $\kappa$ itself is a regular uncountable cardinal. Choose a regular uncountable cardinal $\tau$ such that $\tau > \kappa$ and let $X = M(\kappa, \tau)$.

First we show that $X$ is star-Lindelöf. To this end, let $U$ be an open cover of $X$. For every $\alpha < \kappa$, there exists an $U_\alpha \in U$ such that $(z_\alpha, \tau) \in U_\alpha$. Choose $\beta_\alpha < \tau$ and an open neighborhood $V_\alpha$ of $z_\alpha$ in $D^\kappa$ such that

$$((V_\alpha \cap Z) \times \{\tau\}) \cup (V_\alpha \times (\beta_\alpha, \tau]) \subseteq U_\alpha.$$
By applying Lemma 2.1 to the family \( \{V_{\alpha} : \alpha < \kappa\} \), then we can find a countable set \( S \subseteq D^k \) such that \( S \cap V_{\alpha} \neq \emptyset \) for all \( \alpha < \kappa \). Let \( \beta' = \sup \{\beta_{\alpha} : \alpha < \kappa\} \). Then \( \beta' < \tau \), since \( \tau \) is regular and \( \tau > \kappa \). Let \( F_0 = S \times \{\beta'\} \). Then \( Z \times \{\tau\} \subseteq \text{St}(F_0, \mathcal{U}) \), since \( U_{\alpha} \cap F_0 \neq \emptyset \) for each \( \alpha < \kappa \). On the other hand, since \( D^k \times \tau \) is countably compact, we can find a finite subset \( F_1 \subseteq D^k \times \tau \) such that \( D^k \times \tau \subseteq \text{St}(F_1, \mathcal{U}) \).

If we put \( F = F_0 \cup F_1 \), then \( F \) is a countable subset of \( X \) such that \( X = \text{St}(F, \mathcal{U}) \), which shows that \( X \) is star-Lindelöf.

Next we show that \( a(X) \geq \kappa \). We can partition \( \kappa \) as \( \kappa = \cup \{A_n : n \in \omega, \gamma < \kappa\} \) such that \( |A_{n\gamma}| = n \) for each \( n \in \omega \) and \( \gamma < \kappa \), \( A_{n\gamma} \cap A_{n'\gamma'} = \emptyset \) for \( \langle n, \gamma \rangle \neq \langle n', \gamma' \rangle \). For each \( \alpha < \kappa \), pick an open neighborhood \( U_\alpha \) of \( \langle z_\alpha, \tau \rangle \) such that \( U_\alpha \cap (Z \times \{\tau\}) = \langle z_\alpha, \tau \rangle \), and \( U_\alpha \cap U_{\alpha'} = \emptyset \) if \( \alpha, \alpha' \in A_{n\gamma} \) and \( \alpha \neq \alpha' \) for each \( n \in \omega \) and \( \gamma < \kappa \).

Let us consider the open cover

\[
\mathcal{U} = \{U_\alpha : \alpha < \kappa\} \cup \{D^k \times \tau\}
\]

of the space \( X \). It remains to show that \( \text{St}(F, \mathcal{U}) \neq X \) for any discrete closed subset of \( X \) with \( |F| < \kappa \). To show this, let \( F \) be any discrete closed subset of \( X \) with \( |F| < \kappa \). Let

\[
\alpha' = \sup \{\gamma : F \cap \{\langle z_\alpha, \tau \rangle : \alpha \in A_{n\gamma}\} \neq \emptyset \text{ for some } n \in \omega \text{ and some } \gamma < \kappa\}.
\]

Then \( \alpha' < \kappa \), since \( \kappa \) is regular and \( |F| < \kappa \). Thus \( F \cap \{\langle z_\alpha, \tau \rangle : \alpha \in A_{n\gamma}\} = \emptyset \) for each \( n \in \omega \) and \( \gamma > \alpha' \). On the other hand, since \( D^k \times \tau \) is countably compact, then \( F \cap (D^k \times \tau) \) is finite. Thus we choose \( n_0 \in \omega \) and \( \gamma_0 > \alpha' \) such that \( \{\langle z_\alpha, \tau \rangle : \alpha \in A_{n_0\gamma_0}\} \cap F = \emptyset \). Therefore \( \langle z_\alpha, \tau \rangle \notin \text{St}(F, \mathcal{U}) \) for each \( \alpha \in A_{n_0\gamma_0} \), which shows \( a(X) \geq \kappa \).

□

For a Tychonoff space \( X \), let \( \beta X \) denote the Čech-Stone compactification of the space \( X \).

**Theorem 2.3.** There is a Tychonoff discretely star-Lindelöf space \( X \) such that \( aa(X) \) does not exist.

**Proof:** The author [10] showed that \( M(\omega_1, \omega) \) is discretely star-Lindelöf.

Let

\[
X = (\beta M(\omega_1, \omega) \times \omega_1) \cup (M(\omega_1, \omega) \times \{\omega_1\})
\]

be the subspace of the product space \( \beta M(\omega_1, \omega) \times (\omega_1 + 1) \).

First we show that \( X \) is discretely star-Lindelöf. To this end, let \( \mathcal{U} \) be an open cover of \( X \). Since \( \beta M(\omega_1, \omega) \times \omega_1 \) is countably compact, we can find a finite subset \( F_1 \subseteq \beta M(\omega_1, \omega) \times \omega_1 \) such that

\[
\beta M(\omega_1, \omega) \times \omega_1 \subseteq \text{St}(F_1, \mathcal{U}).
\]
On the other hand, \( M(\omega_1, \omega) \times \{\omega_1\} \) is discretely star-Lindelöf, since it is homeomorphic to \( M(\omega_1, \omega) \). Thus there exists a countable subset \( F_2 \subseteq M(\omega_1, \omega) \times \{\omega_1\} \) such that \( F_2 \) is discrete closed in \( M(\omega_1, \omega) \times \{\omega_1\} \) and

\[
M(\omega_1, \omega) \times \{\omega_1\} \subseteq \text{St}(F_2, U).
\]

Since \( M(\omega_1, \omega) \times \{\omega_1\} \) is closed in \( X \), then \( F_2 \) is closed in \( X \). If we put \( F = F_1 \cup F_2 \), then \( F \) is a countable discrete closed subset of \( X \) such that \( X = \text{St}(F, U) \), which shows that \( X \) is discretely star-Lindelöf.

Next we show that \( aa(X) \) does not exist. For each \( \alpha < \omega_1 \), let \( U_\alpha = \{ \langle z_\alpha, \omega \rangle \} \cup \langle D^{\omega_1} \times \omega \rangle \). Since \( Z \times \{\omega\} \) is relatively discrete the set \( U_\alpha \) is an open neighborhood of \( \langle z_\alpha, \omega \rangle \) such that \( U_\alpha \cap (Z \times \{\omega\}) = \{ \langle z_\alpha, \omega \rangle \} \).

Let us consider the open cover

\[
U = \{ U_\alpha \times (\alpha, \omega_1) : \alpha < \omega_1 \} \cup \{ \beta M(\omega_1, \omega) \times \omega_1 \}
\]

of the space \( X \) and the dense subset \( \beta M(\omega_1, \omega) \times \omega_1 \) of the space \( X \). It remains to show that \( \text{St}(F, U) \neq X \) for any discrete closed subset \( F \) of \( \beta M(\omega_1, \omega) \times \omega_1 \). To show this, let \( F \) be any discrete closed subset of \( \beta M(\omega_1, \omega) \times \omega_1 \). Then \( F \) is finite subset of \( \beta M(\omega_1, \omega) \times \omega_1 \), since \( \beta M(\omega_1, \omega) \times \omega_1 \) is countably compact. Let \( \alpha' = \sup \{ \alpha : \alpha \in \pi(F) \} \), where \( \pi : \beta M(\omega_1, \omega) \times \omega_1 \to \omega_1 \) is the projection. Then \( \alpha' < \omega_1 \), since \( F \) is finite. If we pick \( \beta > \alpha' \), then \( \langle \langle z_\beta, \omega \rangle, \omega_1 \rangle \notin \text{St}(F, U) \), since \( U_\beta \times \{\beta, \omega_1\} \) is the only element of \( U \) containing \( \langle \langle z_\beta, \omega \rangle, \omega_1 \rangle \) and \( (U_\beta \times \{\beta, \omega_1\}) \cap F = \emptyset \), which shows that \( aa(X) \) does not exist. \( \square \)

**Remark 2.1.** The referee asked whether there is a Tychonoff star-Lindelöf space \( X \) such that \( aa(X) \) does not exist. The author noticed that there is a Tychonoff countably compact (hence, starcompact, star-Lindelöf and discretely star-Lindelöf) space \( X \) such that \( aa(X) \) does not exist. The construction of the example is very much simpler than the construction of the space \( X \) in Theorem 2.3. In fact, let \( X = \omega_1 \times (\omega_1 + 1) \) be the product of \( \omega_1 \) and \( \omega_1 + 1 \). Then \( X \) is Tychonoff countably compact space. Let us show that \( aa(X) \) does not exist. For each \( \alpha < \omega_1 \), let \( U_\alpha = [0, \alpha) \times (\alpha, \omega_1] \). Let us consider the open cover

\[
U = \{ U_\alpha : \alpha < \omega_1 \} \cup \{ D \}
\]

and the dense subspace \( D \) of \( X \), where \( D = \omega_1 \times \omega_1 \). It remains to show that \( \text{St}(F, U) \neq X \) for any discrete closed subset \( F \) of \( D \). To show this, let \( F \) be any discrete closed subset of \( D \). Then \( F \) is finite subset of \( D \), since \( D \) is countably compact. Let \( \alpha_0 = \sup \{ \alpha : \alpha \in \pi(F) \} \), where \( \pi : \omega_1 \times (\omega_1 + 1) \to \omega_1 + 1 \) is the projection. Then \( \alpha_0 < \omega_1 \), since \( F \) is finite. If we pick \( \alpha' > \alpha_0 \), then \( \langle \alpha', \omega_1 \rangle \notin \text{St}(F, U) \). Indeed, for every \( U_\beta \in U \), if \( \langle \alpha', \omega_1 \rangle \in U_\beta \), then \( \beta > \alpha' \). Finally, for each \( \beta > \alpha' \), \( U_\beta \cap F = \emptyset \), which shows that \( aa(X) \) does not exist.

**Theorem 2.4.** For any cardinal \( \kappa \), there exists a pseudocompact \( \sigma \)-starcompact Tychonoff space \( X \) such that \( \text{st}-l(X) \geq \kappa \).
Y.-K. Song

Proof: We may assume that \( \kappa \) is a regular uncountable cardinal, as we have done in Theorem 2.2. Let \( D = \{ d_\alpha : \alpha < \kappa \} \) be a discrete space of the cardinality \( \kappa \) and

\[ Y = (\beta D \times \omega) \cup (D \times \{ \omega \}) \]

be the subspace of the product space \( \beta D \times (\omega + 1) \). Then \( Y \) is \( \sigma \)-starcompact, since \( \beta D \times \omega \) is a \( \sigma \)-compact dense subset of \( Y \).

Let

\[ X = (\beta Y \times \kappa) \cup (Y \times \{ \kappa \}) \]

be the subspace of the product space \( \beta Y \times (\kappa + 1) \). Clearly, \( X \) is a Tychonoff space. Since \( \kappa \) has uncountable cofinality, then \( \beta Y \times \kappa \) is a countably compact dense subset of \( X \), hence \( X \) is pseudocompact.

First we show that \( X \) is \( \sigma \)-starcompact. To this end, let \( \mathcal{U} \) be an open cover of \( X \). Since \( \beta Y \times \kappa \) is countably compact, there exists a finite subset \( F \) of \( \beta Y \times \kappa \) such that

\[ \beta Y \times \kappa \subseteq \text{St}(F, \mathcal{U}). \]

On the other hand, \( Y \times \{ \kappa \} \) is \( \sigma \)-starcompact, since it is homeomorphic to \( Y \). Thus

\[ Y \times \{ \kappa \} \subseteq \text{St}((\beta D \times \omega) \times \{ \kappa \}, \mathcal{U}), \]

since \( (\beta D \times \omega) \times \{ \kappa \} \) is a \( \sigma \)-compact dense subset of \( Y \times \{ \kappa \} \). Since \( Y \times \{ \kappa \} \) is closed in \( X \), then \( (\beta D \times \omega) \times \{ \kappa \} \) is a \( \sigma \)-compact subset of \( X \). If we put

\[ E = F \cup ((\beta D \times \omega) \times \{ \kappa \}). \]

Then \( E \) is a \( \sigma \)-compact subset of \( X \) such that \( X = \text{St}(E, \mathcal{U}) \), which shows that \( X \) is \( \sigma \)-starcompact.

Next we show \( \text{st-l}(X) \geq \kappa \). For each \( \alpha < \kappa \), let \( U'_\alpha = \{ d_\alpha \} \times [0, \omega] \), then \( U'_\alpha \) is a compact subset of \( Y \), hence \( U'_\alpha \) is a clopen subset of \( Y \) and \( U'_\alpha \cap U'_\alpha' = \emptyset \) for \( \alpha \neq \alpha' \). For each \( \alpha < \kappa \), let \( U_\alpha = U'_\alpha \times (\kappa + 1) \), then \( U_\alpha \) is an open subset of \( X \) and \( U_\alpha \cap U_\alpha' = \emptyset \) for \( \alpha \neq \alpha' \). For each \( n \in \omega \), let \( V'_n = \beta D \times \{ n \} \), then \( V'_n \) is a compact subset of \( Y \), hence \( V'_n \) is a clopen subset of \( Y \) and \( V_n \cap V_m = \emptyset \) for \( n \neq m \). For each \( n \in \omega \), let \( V_n = V'_n \times (\kappa + 1) \), then \( V_n \) is an open subset of \( X \). Let us consider the open cover

\[ \mathcal{U} = \{ U_\alpha : \alpha < \kappa \} \cup \{ V_n : n \in \omega \} \cup \{ \beta Y \times [0, \kappa] \} \]

of \( X \). It remains to show that \( \text{St}(F, \mathcal{U}) \neq X \) for any subset \( F \) of \( X \) with \( |F| < \kappa \). To show this, let \( F \) be any subset of \( X \) with \( |F| < \kappa \). Then there exists \( \alpha_0 < \kappa \) such that \( F \cap U_{\alpha_0} = \emptyset \), since \( \kappa \) is regular and \( |F| < \kappa \). Hence \( \langle \langle d_{\alpha_0}, \omega \rangle, \kappa \rangle \notin \text{St}(F, \mathcal{U}) \), since \( U_{\alpha_0} \) is the only element of \( \mathcal{U} \) containing \( \langle \langle d_{\alpha_0}, \omega \rangle, \kappa \rangle \), which shows \( \text{st-l}(X) \geq \kappa \). \( \square \)

For normal spaces, it is well-known that countably compactness is equivalent with pseudocompactness, and countably compact space is starcompact. Thus we have the following result.
Theorem 2.5. For any normal space $X$, the following conditions are equivalent:

1. $X$ is pseudocompact $\sigma$-starcompact;
2. $X$ is $\sigma$-star-Lindelöf.

Remark 2.2. The author does not know if there exists an example of a $\sigma$-star-compact normal space that is not $\sigma$-star-Lindelöf.

Acknowledgments. The author would like to thank Prof. R. Li for his valuable suggestions. He would also like to thank the referee for his/her careful reading of the paper and a number of valuable suggestions which led to improvements on several places. Remark 2.1 is due to his/her suggestion.

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(Received October 2, 2011, revised March 10, 2012)