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GROUPOIDS ASSIGNED TO RELATIONAL SYSTEMS

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Abstract. By a relational system we mean a couple \((A, R)\) where \(A\) is a set and \(R\) is a binary relation on \(A\), i.e. \(R \subseteq A \times A\). To every directed relational system \(A = (A, R)\) we assign a groupoid \(G(A) = (A, \cdot)\) on the same base set where \(xy = y\) if and only if \((x, y) \in R\). We characterize basic properties of \(R\) by means of identities satisfied by \(G(A)\) and show how homomorphisms between those groupoids are related to certain homomorphisms of relational systems.

Keywords: relational system, groupoid, directed system, \(g\)-homomorphism

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The theory of binary relations was settled by J. Riguet [8]. An algebraic approach to relational systems was developed by A. I. Mal’cev [6]. Some particular cases of relational systems and certain homomorphisms of them were treated by the first author and his co-authors in [1]–[5]. The method assigning a groupoid to a given relational system was initiated in [5] where certain concepts of this paper were introduced. We are motivated by the fact that to certain relational systems, in particular to directed posets, a certain directoid (see e.g. [3]) or semilattice can be assigned in such a way that \(a \leq b\) if and only if \(a \lor b = b\). Moreover, every homomorphism of a semilattice induces a certain homomorphism of the poset \((A, \leq)\) as was investigated in [3] or, in the case of \((A, E)\) where \(E\) is an equivalence relation on \(A\), in [2]. For quasiordered sets a similar question was solved in [4].

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We generalize these approaches in a way similar to that of [5] and produce several results which enable us to consider relational systems from a purely algebraic point of view.

Some of our results were already published in [5] but, for the reader’s convenience, these results are repeated.

First, we recall the basic concepts.

**Definition 1.** A relational system is an ordered pair \((A, R)\) consisting of a non-empty set \(A\) and a binary relation \(R\) on \(A\). For \(a, b \in A\) the (upper) cone \(U_R(a, b)\) of \(a\) and \(b\) is defined by

\[
U_R(a, b) := \{x \in A: (a, x), (b, x) \in R\}.
\]

A relational system \((A, R)\) is called directed if \(U_R(a, b) \neq \emptyset\) for all \(a, b \in A\).

**Remark 2.** (i) All considerations which will follow can be dualized for the lower cone \(L_R(a, b) := \{x \in A: (x, a), (x, b) \in R\}\) of \(a\) and \(b\).

(ii) Every relational system \(\mathcal{A} = (A, R)\) can be extended to a directed one by adjoining a new element \(1 \notin A\) and putting \(A_d := A \cup \{1\}\), \(R_d := R \cup (A_d \times \{1\})\). Then \(\mathcal{A}_d := (A_d, R_d)\) is directed and \(\mathcal{A}\) is the restriction of \(\mathcal{A}_d\) to \(A\). Hence we will formulate our results mainly for directed relational systems and this is not an essential constraint.

**Definition 3.** A groupoid \((A, \cdot)\) corresponds to a directed relational system \((A, R)\) if \(ab = b\) provided \((a, b) \in R\) and \(ab \in U_R(a, b)\) otherwise.

**Remark 4.** Although a groupoid \((A, \cdot)\) corresponding to a relational system \(\mathcal{A} = (A, R)\) need not be uniquely determined since for \(a, b \in A\) with \((a, b) \notin R\) and \(|U_R(a, b)| > 1\) there are several possibilities to define \(ab\), we have \(ab = b\) if and only if \((a, b) \in R\) in every groupoid corresponding to \(\mathcal{A}\). Hence every groupoid corresponding to \(\mathcal{A}\) contains complete information concerning the relation \(R\).

**Remark 5.** If \((A, \cdot)\) corresponds to \((A, R)\) and \(a, b \in A\) then \((a, ab) \in R\). This is clear in the case \((a, b) \notin R\), and in the case \((a, b) \in R\) it follows from \((a, ab) = (a, b) \in R\).

First we are interested in the question which groupoids may correspond to a relational system.
Theorem 6. For a groupoid $\mathcal{G} = (G, \cdot)$ the following assertions are equivalent:

(i) There exists a directed relational system $\mathcal{A} = (G, R)$ with a reflexive relation $R$ such that $\mathcal{G}$ corresponds to $\mathcal{A}$.

(ii) $\mathcal{G}$ satisfies the identities $xx = x$ and $x(xy) = y(xy) = xy$.

Proof. Let $a, b \in G$.

(i) $\Rightarrow$ (ii): Since $(a, a) \in R$ we have $aa = a$. If $(a, b) \in R$ then $ab = b$ and hence $a(ab) = ab$ and $ab = ab = ab$. If $(a, b) \notin R$ then $(a, ab), (b, ab) \in R$ and hence $a(ab) = b(ab) = ab$.

(ii) $\Rightarrow$ (i): Put $R := \{(x, y) \in G^2: xy = y\} \cup \{(x, x): x \in G\}$. Then $R$ is reflexive. Since $xy \in U_R(x, y)$ for all $x, y \in G$, $(A, R)$ is directed. Moreover, if $(a, b) \in R$ then $ab = b$ or $a = b$. In the latter case we have $ab = bb = b$. If $(a, b) \notin R$ then $ab \in U_R(a, b)$. □

Remark 7. Every relational system $\mathcal{A} = (A, R)$ can be considered as a graph with vertex-set $A$ and edge-set $R$. It is well-known (see e.g. [7]) that to every graph $\mathcal{A} = (A, R)$ a graph algebra $\mathcal{H}(\mathcal{A}) = (A^+, \circ)$ can be assigned as follows: $A^+ := A \cup \{\infty\}$, $x \circ y := x$ if $(x, y) \in R$ and $x \circ y := \infty$ if $(x, y) \notin R$ $(x, y) \in A^+$. However, there is an essential difference in applications. Contrary to a groupoid corresponding to $\mathcal{A}$, the graph algebra $\mathcal{H}(\mathcal{A})$ is determined uniquely. But if e.g. $\mathcal{A} = (A, \leq)$ is a join-semilattice we may put $ab := a \lor b$ for $a, b \in A$ since $a \lor b \in U_{\leq}(a, b)$, and the corresponding groupoid is just a join-semilattice $(A, \lor)$ which has a nice structure used in numerous applications both in algebra and beyond. On the other hand, a graph algebra can be far from a join-semilattice and need not have so nice properties. This means that our relative “vagueness” in the definition of the operation “.” may be an essential advance in applications.

In what follows we are going to show how the properties of $\mathcal{A} = (A, R)$ can be captured by $\mathcal{G}(\mathcal{A}) = (A, \cdot)$.

Theorem 8. If $\mathcal{A} = (A, R)$ is a directed relational system and $\mathcal{G}(\mathcal{A}) = (A, \cdot)$ a groupoid corresponding to $\mathcal{A}$ then the following assertions hold:

(i) $R$ is reflexive if and only if $\mathcal{G}(\mathcal{A})$ is idempotent.

(ii) $R$ is symmetric if and only if $\mathcal{G}(\mathcal{A})$ satisfies the identity $(xy)x = x$.

(iii) $R$ is transitive if and only if $\mathcal{G}(\mathcal{A})$ satisfies the identity $x((xy)z) = (xy)z$.

(iv) If $\mathcal{G}(\mathcal{A})$ is commutative then $R$ is antisymmetric.

(v) If $\mathcal{G}(\mathcal{A})$ satisfies the identity $(xy)x = xy$ then $R$ is antisymmetric.

(vi) If $\mathcal{G}(\mathcal{A})$ is a semigroup then $R$ is transitive.

Proof. Let $a, b, c \in A$.

(i) is evident.
corresponding groupoid, we have 

\[(a, ab) \in R \text{ whence } (ab, a) \in R, \text{ i.e. } (ab)a = a.\]

\[\Leftarrow:\] If \((a, b) \in R\) then \(ab = b\) and hence \(ba = (ab)a = a\), i.e. \((b, a) \in R\).

(iii) \(\Rightarrow\): According to Remark 5, \((a, ab), (ab, (ab)c) \in R \text{ and hence } (a, (ab)c) \in R, \text{ i.e. } a((ab)c) = (ab)c.\)

\[\Leftarrow:\] If \((a, b), (b, c) \in R\) then \(ab = b\) and \(bc = c\) and hence

\[ac = a(bc) = a((ab)c) = (ab)c = bc = c,\]

i.e. \((a, c) \in R\).

(iv) If \((a, b), (b, a) \in R\) then \(ab = b\) and \(ba = a\) and hence \(a = ba = ab = b\).

(v) If \((a, b), (b, a) \in R\) then \(ab = b\) and \(ba = a\) and hence \(a = ba = (ab)a = ab = b\).

(vi) If \((a, b), (b, c) \in R\) then \(ab = b\) and \(bc = c\) and hence \(ac = a(bc) = (ab)c = bc = c\), i.e. \((a, c) \in R.\)

We can ask also conversely which relational systems can be induced by a given groupoid \(G = (G, \cdot, \circ)\).

**Definition 9.** Let \(G = (G, \cdot, \circ)\) be a groupoid. Define two corresponding relational systems \(A(G) := (G, R(G))\) and \(A^*(G) := (G, R^*(G))\) as follows:

\[
R(G) := \{(x, y) \in G^2 : xy = y\},
R^*(G) := \bigcup_{x, y \in G} \{(x, xy), (y, xy)\}
\]

Obviously, \(R(G) \subseteq R^*(G)\).

**Lemma 10.** If \(G = (G, \cdot, \circ)\) is a groupoid then the following assertions hold:

(i) \(A^*(G)\) is directed.

(ii) If \(G\) satisfies the identities \(x(yz) = y(xz) = xy\) then \(A(G) = A^*(G)\).

**Proof.** (i) We have \(xy \in U_{R^*(G)}(x, y)\) for all \(x, y \in G\).

(ii) We have \(R(G) = R^*(G)\). \qed

**Lemma 11.** If \(A = (A, R)\) is a directed relational system and \(G(A) = (A, \cdot, \circ)\) a corresponding groupoid then \(A(G(A)) = A\).

**Proof.** Let \(a, b \in A\). If \((a, b) \in R(G(A))\) then \(ab = b\) and since \(G(A)\) is a corresponding groupoid, we have \((a, b) \in R\). Conversely, if \((a, b) \in R\) then \(ab = b\) in \(G(A)\) and hence \((a, b) \in R(G(A))\). \qed

In what follows, we will study connections between homomorphisms of relational systems and homomorphisms of the corresponding groupoids.
**Definition 12.** Let \( A = (A, R) \) and \( B = (B, S) \) be relational systems, \( h: A \to B \) and \( \Theta \) an equivalence relation on \( A \). The mapping \( h \) is called a homomorphism from \( A \) to \( B \) if \( (a, b) \in R \) implies \( (h(a), h(b)) \in S \). If, moreover, for all \( (c, d) \in S \) there exists \( (a, b) \in R \) with \( h(a) = c \) and \( h(b) = d \) then \( h \) is called strong. Moreover, the quotient relational system \( A/\Theta := (A/\Theta, R/\Theta) \) is defined by

\[
R/\Theta := \{ ([a]_\Theta, [b]_\Theta) : (a, b) \in R \}.
\]

It is almost evident that if \( R \) is reflexive or symmetric then also \( R/\Theta \) has this property. This need not be true for transitivity (see e.g. [1]).

We can state

**Lemma 13.** If \( (A, R) \) is a relational system with transitive \( R \) and with \( \Theta \) an equivalence relation on \( A \) then \( R/\Theta \) is transitive if and only if \( R \circ \Theta \circ R \subseteq \Theta \circ R \circ \Theta \).

**Proof.** Let \( a, b \in A \). Then \( ([a]_\Theta, [b]_\Theta) \in R/\Theta \) if and only if \( (a, b) \in \Theta \circ R \circ \Theta \).

Hence \( R/\Theta \) is transitive if and only if \( \Theta \circ R \circ \Theta \circ R \circ \Theta \subseteq \Theta \circ R \circ \Theta \). But this is equivalent to \( R \circ \Theta \circ R \subseteq \Theta \circ R \circ \Theta \). \( \Box \)

**Lemma 14.** If \( (A, R) \) is a relational system and \( \Theta \) an equivalence relation on \( A \) then the canonical mapping \( h \) from \( A \) to \( A/\Theta \) is a strong homomorphism from \( A \) to \( A/\Theta \) and \( R/\Theta \) is the least binary relation \( T \) on \( A/\Theta \) such that \( h \) is a homomorphism from \( A \) to \( (A/\Theta, T) \).

**Proof.** This is evident. \( \Box \)

We can define one more modification of the notion of a homomorphism between relational systems by means of corresponding groupoids.

**Definition 15.** A \( g \)-homomorphism from a relational system \( A = (A, R) \) to a relational system \( B = (B, S) \) is a homomorphism \( h \) from \( A \) to \( B \) such that there exists a groupoid \( (A, \cdot) \) corresponding to \( A \) such that for all \( a, b, c, d \in A \) the equalities \( h(a) = h(c) \) and \( h(b) = h(d) \) together imply \( h(ab) = h(cd) \).

**Theorem 16.** If \( A = (A, R) \) and \( B = (B, S) \) are directed relational systems and \( G(A) = (A, \cdot) \) and \( G(B) = (B, \circ) \) are corresponding groupoids then every homomorphism \( h \) from \( G(A) \) to \( G(B) \) is a \( g \)-homomorphism from \( A \) to \( B \).

**Proof.** Let \( a, b, c, d \in A \). If \( (a, b) \in R \) then \( ab = b \) and hence \( h(a) \circ h(b) = h(ab) = h(b) \) showing \( (h(a), h(b)) \in S \). Moreover, if \( a, b, c, d \in A \), \( h(a) = h(c) \) and \( h(b) = h(d) \) then \( h(ab) = h(a) \circ h(b) = h(c) \circ h(d) = h(cd) \). \( \Box \)

The next theorem states that in some sense homomorphisms between relational systems are homomorphisms between corresponding groupoids.
Theorem 17. If $\mathcal{A} = (A, R)$ and $\mathcal{B} = (B, S)$ are directed relational systems and $h$ is a strong $g$-homomorphism from $\mathcal{A}$ onto $\mathcal{B}$ with the groupoid $\mathcal{G}(\mathcal{A}) = (A, \cdot)$ corresponding to $\mathcal{A}$ then there exists a groupoid $\mathcal{G}(\mathcal{B}) = (B, \circ)$ corresponding to $\mathcal{B}$ such that $h$ is a homomorphism from $\mathcal{G}(\mathcal{A})$ to $\mathcal{G}(\mathcal{B})$.

Proof. According to Definition 15 there exists a groupoid $\mathcal{G}(\mathcal{A}) = (A, \cdot)$ corresponding to $\mathcal{A}$ such that for each $a, b, c, d \in A$, if $h(a) = h(c)$ and $h(b) = h(d)$ then $h(ab) = h(cd)$. Define $h(x) \circ h(y) := h(xy)$ for all $x, y \in A$. According to Definition 15, $\circ$ is well-defined. Let $a, b \in A$. If $(h(a), h(b)) \in S$ then, since $h$ is strong, there exists $(c, d) \in R$ with $h(c) = h(a)$ and $h(d) = h(b)$. Now $h(a) \circ h(b) = h(ab) = h(cd) = h(d) = h(b)$ according to Definition 15. If $(h(a), h(b)) \notin S$ then $(a, b) \notin R$ according to Definition 15 and hence $ab \in U_R(a, b)$, i.e. $(a, ab), (b, ab) \in R$ whence $(h(a), h(a) \circ h(b)) = (h(a), h(ab)) \in S$ and $(h(b), h(a) \circ h(b)) = (h(b), h(ab)) \in S$, i.e. $h(a) \circ h(b) \in U_S(h(a), h(b))$. This shows that $\mathcal{G}(\mathcal{A})$ corresponds to $\mathcal{B}$. Finally, $h$ is a homomorphism from $\mathcal{G}(\mathcal{A})$ to $\mathcal{G}(\mathcal{B})$ since $h(xy) = h(x) \circ h(y)$ for all $x, y \in A$. \qed

Our next theorem contains some assertions concerning factor groupoids.

Theorem 18. If $\mathcal{A} = (A, R)$ and $\mathcal{B} = (B, S)$ are directed relational groupoids then the following implications hold:

(i) If $h$ is a $g$-homomorphism from $\mathcal{A}$ to $\mathcal{B}$ then there exists a groupoid $\mathcal{G}(\mathcal{A}) = (A, \cdot)$ corresponding to $\mathcal{A}$ such that $\ker h \in \text{Con} \mathcal{G}(\mathcal{A})$.

(ii) If $\mathcal{G}(\mathcal{A}) = (A, \cdot)$ is a groupoid corresponding to $\mathcal{A}$ and $\Theta \in \text{Con} \mathcal{G}(\mathcal{A})$ then the canonical mapping $h$ from $A$ to $A/\Theta$ is a strong $g$-homomorphism from $\mathcal{A}$ to $\mathcal{A}/\Theta$.

Proof. Let $a, b, c, d \in A$.

(i) Obviously, $\ker h$ is an equivalence relation on $A$. According to Definition 15 there exists a groupoid $\mathcal{G}(\mathcal{A}) = (A, \cdot)$ corresponding to $\mathcal{A}$ such that for each $a, b, c, d \in A$, if $h(a) = h(c)$ and $h(b) = h(d)$ then $h(ab) = h(cd)$, i.e. $(a, c), (b, d) \in \ker h$ implies $(ab, cd) \in \ker h$.

(ii) If $(a, b) \in R$ then $(h(a), h(b)) = ([a]_\Theta, [b]_\Theta) \in R/\Theta$. Moreover, if $h(a) = h(c)$ and $h(b) = h(d)$ then $[a]_\Theta = h(a) = h(c) = [c]_\Theta$ and $[b]_\Theta = h(b) = h(d) = [d]_\Theta$ and hence $h(ab) = [ab]_\Theta = [a]_\Theta \cdot [b]_\Theta = [c]_\Theta \cdot [d]_\Theta = [cd]_\Theta = h(cd)$. If, finally, $([c]_\Theta, [d]_\Theta) \in R/\Theta$ then there exists $(a, b) \in R$ with $([a]_\Theta, [b]_\Theta) = ([c]_\Theta, [d]_\Theta)$, i.e. with $h(a) = [a]_\Theta = [c]_\Theta$ and $h(b) = [b]_\Theta = [d]_\Theta$. \qed

Theorem 19. Every homomorphism $h$ from a groupoid $\mathcal{G} = (G, \cdot)$ to a groupoid $\mathcal{H} = (H, \circ)$ is a homomorphism from $\mathcal{A}(\mathcal{G})$ to $\mathcal{A}(\mathcal{H})$ and from $\mathcal{A}^*(\mathcal{G})$ to $\mathcal{A}^*(\mathcal{H})$.

Proof. Let $a, b \in G$. If $(a, b) \in R(\mathcal{G})$ then $ab = b$ and hence $h(a) \circ h(b) = h(ab) = h(b)$, i.e. $(h(a), h(b)) \in R(\mathcal{H})$. This shows that $h$ is a homomorphism from
\( \mathcal{A}(\mathcal{G}) \) to \( \mathcal{A}(\mathcal{H}) \). If, on the other hand, \((a, b) \in R^*(\mathcal{G})\) then there exist \(c, d \in G\) with \((a, b) \in \{(c, cd), (d, cd)\}\). Now \(h(c), h(d) \in H\) and

\[
(h(a), h(b)) = (h(c), h(cd)) = (h(c), h(c) \circ h(d)) = (h(d), h(c) \circ h(d))
\]

and hence \((h(a), h(b)) \in R^*(\mathcal{H})\) showing that \(h\) is a homomorphism from \(\mathcal{A}^*(\mathcal{G})\) to \(\mathcal{A}^*(\mathcal{H})\). \(\square\)

Remark 20. A homomorphism \(h\) from \(\mathcal{G} = (G, \cdot)\) to \(\mathcal{H} = (H, \circ)\) need not be a \(g\)-homomorphism from \(\mathcal{A}(\mathcal{G})\) to \(\mathcal{A}(\mathcal{H})\) as can be seen from the following example:

Example 21. Put \(\mathcal{G} := \{-1, 0, 1\}, \cdot\) and \(\mathcal{H} := \{0, 1\}, \cdot\) where \(\cdot\) denotes the multiplication of integers, and let \(h\) denote the mapping \(x \mapsto |x|\) from \([-1, 0, 1]\) to \(\{0, 1\}\). Then \(h\) is a homomorphism from \(\mathcal{G}\) to \(\mathcal{H}\) and \(R(\mathcal{G}) = \{(-1, 0), (0, 0), (1, -1), (1, 0), (1, 1)\}\). Let \(\{-1, 0, 1\}, *\) be a groupoid corresponding to \(\mathcal{A}(\mathcal{G})\). Then for \(x, y \in \{-1, 0, 1\}\) we have \(1 * x = x\) and \(x * y = 0\) otherwise. Now \(h(-1) = h(1)\) but \(h((-1) * (-1)) = h(0) = 0 \neq 1 = h(1) = h(1 * 1)\) and hence \(h\) is not a \(g\)-homomorphism from \(\mathcal{A}(\mathcal{G})\) to \(\mathcal{A}(\mathcal{H})\).

The next theorem gives the final answer to the question whether a homomorphism between groupoids is a \(g\)-homomorphism between corresponding relational systems. The groupoids have to satisfy the identity natural for corresponding groupoids of relational systems.

Theorem 22. If \(\mathcal{G} = (G, \cdot)\) is a groupoid satisfying the identities \(xx = x\) and \(x(xy) = y(xy) = xy\) then every homomorphism \(h\) from \(\mathcal{G}\) to a groupoid \(\mathcal{H} = (H, \circ)\) is a \(g\)-homomorphism from \(\mathcal{A}(\mathcal{G})\) to \(\mathcal{A}(\mathcal{H})\).

Proof. Obviously, \(\mathcal{G}\) corresponds to \(\mathcal{A}(\mathcal{G})\) and \(h\) is a homomorphism from \(\mathcal{A}(\mathcal{G})\) to \(\mathcal{A}(\mathcal{H})\). If \(a, b, c, d \in G\), \(h(a) = h(c)\) and \(h(b) = h(d)\) then \(h(ab) = h(a) \circ h(b) = h(c) \circ h(d) = h(cd)\) and hence \(h\) is a \(g\)-homomorphism from \(\mathcal{A}(\mathcal{G})\) to \(\mathcal{A}(\mathcal{H})\). \(\square\)

In the remaining part of the paper we point out a relationship between relation preserving functions and corresponding groupoids. This has an application in the theory of clones since both the set of functions preserving a given relation and the set of functions commuting with a given operation are clones.

Definition 23. An \(m\)-ary operation \(f\) on \(A\) is said to preserve a binary relation \(R\) on \(A\) if \((a_1, b_1), \ldots, (a_m, b_m) \in R\) implies

\[
(f(a_1, \ldots, a_m), f(b_1, \ldots, b_m)) \in R.
\]
An $m$-ary operation $f$ and an $n$-ary operation $g$ on $A$ are said to commute with each other if

$$f(g(x_{11}, \ldots, x_{1n}), \ldots, g(x_{m1}, \ldots, x_{mn})) = g(f(x_{11}, \ldots, x_{m1}), \ldots, f(x_{1n}, \ldots, x_{mn}))$$

for all $x_{11}, \ldots, x_{1n}, \ldots, x_{m1}, \ldots, x_{mn} \in A$.

Remark 24. If $m = 2$ and $n = 1$ then $f$ and $g$ commute with each other if and only if $g$ is an endomorphism of the groupoid $(A, f)$.

Lemma 25. If $(A, R)$ is a directed relational system and $(A, \cdot)$ a corresponding groupoid then every $m$-ary operation $f$ on $A$ commuting with $\cdot$ preserves $R$.

Proof. If $(a_1, b_1), \ldots, (a_m, b_m) \in R$ then $a_i b_i = b_i$ for $i = 1, \ldots, m$ and hence

$$f(a_1, \ldots, a_m) f(b_1, \ldots, b_m) = f(a_1 b_1, \ldots, a_m b_m) = f(b_1, \ldots, b_m)$$

whence $(f(a_1, \ldots, a_m), f(b_1, \ldots, b_m)) \in R$. □

Example 26. If $A = (A, \leq)$ is a poset which is a join-semilattice $(A, \lor)$ then $(A, \lor)$ is a groupoid corresponding to $(A, \leq)$ and Lemma 25 witnesses the fact that every join-preserving operation (i.e. every operation commuting with $\lor$) is order-preserving. However, the assumption that $f$ commutes with the operation of a corresponding groupoid is only sufficient but not necessary. It is e.g. elementary to show that an order preserving function on a join-semilattice need not commute with $\lor$. Hence we ask for a necessary and sufficient condition formulated in terms of a corresponding groupoid which ensures that a given operation preserves $R$. The answer is as follows:

Theorem 27. If $(A, R)$ is a directed relational system, $(A, \cdot)$ a corresponding groupoid and $f$ an $m$-ary operation on $A$ then the following conditions are equivalent:

(i) $f$ preserves $R$.

(ii) $f$ satisfies the identity

$$f(x_1, \ldots, x_m) f(x_1 y_1, \ldots, x_m y_m) = f(x_1 y_1, \ldots, x_m y_m).$$

Proof. Let $a_1, \ldots, a_m, b_1, \ldots, b_m \in A$.

(i) $\Rightarrow$ (ii): Since $(a_1, a_1 b_1), \ldots, (a_m, a_m b_m) \in R$ according to Remark 5, we have

$$(f(a_1, \ldots, a_m), f(a_1 b_1, \ldots, a_m b_m)) \in R$$

whence $f(a_1, \ldots, a_m) f(a_1 b_1, \ldots, a_m b_m) = f(a_1 b_1, \ldots, a_m b_m)$. 22
(ii) ⇒ (i): If \((a_1, b_1), \ldots, (a_m, b_m) \in R\) then \(a_1 b_1 = b_1, \ldots, a_m b_m = b_m\) and hence
\[
f(a_1, \ldots, a_m)f(b_1, \ldots, b_m) = f(a_1, \ldots, a_m)f(a_1b_1, \ldots, a_mb_m) \\
= f(a_1b_1, \ldots, a_mb_m) = f(b_1, \ldots, b_m)
\]
whence \((f(a_1, \ldots, a_m), f(b_1, \ldots, b_m)) \in R\).

\[\square\]

References


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