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N. Rajesh; V. Vijayabharathi

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ON STRONGLY PREIRRESOLUTE TOPOLOGICAL VECTOR SPACES

N. Rajesh, Thanjavur, V. Vijayabharathi, Tiruchirappalli

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Abstract. In the paper we obtain several characteristics of pre-T_2 of strongly preirresolute topological vector spaces and show that the extreme point of a convex subset of a strongly preirresolute topological vector space X lies on the boundary.

Keywords: topological vector space, strongly preirresolute vector space

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1. Introduction

Generalized open sets play a very important role in General Topology and they are now the research topics of many topologists worldwide. Indeed, a significant theme in General Topology and Real Analysis concerns the various modified forms of continuity, separation axioms etc. by utilizing generalized open sets. One of the best known notions and also an inspiration source is the notion of preopen sets [3] introduced by Mashhour et al. in 1982. A subset A of a space X is called a preopen set [3] if it is a set which is contained in the interior of its closure. So, many mathematicians turned their attention to the generalizations of various concepts of topology by considering preopen sets instead of open sets. Throughout this paper, (X,τ) and (Y,σ) stand for topological spaces with no separation axioms assumed, unless otherwise stated. If $A \subset X$, $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$ will denote the closure and interior of A in (X,τ), respectively. The complement of a preopen set is called a preclosed set. The intersection of all preclosed sets containing S is called the preclosure of S and is denoted by $p\,\mathrm{Cl}(S)$. The preinterior of S is defined as the union of all preopen sets contained in S and is denoted by $p\,\mathrm{Int}(S)$. The family of all preopen subsets of X is denoted by $\mathrm{PO}(X)$. For each $x \in X$, the family of all preopen sets containing x is denoted by $\mathrm{PO}(X,x)$. In this paper, we obtain several characteristics of pre-T_2 of strongly preirresolute topological vector spaces and show that the extreme point of convex subset of strongly preirresolute topological vector space X lies on the boundary.

2. Preliminaries

Definition 2.1. A function $f\colon X \to Y$ is said to be
(1) preirresolute [6] if $f^{-1}(B) \in \mathrm{PO}(X)$ for every $B \in \mathrm{PO}(Y)$.
(2) p-continuous if $f^{-1}(B)$ is open in X for every $B \in \mathrm{PO}(Y)$.
(3) M-preopen [5] if $f(B) \in \mathrm{PO}(Y)$ for every $B \in \mathrm{PO}(X)$.

Definition 2.2. A topological space X is called pre-T_2 [2] if for each two distinct points x and y in X, there exist disjoint sets $U, V \in \mathrm{PO}(X)$ such that $x \in U$ and $y \in V$.

Definition 2.3. A topological space X is said to be strongly compact [4] if every cover of X by preopen sets has a finite subcover.

Lemma 2.4 [3]. *Let $\{U_\alpha\colon \alpha \in \Lambda\}$ be a collection of preopen sets in a topological space X. Then $\bigcup\limits_{\alpha\in\Lambda} U_\alpha$ is preopen.*

Lemma 2.5 [3]. *If U is preopen and $U \cap A = \emptyset$, then $U \cap p\,\mathrm{Cl}(A) = \emptyset$.*

3. Strongly preirresolute topological vector spaces

Definition 3.1. Let τ be a topology on a real vector space X such that
(1) the addition map $S\colon X \times X \to X$,
(2) the scalar multiplication $M\colon \mathbb{R} \times X \to X$
are both p-continuous. Then the pair $(X, \mathrm{PO}(X))$ is called a strongly preirresolute topological vector space (SPITVS).

Definition 3.2. A subset A of a SPITVS $(X, \mathrm{PO}(X))$ is called a preneighbourhood (a neighbourhood) of $x \in X$, if there exists $U \in \mathrm{PO}(X)$ $(U \in \tau)$ such that $x \in U \subset A$. The set of all preneighbourhoods of $x \in X$ is denoted by $N_x(X)$ or simply N_x. In particular, the set of all preneighbourhoods of the zero vector of X is denoted by $N_0(X)$ or N_0.

The proof of the following theorem is obvious and hence omitted.

Theorem 3.3. *Let $(X, \mathrm{PO}(X))$ be a* SPITVS. *For $x \in X$, the following assertions hold:*

(1) *If $U \in N_x$, then $x \in U$.*
(2) *If $U \in N_x$ and V is a neighbourhood of x, then $U \cap V \in N_x$.*
(3) *If $U \in N_x$, then there exists $V \in N_x$ such that $U \in N_y$ for all $y \in V$.*
(4) *If $U \in N_x$ and $U \subset V$, then $V \in N_x$.*
(5) *If $U \in N_0$, then $\alpha U \in N_0$ for all $\alpha \in \mathbb{R}$, $\alpha \neq 0$.*
(6) *$U \in N_0$ if and only if $x + U \in N_x$.*

Recall that a subset A of a vector space X is called balanced if $\alpha A \subset A$ for $|\alpha| \leqslant 1$ and absorbing if for every $x \in X$ there exists $\varepsilon > 0$ such that $\alpha x \in A$ for $|\alpha| \leqslant \varepsilon$. It is called absolutely convex if it is both convex and balanced.

Theorem 3.4. *Let $(X, \mathrm{PO}(X))$ be a* SPITVS. *Then*

(1) *every $U \in N_0$ is absorbing;*
(2) *for every $U \in N_0$ there exists a balanced $V \in N_0$ such that $V \subset U$.*

P r o o f. The proof is clear. □

Theorem 3.5. *Let $(X, \mathrm{PO}(X))$ be a* SPITVS. *If $A \subset X$, then $p\,\mathrm{Cl}(A) = \bigcap\limits_{U \in N_0} (A + U)$. In particular, $p\,\mathrm{Cl}(A) \subset A + U$ for all $U \in N_0$.*

Theorem 3.6. *Let $(X, \mathrm{PO}(X))$ be a* SPITVS. *Then*

(1) *for every $U \in N_0$ there exists $V \in N_0$ such that $V + V \subset U$;*
(2) *for every $U \in N_0$ there exists a preclosed balanced $V \in N_0$ such that $V \subset U$.*

P r o o f. The proof is clear. □

Definition 3.7. A SPITVS $(X, \mathrm{PO}(X))$ is called locally convex, if for all $x \in X$ every $V \in N_x$ contains a convex $U \in N_x$.

Theorem 3.8. *A* SPITVS *$(X, \mathrm{PO}(X))$ is called locally convex if and only if every $V \in N_0$ contains a convex $U \in N_0$.*

The following result provides a characterization for pre-T_2 of SPITVS.

Theorem 3.9. *Let X be a* SPITVS. *Then the following statements are equivalent:*
(1) *X is pre-T_2.*
(2) *If $x \in X$, $x \neq 0$, then there exists $U \in N_0$ such that $x \in U$.*
(3) *If $x, y \in X$, $x \neq y$, then there exists $V \in N_x$ such that $y \notin V$.*

Proof. By continuity of translation, it is suficient to prove the equivalence between (1) and (2) only. Let $x \in X, x \neq 0$ by assumption; there exist $U, V \in \mathrm{PO}(X)$ such that $0 \in U, x \in V$ and $U \cap V = \emptyset$. Thus, $U \in N_0$, $V \in N_x$ and $x \notin U$. Conversely, let $x, y \in X$ be such that $x - y \neq 0$. Then there exists $U \in N_0$ such that $x - y \notin U$. But by part (1) of Theorem 3.6, there exists $w \in N_0$ such that $w + w \subset U$. By part (2) of Theorem 3.6, w can be assumed to be balanced. Let $V_1 = x + w$ and $V_2 = y + w$ and note that $V_1 \in N_x$, $V_2 \in N_y$ and $V_1 \cap V_2 = \emptyset$, since if $a \in V_1 \cap V_2$ then $-(z - x) \in w$, as w is balanced and $z - y \in w$. It follows that $x - y = (z - y) + (-(z - x)) \in w + w \subset U$, which is a contradiction. So, we must have $V_1 \cap V_2 = \emptyset$. Finally, by definition of the preneighbourhood, there exist $V_1', V_2' \in \mathrm{PO}(X)$ such that $x \in V_1' \subset V_1$, $y \in V_2' \subset V_2$ and $V_1' \cap V_2' = \emptyset$. This shows that X is pre-T_2. □

The following result follows from Theorem 3.9.

Corollary 3.10. *Let X be a* SPITVS. *Then the following statements are equivalent:*
(1) *X is pre-T_2.*
(2) $\bigcap\{U \colon U \in N_0\} = \{0\}$.
(3) $\bigcap\{U \colon U \in N_x\} = \{x\}$.

Theorem 3.11. *A* SPITVS *X is pre-T_2 if and only if every one-point set in X is preclosed.*

Proof. Let $x \in X$ and $y \in X - \{x\}$. Then $y - x \neq 0$, and by assumption, there exists $U \in N_0$ such that $y - x \in U$. By part (2) of Theorem 3.6, there exists a preclosed and balanced $V \in N_0$ such that $V \subset U$. It follows that $y - x \notin V$ that is $y - x \in X - V$. Thus $y \in (X - V) + \{x\}$. But $(x - V) + \{x\} \in \mathrm{PO}(X)$, since V is preclosed, and $(X - V) + \{x\} \subset X - \{x\}$. This shows that $X - \{x\} \in \mathrm{PO}(X)$, hence $\{x\}$ is preclosed. For the converse, let $x \in X$ and assume that $\{x\}$ is preclosed. Then by Theorem 3.5 $\{x\} = p\,\mathrm{Cl}\{x\} = \bigcap\{U + \{x\} \colon U \in N_0\} = \{V \colon V \in N_x\}$, where $V = U + \{x\} \in N_x$. By Corollary 3.10, X is pre-T_2. □

Since translation is a prehomeomorphism and as a consequence of Theorem 3.11, we have the following

Corollary 3.12. *A* SPITVS X *is pre-*T_2 *if and only if* $\{0\}$ *is preclosed.*

Theorem 3.13. *Let* C, K *be disjoint sets in a* SPITVS X *with* C *preclosed,* K *strongly compact. Then there exists* $U \in N_0(X)$ *with* $(K+U) \cap (C+U) = \emptyset$.

P r o o f. If $K = \emptyset$, then there is nothing to prove. Otherwise, let $x \in K$ by the invariance with translation; we can assume $x = 0$. Then $X \setminus C$ is a preopen set of 0. Since addition is preirresolute and p-continuous, by $0 = 0+0+0$, there is a neighbourhood U and $U \in N_0(X)$ such that $3U = U+U+U \subset X - C$. By defining $\tilde{U} = U \cap (-U) \subset U$ we have that $\tilde{U}$ is open, and hence preopen symmetric and $3\tilde{U} = \tilde{U}+\tilde{U}+\tilde{U} \subset X - C$. This means that $\emptyset = \{3x, x \in \tilde{U}\} \cap C = \{2x, x \in U\tilde{U}\} \cap \{y - x, y \in C, x \in \tilde{U}\} \supset \tilde{U} \cap (C + \tilde{U})$. This concludes the proof for a single point. Since K is strongly compact, repeating the above argument for all $x \in K$ we obtain symmetric preopen sets V_x such that $(x + 2V_x) \cap (C + V_x) = \emptyset$. The sets $\{V_x\colon x \in K\}$ are a preopen (and open) covering of K, but K is strongly compact hence there is a finite number of points $x_i \in K$, $i = 1, \ldots, n$ such that $K \subset \bigcup\limits_{i=1}^{n} (x_i + V_{x_i})$. Define the preneighourhood V of 0 by $V = \bigcap\limits_{i=1}^{n} V_{x_i}$. Then $(K+V) \cap (C+V) \subset \bigcup\limits_{i=1}^{n} (x_i + V_{x_i} + V) \cap (C+V) \subset \bigcup\limits_{i=1}^{n} ((x_i + 2V_{x_i}) \cap (C \cap V_{x_i})) = \emptyset$.
□

Corollary 3.14. *Let* C, K *be disjoint sets in a* SPITVS X *with* C *preclosed,* K *strongly compact. Then there exists* $U \in N_0(X)$ *with* $p\,\mathrm{Cl}(K+U) \cap (C+U) = \emptyset$.

P r o o f. Since $C+U$ is a union of preopen sets $y+U$ for $y \in C$ the proof follows directly from Theorems 3.5 and 3.4. □

Corollary 3.15. *A* SPITVS X *is a pre-*T_2 *space.*

P r o o f. Take $K = \{x\}$ and $C = \{y\}$ in Theorem 3.13. □

Let $(X, \|\cdot\|)$ be a normed space over K. We denote by X^* the vector space of all linear maps from X to X. Space X^* is called the algebraic dual of X. Note that for any $f \in X^*$ and $x \in X$ we write $\langle x, f\rangle = f(x)$ [1].

Theorem 3.16. *Let* $(X, \mathrm{PO}(X))$ *be a* SPITVS *and* $0 \neq f \in X^*$. *Then* $f(G)$ *is preopen in* K *whenever* G *is preopen in* X.

P r o o f. Let G be a nonempty preopen set. Then one can assume that there is $0 \neq x_0 \in X$ such that $f(x_0) = 1$. For any $a \in G$, it is required to show that $f(a) \in p\,\mathrm{Int}(f(G))$. Since $G \in N_a(X)$ by Theorem 3.3 we have $G - a \in N_0(X)$. Also by Theorem 3.4 $G - a$ is absorbing, that is, absorbs x_0, namely there exists an

$\varepsilon > 0$ such that $\lambda x_0 \in G - a$ whenever $\lambda \in \Re$ with $|\lambda| \leqslant \varepsilon$. Now for any $\beta \in \Re$ with $|\beta - f(a)| \leqslant \varepsilon$ we have $(\beta - f(A))x_0 \in G - a$, hence $f((\beta - f(a)x_0)) \in f(G - a) \Rightarrow (\beta - f(a))f(x_0) \in f(G - a) \Rightarrow (\beta - f(a))(1) \in f(G - a) = f(G) - f(a)$. This imples that $\beta \in f(G)$ and $f(a) \in [\beta - \varepsilon, \beta + \varepsilon]$. Thus $f(a) \in \operatorname{Int}(f(G)) \subset p\operatorname{Int}(f(G))$; hence $f(G) = p\operatorname{Int}(f(G))$. □

Lemma 3.17 [1]. *Let X be a vector space and $\emptyset \neq K \subset X$. For $a \in K$, the following statements are equivalent:*

(1) *a is an extreme point of K,*

(2) *if $x, y \in k$ are such that $a = \frac{1}{2}(x + y)$, then $a = x = y$,*

(3) *let $x, y \in k$ be such that $x \neq y$, let $\lambda \in (0, 1)$ and $a = \lambda x + (1 - \lambda)y$. Then we have either $\lambda = 0$ or $\lambda = 1$.*

Theorem 3.18. *Let $(X, \operatorname{PO}(X))$ be a* SPITVS *and K a convex subset of X. Then $(p\operatorname{Int}(K)) \cap (\eth K) = \emptyset$.*

P r o o f. If $p\operatorname{Int}(K) = \emptyset$, the result is trivial. Suppose that $p\operatorname{Int}(K) \neq \emptyset$ and $x \in p\operatorname{Int}(K)$. Then there exists $V \in N_0(X)$ (a preneighbourhood) such that $x + V \subset K$. As the map $\Phi\colon \Re \to X$, where $\Phi(\mu) = \mu x$ is continuous at $\mu = 1$, for this preneighbourhood $x + V$ there is an $r > 0$ such that $\mu x \in x + V$ whenever $|\mu - 1| \leqslant r$. In particular, we have $(1 + r)x \in x + V \subset K$ and $(1 - r)x \in x + V \subset K$. Now consider $x = \lambda(1 + r)x + (1 - \lambda)(1 - r)x$ and set $\lambda = 1/2$. Consequently, we have $x = 1/2(1 + r)x + (1 - 1/2)(1 - r)x$, which implies that x is not an extreme point of K. □

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Authors' addresses: *N. Rajesh*, Department of Mathematics, Rajah Serfoji Govt. College, Thanjavur-613005, Tamilnadu, India, e-mail: `nrajesh_topology@yahoo.co.in`; *V. Vijayabharathi*, Department of Mathematics, National Institute of Technology, Tiruchirappalli, Tamilnadu, India, e-mail: `vijayabharathi_v@yahoo.com`.