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[Mathematica Bohemica, Vol. 138 (2013), No. 1, 1--13]

Persistent URL: http://dml.cz/dmlcz/143223

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$m^*$-FUZZY BASICALLY DISCONNECTED SPACES IN SMOOTH FUZZY TOPOLOGICAL SPACES

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(Received April 20, 2011)

Abstract. In this paper, the concepts of $m^*$-$r$-fuzzy $\tilde{g}$-open $F_\sigma$ sets and $m^*$-fuzzy basically disconnected spaces are introduced in the sense of Šostak and Ramadan. Some interesting properties and characterizations are studied. Tietze extension theorem for $m^*$-fuzzy basically disconnected spaces is discussed.

Keywords: $m^*$-$r$-fuzzy $\tilde{g}$-open $F_\sigma$ set, $m^*$-fuzzy basically disconnected space, $m^*$-$r$-fuzzy open function

MSC 2010: 54A40, 03E72

1. INTRODUCTION

The concept of fuzzy sets was introduced by Zadeh [13] in his classical paper. Fuzzy sets have applications in many fields such as information [9] and control [10]. In 1985, Šostak [11] introduced a new form of topological structure. In 1992, Ramadan [8] studied the concept of smooth fuzzy topological spaces. The concept of $\tilde{g}$-open sets was discussed by Rajesh and Erdal Ekici [7]. The concept of fuzzy basically disconnected spaces was introduced and studied in [12]. The notions of $m$-structures, $m$-spaces and $m$-continuity were introduced by Popa and Noiri [5], [6]. The concepts of $r$-fuzzy $G_\delta$ sets and $r$-fuzzy $F_\sigma$ sets were introduced in [3]. In this paper, the concepts of $m^*$-$r$-fuzzy $\tilde{g}$-open $F_\sigma$ sets and $m^*$-fuzzy basically disconnected spaces are introduced in the sense of Šostak [11] and Ramadan [8]. Some interesting properties and characterizations are studied. Tietze extension theorem for $m^*$-fuzzy basically disconnected spaces is discussed as in [1].

Throughout this paper, let $X$ be a nonempty set, $I = [0, 1]$ and $I_0 = (0, 1]$. For $\zeta \in I$, $\zeta(x) = \zeta$ for all $x \in X$. The characteristics function of $\lambda \in I^X$ is the function $1_\lambda: X \to I^X$ defined by $1_\lambda(x) = \lambda(x)$, $x \in X$, $r \in I_0$. 

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Definition 1.1 [11]. A function $T: I^X \to I$ is called a smooth fuzzy topology on $X$ if it satisfies the following conditions:

1. $T(\emptyset) = T(\bar{1}) = 1$.
2. $T(\mu_1 \land \mu_2) \geq T(\mu_1) \land T(\mu_2)$ for any $\mu_1, \mu_2 \in I^X$.
3. $T\left( \bigvee_{i \in \Gamma} \mu_i \right) \geq \bigwedge_{j \in \Gamma} T(\mu_j)$ for any $\{\mu_j\}_{j \in \Gamma} \subset I^X$.

The pair $(X, T)$ is called a smooth fuzzy topological space.

Remark 1.1. Let $(X, T)$ be a smooth fuzzy topological space. Then, for each $r \in I_0$, $T_r = \{\mu \in I^X : T(\mu) \geq r\}$ is Chang’s fuzzy topology on $X$.

Definition 1.2 [2]. Let $(X, T)$ be a smooth fuzzy topological space. For each $\lambda \in I^X$, $r \in I_0$, an operator $C_T : I^X \times I_0 \to I^X$ is defined as follows: $C_T(\lambda, r) = \bigwedge_{i \in \Gamma} \{\mu : \mu \geq \lambda, T(\bar{1} - \mu) \geq r\}$. For each $\lambda, \mu \in I^X$ and $r, s \in I_0$, it satisfies the following conditions:

1. $C_T(\emptyset, r) = \bar{0}$.
2. $\lambda \leq C_T(\lambda, r)$.
3. $C_T(\lambda, r) \lor C_T(\mu, r) = C_T(\lambda \lor \mu, r)$.
4. $C_T(\lambda, r) \leq C_T(\lambda, s)$, if $r \leq s$.
5. $C_T(C_T(\lambda, r), r) = C_T(\lambda, r)$.

Proposition 1.1 [2]. Let $(X, T)$ be a smooth fuzzy topological space. For each $\lambda \in I^X$, $r \in I_0$, an operator $I_T : I^X \times I_0 \to I^X$ is defined as follows: $I_T(\lambda, r) = \bigvee_{i \in \Gamma} \{\mu : \mu \leq \lambda, T(\mu) \geq r\}$. For $\lambda, \mu \in I^X$ and $r, s \in I_0$, it satisfies the following conditions:

1. $I_T(\bar{1} - \lambda, r) = \bar{1} - C_T(\lambda, r)$.
2. $I_T(\bar{1}, r) = \bar{1}$.
3. $\lambda \geq I_T(\lambda, r)$.
4. $I_T(\lambda, r) \land I_T(\mu, r) = I_T(\lambda \land \mu, r)$.
5. $I_T(\lambda, r) \geq I_T(\lambda, s)$, if $r \leq s$.
6. $I_T(I_T(\lambda, r), r) = I_T(\lambda, r)$.

Definition 1.3 [3]. Let $(X, T)$ be a smooth fuzzy topological space, $r \in I_0$. For any $\lambda \in I^X$ and $r \in I^0$, $\lambda$ is called

1. an $r$-fuzzy $G_\delta$ set if $\lambda = \bigwedge_{i=1}^{\infty} \lambda_i$ where each $\lambda_i$ is such that $T(\lambda_i) \geq r$;
2. an $r$-fuzzy $F_\sigma$ set if $\lambda = \bigvee_{i=1}^{\infty} \lambda_i$ where each $\bar{1} - \lambda_i$ is such that $T(\bar{1} - \lambda_i) \geq r$.
**Definition 1.4** [8]. Let \((X, T)\) be a smooth fuzzy topological space. For \(\lambda \in I^X\) and \(r \in I_0\),

1. \(\lambda\) is called \(r\)-fuzzy semi-closed (briefly, \(r\)-fsc) if \(\lambda \geq I_T(C_T(\lambda, r), r)\);
2. \(\lambda\) is called \(r\)-fuzzy semi-open (briefly, \(r\)-fso) if \(\lambda \leq C_T(I_T(\lambda, r), r)\).

**Definition 1.5** [8]. Let \((X, T)\) be a smooth fuzzy topological space. For \(\lambda \in I^X\) and \(r \in I_0\),

1. \(SC_T(\lambda, r) = \bigwedge \{\mu \in I^X : \mu \geq \lambda, \mu\) is \(r\)-fuzzy semi-closed\}\) is called the \(r\)-fuzzy semi-closure of \(\lambda\);
2. \(SI_T(\lambda, r) = \bigvee \{\mu \in I^X : \mu \leq \lambda, \mu\) is \(r\)-fuzzy semi-open\}\) is called the \(r\)-fuzzy semi-interior of \(\lambda\).

**Definition 1.6** [1]. Let \((X, T)\) be a smooth fuzzy topological space. For any \(\lambda \in I^X\) and \(r \in I_0\), \(\lambda\) is called

1. \(r\)-fuzzy \(g\)-closed if \(C_T(\lambda, r) \leq \mu\) whenever \(\lambda \leq \mu\) and \(\mu\) is \(r\)-fuzzy semi-open. The complement of an \(r\)-fuzzy \(g\)-closed set is said to be an \(r\)-fuzzy \(g\)-open set;
2. \(r\)-fuzzy \(\ast g\)-closed if \(C_T(\lambda, r) \leq \mu\) whenever \(\lambda \leq \mu\) and \(\mu\) is \(r\)-fuzzy \(g\)-open. The complement of an \(r\)-fuzzy \(\ast g\)-closed set is said to be an \(r\)-fuzzy \(\ast g\)-open set;
3. \(r\)-fuzzy \(#)\(g\)-semiclosed (briefly \(r\)-f\#gs\)-closed) if \(SC_T(\lambda, r) \leq \mu\) whenever \(\lambda \leq \mu\) and \(\mu\) is \(r\)-fuzzy \(\ast g\)-open. The complement of an \(r\)-fuzzy \(# g\)-semiclosed set is said to be an \(r\)-fuzzy \(\# g\)-semiclosed set (briefly \(r\)-f\#gs\)-open set);
4. \(r\)-fuzzy \(\tilde{g}\)-closed if \(C_T(\lambda, r) \leq \mu\) whenever \(\lambda \leq \mu\) and \(\mu\) is \(r\)-fuzzy \(# fgs\)-open. The complement of an \(r\)-fuzzy \(\tilde{g}\)-closed set is said to be an \(r\)-fuzzy \(\tilde{g}\)-open set;

**Definition 1.7** [5], [6]. A subfamily \(m_X\) of the power set \(\mathcal{P}(X)\) of a nonempty set \(X\) is called a minimal structure (briefly, \(m\)-structure) on \(X\) if \(\varphi \in m_X\) and \(X \in m_X\). By \((X, m_X)\) we denote a non-empty subset \(X\) with a minimal structure \(m_X\) on \(X\) and call it an \(m\)-space. Each member of \(m_X\) is said to be \(m_X\)-open (or briefly, \(m\)-open) and the complement of an \(m_X\)-open set is said to be \(m_X\)-closed (or briefly, \(m\)-closed).

**Notation 1.1.** Let \((X, T)\) be a smooth fuzzy topological space, \(r \in I_0\).

1. The family of \(r\)-fuzzy \(\tilde{g}\) open sets in \((X, T)\) is denoted by \(\tilde{g}O(X, T)\).
2. The family of \(r\)-fuzzy \(F_\sigma\) sets in \((X, T)\) is denoted by \(F_\sigma(X, T)\).
2. $m^*\text{-fuzzy basically disconnected spaces}$

In this section, the concepts of $m^*r\text{-fuzzy} \, \tilde{g}\text{-open} \, F_\sigma$ sets and $m^*\text{-fuzzy basically disconnected spaces}$ are introduced. Some interesting properties and characterizations are studied.

**Definition 2.1.** Let $X$ be a nonempty set and $I^X$ a collection of all fuzzy sets in $X$. A subfamily $m_X$ of $I^X$ is called a minimal structure (briefly, $m$-structure) on $X$ if $0 \in m_X$ and $1 \in m_X$.

**Definition 2.2.** Let $(X, T)$ be a smooth fuzzy topological space, $r \in I_0$. Then the collection of the families $\tilde{g}O(X, T)$ and $F_\sigma(X, T)$ which is finer than the smooth fuzzy topology $T$ on $X$ is a minimal* structure (briefly, $m^*$-structure) on $X$, denoted by $m^*_X$. A nonempty set $X$ with an $m^*$-structure $m^*_X$ on $X$ is denoted by $(X, m^*_X)$ (or briefly, $(X, m^*)$) and it is called an $m^*$-smooth fuzzy space. Each member of $m^*_X$ is said to be $m^*r\text{-fuzzy} \, \tilde{g}\text{-open} \, F_\sigma$ and the complement of an $m^*r\text{-fuzzy} \, \tilde{g}\text{-open} \, F_\sigma$ set is said to be $m^*r\text{-fuzzy} \, \tilde{g}\text{-closed} \, G_\delta$.

**Definition 2.3.** A minimal structure $m^*_X$ on a nonempty set $X$ is said to have property $B$ if the union of any family of $m^*r\text{-fuzzy} \, \tilde{g}\text{-open} \, F_\sigma$ sets belonging to $m^*_X$, $r \in I_0$.

**Definition 2.4.** Let $(X, T)$ be a smooth fuzzy topological space with an $m^*$-structure $m^*_X$ determined by $T$ and let $m^*_X$ have property $B$. For any $\lambda \in I^X$ and $r \in I_0$, the $m^*_Xr\text{-fuzzy} \, \tilde{g}\text{G}_\delta\text{-closure}$ of $\lambda$ and the $m^*_Xr\text{-fuzzy} \, \tilde{g}\text{F}_\sigma$ interior of $\lambda$ are defined as follows:

1. $C_{m^*}(\lambda, r) = \bigwedge\{\mu: \lambda \leq \mu, \mu \text{ is } m^*r\text{-fuzzy } \tilde{g}\text{-closed } G_\delta\}$;
2. $I_{m^*}(\lambda, r) = \bigvee\{\mu: \lambda \geq \mu, \mu \text{ is } m^*r\text{-fuzzy } \tilde{g}\text{-open } F_\sigma\}$.

**Remark 2.1.** Let $(X, T)$ be a smooth fuzzy topological space, $r \in I_0$. For any $\lambda \in I^X$, if $m^*_X = T$, then

1. $C_{m^*}(\lambda, r) = C_T(\lambda, r)$;
2. $I_{m^*}(\lambda, r) = I_T(\lambda, r)$.

**Notation 2.1.** Let $(X, T)$ be a smooth fuzzy topological space with an $m^*$-structure determined by $T$. For $r \in I_0$, any $\lambda \in I^X$ which is both $m^*r\text{-fuzzy} \, \tilde{g}\text{-open} \, F_\sigma$ and $m^*r\text{-fuzzy} \, \tilde{g}\text{-closed} \, G_\delta$ is denoted by $m^*r\text{-fuzzy} \, \tilde{g}\text{-COGF}$.

**Definition 2.5.** Let $(X, T)$ be a smooth fuzzy topological space with an $m^*$-structure $m^*_X$ determined by $T$ and let $m^*_X$ have property $B$. The $m^*$-smooth fuzzy space $(X, m^*)$ is said to be $m^*$-fuzzy basically disconnected if the $m^*r\text{-fuzzy} \, \tilde{g}\text{G}_\delta\text{-closure}$ of every $m^*r\text{-fuzzy} \, \tilde{g}\text{-open} \, F_\sigma$ set is $m^*r\text{-fuzzy} \, \tilde{g}\text{-open} \, F_\sigma$, $r \in I_0$. 

Proposition 2.1. For a smooth fuzzy topological space with an $m^*$-structure on $X$ determined by $T$ where $m^*_X$ has property $B$, the following conditions are equivalent:

(a) $(X, m^*)$ is an $m^*$-fuzzy basically disconnected space.

(b) For each $m^*$-r-fuzzy $g$-closed $G_\delta$ set $\lambda$, $I_{m^*}(\lambda, r)$ is $m^*$-r-fuzzy $g$-closed $G_\delta$, $r \in I_0$.

(c) For each $m^*$-r-fuzzy $g$-open $F_\sigma$ set $\lambda$,

\[ C_{m^*}(\lambda, r) + C_{m^*}(\bar{I} - C_{m^*}(\lambda, r)), r) = \bar{1}, r \in I_0. \]

(d) For every pair of $m^*$-r-fuzzy $g$-open $F_\sigma$ sets $\lambda$ and $\mu$ with $C_{m^*}(\lambda, r) + \mu = \bar{1}$, we have $C_{m^*}(\lambda, r) + C_{m^*}(\mu, r) = \bar{1}, r \in I_0$.

Proof. (a) $\Rightarrow$ (b). Let $\lambda \in I^X$ be any $m^*$-r-fuzzy $g$-closed $G_\delta$ set. Then $\bar{I} - \lambda$ is $m^*$-r-fuzzy $g$-open $F_\sigma$. Now, $C_{m^*}(\bar{I} - \lambda, r) = \bar{I} - C_{m^*}(\lambda, r)$. By (a), $C_{m^*}(\bar{I} - \lambda, r)$ is $m^*$-r-fuzzy $g$-open $F_\sigma$, which implies that $I_{m^*}(\lambda, r)$ is $m^*$-r-fuzzy $g$-closed $G_\delta$.

(b) $\Rightarrow$ (c). Let $\lambda$ be any $m^*$-r-fuzzy $g$-open $F_\sigma$ set. Then

\[ (2.1) \quad C_{m^*}(\lambda, r) + C_{m^*}(\bar{I} - C_{m^*}(\lambda, r)), r) = C_{m^*}(\lambda, r) + C_{m^*}((I_{m^*}(\bar{I} - \lambda, r)), r). \]

Since $\lambda$ is $m^*$-r-fuzzy $g$-open $F_\sigma$, $\bar{I} - \lambda$ is $m^*$-r-fuzzy $g$-closed $G_\delta$. Hence by (b), $I_{m^*}(\bar{I} - \lambda, r)$ is $m^*$-r-fuzzy $g$-closed $G_\delta$. Therefore, by (2.1)

\[ C_{m^*}(\lambda, r) + C_{m^*}(\bar{I} - C_{m^*}(\lambda, r)), r) = C_{m^*}(\lambda, r) + (I_{m^*}(\bar{I} - \lambda, r)) \]

\[ = C_{m^*}(\lambda, r) + \bar{I} - C_{m^*}(\lambda, r) \]

\[ = \bar{1}. \]

Therefore, $C_{m^*}(\lambda, r) + C_{m^*}(\bar{I} - C_{m^*}(\lambda, r)), r) = \bar{1}$.

(c) $\Rightarrow$ (d). Let $\lambda$ and $\mu$ be $m^*$-r-fuzzy $g$-open $F_\sigma$ sets with

\[ (2.2) \quad C_{m^*}(\lambda, r) + \mu = \bar{1}. \]

Then by (c) we have $\bar{I} = C_{m^*}(\lambda, r) + C_{m^*}((I - C_{m^*}(\lambda, r)), r) = C_{m^*}(\lambda, r) + C_{m^*}(\mu, r)$.

Therefore, $C_{m^*}(\lambda, r) + C_{m^*}(\mu, r) = \bar{1}$.

(d) $\Rightarrow$ (a). Let $\lambda$ be an $m^*$-r-fuzzy $g$-open $F_\sigma$ set. Put $\mu = \bar{I} - C_{m^*}(\lambda, r)$. Then $C_{m^*}(\lambda, r) + \mu = \bar{1}$. Therefore by (d), $C_{m^*}(\lambda, r) + C_{m^*}(\mu, r) = \bar{1}$. This implies that $C_{m^*}(\lambda, r)$ is $m^*$-r-fuzzy $g$-open $F_\sigma$ and so $(X, T)$ is $m^*$-fuzzy basically disconnected.

$\square$
Proposition 2.2. Let \((X, T)\) be a smooth fuzzy topological space with an \(m^*\)-structure \(m_X^*\) determined by \(T\) and let \(m_X^*\) possess property \(B\). Then \((X, m^*)\) is \(m^*\)-fuzzy basically disconnected if and only if for all \(m^*\)-r-fuzzy \(\tilde{g}\)-open \(F_\sigma\) sets \(\lambda\) and \(m^*\)-r-fuzzy \(\tilde{g}\)-closed sets \(\mu\) such that \(\lambda \leq \mu\), \(C_{m^*}(\lambda, r) \leq I_{m^*}(\mu, r), r \in I_0\).

Proof. Let \(\lambda\) be \(m^*\)-r-fuzzy \(\tilde{g}\)-open \(F_\sigma\) and let \(\mu\) be \(m^*\)-r-fuzzy \(\tilde{g}\)-closed \(G_\delta\) with \(\lambda \leq \mu\). Then by (b) of Proposition 2.1, \(I_{m^*}(\mu, r)\) is \(m^*\)-r-fuzzy \(\tilde{g}\)-closed \(G_\delta\). Also, since \(\lambda\) is \(m^*\)-r-fuzzy \(\tilde{g}\)-open \(F_\sigma\), \(C_{m^*}(\lambda, r) \leq I_{m^*}(\mu, r)\).

Conversely, let \(\mu\) be any \(m^*\)-r-fuzzy \(\tilde{g}\)-closed \(G_\delta\) set. Then \(I_{m^*}(\mu, r) \in I^X\) is \(m^*\)-r-fuzzy \(\tilde{g}\)-open \(F_\sigma\) and \(I_{m^*}(\mu, r) \leq \mu\). Therefore by assumption, \(C_{m^*}((I_{m^*}(\mu, r), r)) \leq I_{m^*}(\mu, r)\). This implies that \(I_{m^*}(\mu, r)\) is \(m^*\)-r-fuzzy \(\tilde{g}\)-closed \(G_\delta\). Hence by (b) of Proposition 2.1, it follows that \((X, m^*)\) is \(m^*\)-fuzzy basically disconnected. \(\Box\)

Remark 2.2. Let \((X, m^*)\) be any \(m^*\)-fuzzy basically disconnected space. Let \(\{\lambda_i, \tilde{1} - \mu_i/i \in \mathbb{N}\}\) be collection such that \(\lambda^*_i\)s are \(m^*\)-r-fuzzy \(\tilde{g}\)-open \(F_\sigma\) and \(\mu^*_i\)s are \(m^*\)-r-fuzzy \(\tilde{g}\)-closed \(G_\delta\), and let \(\lambda\) and \(\mu\) be \(m^*\)-r-fuzzy \(\tilde{g}\)-COGF sets. If \(\lambda_i \leq \lambda \leq \mu_j\) and \(\lambda_i \leq \mu \leq \mu_j\) for all \(i, j \in \mathbb{N}\), then there exists an \(m^*\)-r-fuzzy \(\tilde{g}\)-COGF set \(\gamma\) such that \(C_{m^*}(\lambda_i, r) \leq \gamma \leq I_{m^*}(\mu_j, r)\) for all \(i, j \in \mathbb{N}, r \in I_0\).

Proof. By Proposition 2.2, \(C_{m^*}(\lambda_i, r) \leq C_{m^*}(\lambda, r) \land I_{m^*}(\mu, r) \leq I_{m^*}(\mu_j, r)\) for all \(i, j \in \mathbb{N}\). Therefore, \(\gamma = C_{m^*}(\lambda, r) \land I_{m^*}(\mu, r)\) is an \(m^*\)-r-fuzzy \(\tilde{g}\)-COGF set satisfying the required conditions. \(\Box\)

Proposition 2.3. Let \((X, m^*)\) be any \(m^*\)-fuzzy basically disconnected space. Let \(\{\lambda_i\}_{i \in \mathbb{Q}}\) and \(\{\mu_i\}_{i \in \mathbb{Q}}\) be monotone increasing collections of \(m^*\)-r-fuzzy \(\tilde{g}\)-open \(F_\sigma\) sets and \(m^*\)-r-fuzzy \(\tilde{g}\)-closed \(G_\delta\) sets of \((X, m^*)\) and suppose that \(\lambda_{q_1} \leq \mu_{q_2}\) whenever \(q_1 < q_2\) (\(Q\) is the set of all rational numbers). Then there exists a monotone increasing collection \(\{\gamma_i\}_{i \in \mathbb{Q}}\) of \(m^*\)-r-fuzzy \(\tilde{g}\)-COGF sets of \((X, m^*)\) such that \(C_{m^*}(\lambda_{q_1}, r) \leq \gamma_{q_2}\) and \(\gamma_{q_1} \leq I_{m^*}(\mu_{q_2}, r)\) whenever \(q_1 < q_2, r \in I_0\).

Proof. Let us arrange all rational numbers into a sequence \(\{q_n\}\) (without repetitions). For every \(n \geq 2\), we shall define inductively a collection \(\{\gamma_{q_i}/1 \leq i \leq n\} \subset I^X\) such that

\[
(S_n) \quad C_{m^*}(\lambda_{q_i}, r) \leq \gamma_{q_i} \quad \text{if} \quad q < q_i, \quad \gamma_{q_i} \leq I_{m^*}(\mu_{q_i}, r) \quad \text{if} \quad q_i < q, \quad \text{for all} \quad i \leq n
\]

By Proposition 2.2, the countable collections \(\{C_{m^*}(\lambda_{q_i}, r)\}\) and \(\{I_{m^*}(\mu_{q_i}, r)\}\) satisfy \(C_{m^*}(\lambda_{q_1}, r) \leq I_{m^*}(\mu_{q_2}, r)\) if \(q_1 < q_2\). By Remark 2.2, there exists an \(m^*\)-r-fuzzy \(\tilde{g}\)-COGF set \(\delta_1\) such that \(C_{m^*}(\lambda_{q_1}, r) \leq \delta_1 \leq I_{m^*}(\mu_{q_2}, r)\). Setting \(\gamma_{q_1} = \delta_1\), we get \((S_2)\).

Define \(\psi = \bigvee\{\gamma_{q_i} : i < n, q_i < q_n\} \lor \lambda_{q_n}\) and \(\varphi = \bigwedge\{\gamma_{q_j} : j < n, q_j > q_n\} \land \mu_{q_n}\). Then we have \(C_{m^*}(\gamma_{q_1}, r) \leq C_{m^*}(\psi, r) \leq I_{m^*}(\gamma_{q_2}, r)\) and \(C_{m^*}(\gamma_{q_1}, r) \leq I_{m^*}(\varphi, r) \leq \).
\[ I_m(\gamma_{q_i}, r) \] whenever \( q_i < q_n < q_j \) \((i, j < n)\), as well as \( \lambda_q \leq C_m(\psi, r) \leq \mu_q \) and 
\( \lambda_q \leq I_m(\varphi, r) \leq \mu' \) whenever \( q < q_n < q' \). This shows that the countable collections 
\( \{ \gamma_{q_i}: i < n, q_i < q_n \} \cup \{ \lambda_q: q < q_n \} \) and \( \{ \gamma_{q_j}: j < n, q_j > q_n \} \cup \{ \mu_q: q > q_n \} \)
together with \( \psi \) and \( \varphi \) fulfil all the conditions of Remark 2.2. Hence, there exists an 
\( m^*r \)-fuzzy \( g \)-COGF set \( \delta_n \) such that \( C_m(\delta_n, r) \leq \mu_q \) if \( q_n < q \), \( \lambda_q \leq I_m(\delta_n, r) \) if
\( q < q_n, C_m(\gamma_{q_i}, r) \leq I_m(\delta_n, r) \) if \( q_i < q_n \), \( C_m(\delta_n, r) \leq I_m(\gamma_{q_i}, r) \) if \( q_n < q_j \) where
\( 1 \leq i, j \leq n - 1 \). Now, setting \( \gamma_{q_n} = \delta_n \) we obtain the fuzzy sets \( \gamma_{q_1}, \gamma_{q_2}, \gamma_{q_3}, \ldots, \gamma_{q_n} \)
that satisfy \((S_n+1)\). Therefore, the collection \( \{ \gamma_{q_i}: i = 1, 2, \ldots \} \) has the required property. \( \square \)

3. Properties and characterizations of \( m^* \)-fuzzy basically disconnected spaces

In this section, the concept of \( m^* \)-fuzzy continuous functions is introduced. In this
regard, various properties and characterizations are discussed.

**Definition 3.1.** Let \((X, T)\) and \((Y, S)\) be any two smooth fuzzy topological
spaces with the \( m^* \)-structures \( m_1^* \) and \( m_2^* \) determined by \( T \) and \( S \) respectively, and
let both \( m_1^* \) and \( m_2^* \) have property \( \mathcal{B} \). A function \( f: (X, m_1^*) \to (Y, m_2^*) \) is called 
\( m^* \)-fuzzy irresolute if \( f^{-1}(\lambda) \in I^X \) is \( m^*r \)-fuzzy \( g \)-closed \( G_\delta \), for every \( m^*r \)-fuzzy closed \( g \)-\( G_\delta \) set \( \lambda \in I^Y, r \in I_0 \).

**Definition 3.2.** Let \((X, T)\) and \((Y, S)\) be any two smooth fuzzy topological
spaces with the \( m^* \)-structures \( m_1^* \) and \( m_2^* \) determined by \( T \) and \( S \) respectively, and
let both \( m_1^* \) and \( m_2^* \) have property \( \mathcal{B} \). A function \( f: (X, m_1^*) \to (Y, m_2^*) \) is called 
\( m^* \)-fuzzy open if \( f(\lambda) \in I^Y \) is \( m^*r \)-fuzzy \( g \)-open \( F_\sigma \), for every \( m^*r \)-fuzzy \( g \)-open \( F_\sigma \)
set \( \lambda \in I^X, r \in I_0 \).

**Proposition 3.1.** Let \((X, T)\) and \((Y, S)\) be any two smooth fuzzy topological
spaces with the \( m^* \)-structures \( m_1^* \) and \( m_2^* \) determined by \( T \) and \( S \) respectively, and
let both \( m_1^* \) and \( m_2^* \) have property \( \mathcal{B} \). A function \( f: (X, m_1^*) \to (Y, m_2^*) \) is \( m^* \)-fuzzy
irresolute iff \( f(C_m(\lambda, r)) \leq C_m(f(\lambda), r) \), for every \( \lambda \in I^X, r \in I_0 \).

**Proof.** Suppose that \( f \) is \( m^* \)-fuzzy irresolute and \( \lambda \in I^X \). Then \( C_m(f(\lambda), r) \in I^Y \) is \( m^*r \)-fuzzy \( g \)-closed \( G_\delta \). By hypothesis, \( f^{-1}(C_m(f(\lambda), r)) \in I^X \) is \( m^*r \)-fuzzy
\( g \)-closed \( G_\delta \). Also, \( \lambda \leq f^{-1}(f(\lambda)) \leq f^{-1}(C_m(f(\lambda), r)) \). Hence by the definition of
the \( m^*r \)-fuzzy \( g \)-\( G_\delta \) closure, \( C_m(\lambda, r) \leq f^{-1}(C_m(f(\lambda), r)) \). That is, \( f(C_m(\lambda, r)) \leq C_m(f(\lambda), r) \).

Conversely, suppose that \( \lambda \in I^Y \) is \( m^*r \)-fuzzy \( g \)-closed \( G_\delta \). Now by hypothesis,
\( f(C_m(f^{-1}(\lambda), r)) \leq C_m(f(f^{-1}(\lambda)), r) \). This implies \( C_m(f^{-1}(\lambda), r) \leq f^{-1}(\lambda) \). So
$f^{-1}(\lambda) = C_{m^*}(f^{-1}(\lambda), r)$. That is, $f^{-1}(\lambda) \in I^X$ is $m^*$-fuzzy $\tilde{g}$-closed $G_{\delta}$ and so $f$ is $m^*$-fuzzy irresolute. 

\[\qed\]

**Proposition 3.2.** Let $(X, T)$ and $(Y, S)$ be any two smooth fuzzy topological spaces with the $m^*$-structures $m_1^*$ and $m_2^*$ determined by $T$ and $S$ respectively, and let both $m_1^*$ and $m_2^*$ have property $B$. Let $f: (X, m_1^*) \rightarrow (Y, m_2^*)$ be an $m^*$-fuzzy open surjective function. Then $f^{-1}(C_{m^*}(\lambda, r)) \subseteq C_{m^*}(f^{-1}(\lambda), r)$ for every $\lambda \in I^Y$, $r \in I_0$.

**Proof.** Let $\lambda \in I^Y$ and let $\mu = f^{-1}(\tilde{1} - \lambda)$. Then $I_{m^*}(f^{-1}(\tilde{1} - \lambda), r) = I_{m^*}(\tilde{1} - \lambda, r) \in I^X$ is $m^*$-fuzzy $\tilde{g}$-open $F_{\sigma}$. Now, $I_{m^*}(\mu, r) \subseteq \mu$. Hence, $f(I_{m^*}(\mu, r)) \subseteq f(\mu)$. That is, $I_{m^*}(f(I_{m^*}(\mu, r)), r) \leq I_{m^*}(f(\mu), r)$. Since $f$ is $m^*$-fuzzy open, $f(I_{m^*}(\mu, r))$ is $m^*$-fuzzy $\tilde{g}$-open $F_{\sigma}$. Therefore, $f(I_{m^*}(\mu, r)) \subseteq f(I_{m^*}(\mu), r) = I_{m^*}(\tilde{1} - \lambda, r)$. Hence, $I_{m^*}(f^{-1}(\tilde{1} - \lambda), r) = I_{m^*}(\tilde{1} - \lambda, r) \subseteq I_{m^*}(f^{-1}(\tilde{1} - \lambda), r)$. Therefore, $f^{-1}(C_{m^*}(\lambda, r)) \subseteq C_{m^*}(f^{-1}(\tilde{1} - \lambda), r)$. Therefore, $f^{-1}(C_{m^*}(\lambda, r)) \subseteq C_{m^*}(f^{-1}(\lambda), r)$. 

\[\qed\]

**Proposition 3.3.** Let $(X, m_1^*)$ be any $m^*$-fuzzy basically disconnected space and let $(Y, S)$ be any smooth fuzzy topological space with an $m^*$-structure $m_2^*$ determined by $S$ where $m_2^*$ has property $B$. Let $f: (X, m_1^*) \rightarrow (Y, m_2^*)$ be an $m^*$-fuzzy irresolute, $m^*$-fuzzy open and surjective function. Then $(Y, m_2^*)$ is $m^*$-fuzzy basically disconnected.

**Proof.** Let $\lambda \in I^Y$ be any $m^*$-fuzzy $\tilde{g}$-open $F_{\sigma}$ set. Since $f$ is $m^*$-fuzzy irresolute, $f^{-1}(\lambda) \in I^X$ is $m^*$-fuzzy $\tilde{g}$-open $F_{\sigma}$. Therefore by an assumption on $(X, m_1^*)$, it follows that $C_{m^*}(f^{-1}(\lambda), r) \in I^X$ is $m^*$-fuzzy $\tilde{g}$-open $F_{\sigma}$. As $f$ is $m^*$-fuzzy open, $f(C_{m^*}(f^{-1}(\lambda), r)) \in I^Y$ is $m^*$-fuzzy $\tilde{g}$-open $F_{\sigma}$. By Proposition 3.2, $f^{-1}(C_{m^*}(\lambda, r)) \subseteq C_{m^*}(f^{-1}(\lambda), r)$ and hence, $f(\tilde{1} - f^{-1}(C_{m^*}(\lambda, r))) = C_{m^*}(\lambda, r) \subseteq f(C_{m^*}(f^{-1}(\lambda), r)) \subseteq C_{m^*}(f^{-1}(\lambda), r) = C_{m^*}(\lambda, r)$ by Proposition 3.1. Thus $C_{m^*}(\lambda, r) = f(C_{m^*}(f^{-1}(\lambda), r))$ and therefore, $C_{m^*}(\lambda, r) \in I^Y$ is $m^*$-fuzzy $\tilde{g}$-open $F_{\sigma}$, proving that $(Y, m_2^*)$ is $m^*$-fuzzy basically disconnected. 

\[\qed\]

**Definition 3.3.** Let $(X, T)$ be a smooth fuzzy topological space with an $m^*$-structure $m_X^*$ determined by $T$ and let $m_X^*$ possess property $B$. A function $f: X \rightarrow R(I)$ is called lower (upper) $m^*$-fuzzy continuous if $f^{-1}(R_t)(f^{-1}(L_t))$ is $m^*$-fuzzy $\tilde{g}$-open $F_{\sigma}$ $(m^*$-fuzzy $\tilde{g}$-open $F_{\sigma}/m^*$-fuzzy $\tilde{g}$-closed $G_{\delta}$), for each $t \in \mathbb{R}$, $r \in I_0$. 

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Proposition 3.4. Let \((X, T)\) be a smooth fuzzy topological space with an \(m^*\)-structure \(m_X^*\) determined by \(T\) and let \(m_X^*\) have property \(B\). For \(\lambda \in I^X\) and \(r \in I_0\), let \(f: X \to R(I)\) be such that

\[
f(x)(t) = \begin{cases} 
1 & \text{if } t < 0, \\
\lambda(x) & \text{if } 0 \leq t \leq 1, \\
0 & \text{if } t > 0 
\end{cases}
\]

for all \(x \in X\). Then \(f\) is lower (upper) \(m^*\)-fuzzy continuous iff \(\lambda\) is \(m^r\)-fuzzy \(\tilde{g}\)-open \(F_\sigma\) (\(m^r\)-fuzzy \(\tilde{g}\)-open \(F_\sigma/m^r\)-fuzzy \(\tilde{g}\)-closed \(G_\delta\)), \(r \in I_0\).

Proof. The proof follows from Proposition 3.4. \(\square\)

Proposition 3.5. Let \((X, T)\) be a smooth fuzzy topological space with an \(m^*\)-structure \(m_X^*\) determined by \(T\) and let \(m_X^*\) have property \(B\); let \(\lambda \in I^X\). Then \(1_\lambda\) is lower (upper) \(m^*\)-fuzzy continuous iff \(\lambda\) is \(m^r\)-fuzzy \(\tilde{g}\)-open \(F_\sigma\) (\(m^r\)-fuzzy \(\tilde{g}\)-open \(F_\sigma/m^r\)-fuzzy \(\tilde{g}\)-closed \(G_\delta\)), \(r \in I_0\).

Proof. The proof follows from Proposition 3.4. \(\square\)

Definition 3.4. Let \((X, T)\) and \((Y, S)\) be any two smooth fuzzy topological spaces with the \(m^*\)-structures \(m_1^*\) and \(m_2^*\) determined by \(T\) and \(S\) respectively and let both \(m_1^*\) and \(m_2^*\) have property \(B\). A function \(f: (X, m_1^*) \to (Y, m_2^*)\) is called strongly \(m^*\)-fuzzy continuous if \(f^{-1}(\lambda) \in I^X\) is \(m^r\)-fuzzy \(\tilde{g}\)-open \(F_\sigma/m^r\)-fuzzy \(\tilde{g}\)-closed \(G_\delta\), for every \(m^r\)-fuzzy \(\tilde{g}\)-open \(F_\sigma\) set \(\lambda \in I^Y\), \(r \in I_0\).

Proposition 3.6. Let \((X, T)\) be a smooth fuzzy topological space with an \(m^*\)-structure \(m_X^*\) determined by \(T\) and let \(m_X^*\) possess property \(B\). Then for \(r \in I_0\), the following conditions are equivalent:

(a) \((X, m^*)\) is an \(m^*\)-fuzzy basically disconnected space.
(b) If \( g, h : X \to R(I) \) where \( g \) is lower \( m^* \)-fuzzy continuous, \( h \) is upper \( m^* \)-fuzzy continuous, then there exists \( f \in C_{S_{m^*}}(X, m^*) \) such that \( g \leq f \leq h \).

\([C_{S_{m^*}}(X, m^*)] = \) collection of all strongly \( m^* \)-fuzzy continuous functions on \( X \) with values in \( R(I) \).

(c) If \( \bar{1} - \lambda, \mu \) are \( m^*r \)-fuzzy \( \tilde{g} \)-open \( F_\sigma \) sets such that \( \mu \leq \lambda \), then there exists a strongly \( m^* \)-fuzzy continuous \( f : X \to I^X \) such that \( \mu \leq (\bar{1} - L_1)f \leq R_0f \leq \lambda \).

**Proof.** (a) \( \Rightarrow \) (b). Define \( H_k = L_kh \) and \( \tilde{g} = (\bar{1} - R_k)g \), \( k \in Q \). Thus we have two monotone increasing families of \( m^*r \)-fuzzy \( \tilde{g} \)-open \( F_\sigma \) sets and \( m^*r \)-fuzzy \( \tilde{g} \)-closed sets of \( (X, m^*) \). Moreover \( H_k \leq G_s \) if \( k < s \). By Proposition 2.3, there exists a monotone increasing family \( \{ F_k \}_{k \in Q} \) of \( m^*r \)-fuzzy \( \tilde{g} \)-COGF sets of \( (X, m^*) \) sets such that \( C_{m^*}(H_k, r) \leq F_s \) and \( F_k \leq I_{m^*}(G_s, r) \) whenever \( k < s \). Letting \( V_t = \bigwedge_{k < t} (\bar{1} - F_k) \) for all \( t \leq R \), we define a monotone decreasing family \( \{ V_t : t \in R \} \subset I^X \). Moreover, we have \( C_{m^*}(V_t, r) \leq I_{m^*}(V_s, r) \) whenever \( s < t \). We have

\[
\bigvee_{t \in R} V_t = \bigvee_{t \in R} \bigwedge_{k < t} (\bar{1} - F_k) = \bigvee_{t \in R} \bigwedge_{k < t} (\bar{1} - G_k)
\]

\[
= \bigvee_{t \in R} \bigwedge_{k < t} g^{-1}(R_k) = g^{-1}\left( \bigvee_{t \in R} R_t \right) = \bar{1}.
\]

Similarly, \( \bigwedge_{t \in R} V_t = 0 \). We now define a function \( f : X \to R(I) \) possessing the required properties. Let \( f(x)(t) = V_t(x) \) for all \( x \in X \) and \( t \in R \). By the above discussion it follow that \( f \) is well defined. To prove \( f \) is strongly \( m^* \)-fuzzy continuous, we observe that

\[
\bigvee_{s > t} V_s = \bigvee_{s > t} I_{m^*}(V_s, r) \quad \text{and} \quad \bigwedge_{s < t} V_s = \bigwedge_{s < t} C_{m^*}(V_s, r).
\]

Then \( f^{-1}(R_t) = \bigvee_{s > t} V_s = \bigvee_{s > t} I_{m^*}(V_s, r) \) is \( m^*r \)-fuzzy \( \tilde{g} \)-COGF. And \( f^{-1}(L'_t) = \bigwedge_{s < t} V_s = \bigwedge_{s < t} C_{m^*}(V_s, r) \) is \( m^*r \)-fuzzy \( \tilde{g} \)-COGF. Therefore, \( f \) is strongly \( m^* \)-fuzzy continuous. To conclude the proof it remains to show that \( g \leq f \leq h \). That is, \( g^{-1}(\bar{1} - L_t) \leq f^{-1}(\bar{1} - L_t) \leq h^{-1}(\bar{1} - L_t) \) and \( g^{-1}(R_t) \leq f^{-1}(R_t) \leq h^{-1}(R_t) \) for each \( t \in R \). We have

\[
g^{-1}(\bar{1} - L_t) = \bigwedge_{s < t} g^{-1}(\bar{1} - L_s) = \bigwedge_{s < t} \bigwedge_{k < s} g^{-1}(R_k)
\]

\[
= \bigwedge_{s < t} \bigwedge_{k < s} (\bar{1} - G_k) \leq \bigwedge_{s < t} \bigwedge_{k < s} (\bar{1} - F_k)
\]

\[
= \bigwedge_{s < t} V_s = f^{-1}(\bar{1} - L_t)
\]

and
\[ f^{-1}(\bar{1} - L_t) = \bigwedge_{s < t} V_s = \bigwedge_{s < t, k < s} (\bar{1} - F_k) \]
\[ \leq \bigwedge_{s < t} \bigcap_{k < s} (\bar{1} - H_k) = \bigwedge_{s < t} h^{-1}(\bar{1} - L_k) \]
\[ = \bigwedge_{s < t} h^{-1}(\bar{1} - L_s) = h^{-1}(\bar{1} - L_t). \]

Similarly, we obtain
\[ g^{-1}(R_t) = \bigvee_{s > t} g^{-1}(R_s) = \bigvee_{s > t} \bigvee_{k > s} g^{-1}(R_k) \]
\[ = \bigvee_{s > t} \bigvee_{k > s} (\bar{1} - G_k)) \leq \bigvee_{s > t} \bigcap_{k < s} (\bar{1} - F_k) \]
\[ = \bigvee_{s > t} V_s = f^{-1}(R_t) \quad \text{and} \]
\[ f^{-1}(R_t) = \bigvee_{s > t} V_s = \bigvee_{s > t} \bigcap_{k < s} (\bar{1} - F_k) \]
\[ \leq \bigvee_{s > t} \bigvee_{k > s} (\bar{1} - H_k) = \bigvee_{s > t} h^{-1}(\bar{1} - L_k) \]
\[ = \bigvee_{s > t} h^{-1}(R_s) = h^{-1}(R_t). \]

Thus, (b) is proved.

(b) ⇒ (c). Suppose that \( \lambda \) is \( m^*r \)-fuzzy \( \tilde{g} \)-closed \( G_\delta \) and \( \mu \) is \( m^*r \)-fuzzy \( \tilde{g} \)-open \( F_\sigma \) such that \( \mu \leq \lambda \). Then \( 1_\mu \leq 1_\lambda \) where \( 1_\mu, 1_\lambda \) are lower and upper \( m^* \)-fuzzy continuous functions, respectively. Hence by (b), there exists a strong \( m^* \)-fuzzy continuous function \( f: X \to R(I) \) such that \( 1_\mu \leq f \leq 1_\lambda \). Clearly, \( f(x) \in I^X \) for all \( x \in X \) and \( \mu = (\bar{1} - L_1)1_\mu \leq (\bar{1} - L_1)f \leq R_0f \leq R_01_\lambda = \lambda \). Therefore, \( \mu \leq (\bar{1} - L_1)f \leq R_0f \leq \lambda \).

(c) ⇒ (a). \( (\bar{1} - L_1)f \) and \( R_0f \) are \( m^*r \)-fuzzy \( \tilde{g} \)-COGF sets. By Proposition 2.2, \( (X, m^*) \) is an \( m^* \)-fuzzy basically disconnected space. \( \square \)

4. Tietze extension theorem

In this section, Tietze Extension Theorem for \( m^* \)-fuzzy basically disconnected spaces is studied.
Proposition 4.1. Let \((X, m^*)\) be an \(m^*\)-fuzzy basically disconnected space and let \(A \subseteq X\) be such that \(1_A\) is \(m^* r\)-fuzzy \(\tilde{g}\)-open \(F_\sigma\). Let \(f: (A, m^*/A) \to I^X\) be strong \(m^*\)-fuzzy continuous. Then \(f\) has a strong \(m^*\)-fuzzy continuous extension over \((X, m^*)\), \(r \in I_0\).

**Proof.** Let \(g, h: X \to I^X\) be such that \(g = f = h\) on \(A\) and \(g(x) = 0, h(x) = 1\) if \(x \not\in A\). We now have

\[
R_t g = \begin{cases} 
\mu_t \land 1_A & \text{if } t \leq 0, \\
1 & \text{if } t < 0
\end{cases}
\]

where \(\mu_t\) is \(m^* r\)-fuzzy \(\tilde{g}\)-open \(F_\sigma\) and is such that \(\mu_t/A = R_t f\) and

\[
L_t h = \begin{cases} 
\lambda_t \land 1_A & \text{if } t \leq 1, \\
1 & \text{if } t > 1
\end{cases}
\]

where \(\lambda_t\) is \(m^* r\)-fuzzy \(\tilde{g}\)-COGF and is such that \(\lambda_t/A = L_t f\). Thus, \(g\) is lower \(m^*\)-fuzzy continuous and \(h\) is upper \(m^*\)-fuzzy continuous with \(g \leq h\). By Proposition 3.6, there is a strong \(m^*\)-fuzzy continuous function \(F: X \to I^X\) such that \(g \leq F \leq h\). Hence \(F \equiv f\) on \(A\). \(\square\)

Acknowledgement. The authors would like to convey their sincere thanks to the referees for their contributions towards the improvement of this paper.

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