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DOMINATION WITH RESPECT TO NONDEGENERATE PROPERTIES: VERTEX AND EDGE REMOVAL

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Abstract. In this paper we present results on changing and unchanging of the domination number with respect to the nondegenerate property $\mathcal{P}$, denoted by $\gamma_{\mathcal{P}}(G)$, when a graph G is modified by deleting a vertex or deleting edges. A graph G is $(\gamma_{\mathcal{P}}(G), k)_{\mathcal{P}}$-critical if $\gamma_{\mathcal{P}}(G-S) < \gamma_{\mathcal{P}}(G)$ for any set $S \subsetneq V(G)$ with $|S| = k$. Properties of $(\gamma_{\mathcal{P}}, k)_{\mathcal{P}}$-critical graphs are studied. The plus bondage number with respect to the property $\mathcal{P}$, denoted $b^+_{\mathcal{P}}(G)$, is the cardinality of the smallest set of edges $U \subseteq E(G)$ such that $\gamma_{\mathcal{P}}(G-U) > \gamma_{\mathcal{P}}(G)$. Some known results for ordinary domination and bondage numbers are extended to $\gamma_{\mathcal{P}}(G)$ and $b^+_{\mathcal{P}}(G)$. Conjectures concerning $b^+_{\mathcal{P}}(G)$ are posed.

Keywords: dominating set, domination number, bondage number, additive graph property, hereditary graph property, induced-hereditary graph property

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1. Introduction

All graphs considered in this article are finite, undirected, without loops or multiple edges. For the graph theory terminology not presented here, we follow Haynes et al. [10]. We denote the vertex set and the edge set of a graph G by $V(G)$ and $E(G)$, respectively. The subgraph induced by $S \subseteq V(G)$ is denoted by $\langle S, G\rangle$. For a vertex x of G, $N(x,G)$ denotes the set of all neighbors of x in G, $N[x,G] = N(x,G) \cup \{x\}$ and the degree of x is $\deg(x,G) = |N(x,G)|$. The maximum and minimum degrees of vertices in the graph G are denoted by $\Delta(G)$ and $\delta(G)$, respectively.

Let $\mathcal{G}$ denote the set of all mutually nonisomorphic graphs. A *graph property* is any nonempty subset of $\mathcal{G}$. We say that a *graph G has the property $\mathcal{P}$* whenever there exists a graph $H \in \mathcal{P}$ which is isomorphic to G. For example, we list some graph properties:

▷ $\mathcal{I} = \{H \in \mathcal{G}\colon\ H \text{ is totally disconnected}\}$;

▷ $\mathcal{F} = \{H \in \mathcal{G}\colon\ H$ is a forest$\}$;

▷ $\mathcal{UK} = \{H \in \mathcal{G}$: each component of H is complete$\}$.

A graph property $\mathcal{P}$ is called: (a) *hereditary (induced-hereditary)*, if from the fact that a graph G has property $\mathcal{P}$, it follows that all subgraphs (induced subgraphs) of G also belong to $\mathcal{P}$; (b) *nondegenerate* if $\mathcal{I} \subseteq \mathcal{P}$, and (c) *additive* if it is closed under taking disjoint unions of graphs. Note that: (i) $\mathcal{I}$ and $\mathcal{F}$ are nondegenerate, additive and hereditary properties, and (ii) $\mathcal{UK}$ is nondegenerate, additive, induced-hereditary and is not hereditary.

A *dominating set* for a graph G is a set of vertices $D \subseteq V(G)$ such that every vertex of G is either in D or is adjacent to an element of D. The *domination number* $\gamma(G)$ of a graph G is the minimum cardinality of a dominating set of G. A dominating set D is called an *efficient dominating set* if the distance between any two vertices in D is at least three. Not all graphs have efficient dominating sets; however, if a graph G has an efficient dominating set, then the cardinality of any efficient dominating set equals the domination number of G [2].

Any set $S \subseteq V(G)$ such that the subgraph $\langle S, G\rangle$ possesses the property $\mathcal{P}$ is called a $\mathcal{P}$-set. The *domination number with respect to the property* $\mathcal{P}$, denoted by $\gamma_{\mathcal{P}}(G)$, is the smallest cardinality of a dominating $\mathcal{P}$-set of G. Observe that if $\mathcal{I} \subseteq \mathcal{P}_2 \subseteq \mathcal{P}_1 \subseteq \mathcal{G}$ then [8] $\gamma(G) = \gamma_{\mathcal{G}}(G) \leqslant \gamma_{\mathcal{P}_1}(G) \leqslant \gamma_{\mathcal{P}_2}(G) \leqslant \gamma_{\mathcal{I}}(G) = i(G)$, where $i(G)$ is the independent domination number of G. The concept of domination with respect to any property $\mathcal{P}$ was introduced by Goddard et al. [8]. Michalak [11] has considered this parameter when the property is additive and induced-hereditary.

It is often of interest to know how the value of a graph parameter is affected when a small change is made in a graph. In this connection, in [14], the present author began an investigation on effects on $\gamma_{\mathcal{P}}$ when a graph is modified by deleting a vertex or by adding an edge. We continue this work here and present results on changing $\gamma_{\mathcal{P}}(G)$ when an edge or a vertex is removed from G.

2. Definitions and known results

Let G be a graph and let $\mathcal{P} \subseteq \mathcal{G}$ be nondegenerate. Any minimum dominating $\mathcal{P}$-set of G is called a $\gamma_{\mathcal{P}}(G)$*-set*. Let G be a graph and $v \in V(G)$. A vertex v of the graph G is said to be

(a) [6] $\gamma_{\mathcal{P}}$*-good*, if v belongs to some $\gamma_{\mathcal{P}}(G)$-set;

(b) [6] $\gamma_{\mathcal{P}}$*-bad*, if v belongs to no $\gamma_{\mathcal{P}}(G)$-set;

(c) [18] $\gamma_{\mathcal{P}}$*-fixed* if v belongs to every $\gamma_{\mathcal{P}}$(G)-set;

(d) [18] $\gamma_{\mathcal{P}}$*-free* if v belongs to some $\gamma_{\mathcal{P}}(G)$-set but not to all $\gamma_{\mathcal{P}}(G)$-sets.

We also need the following sets:

$\mathbf{G}_{\mathcal{P}}(G) = \{x \in V(G)\colon\ x \text{ is } \gamma_{\mathcal{P}}\text{-good}\}$;
$\mathbf{B}_{\mathcal{P}}(G) = \{x \in V(G)\colon\ x \text{ is } \gamma_{\mathcal{P}}\text{-bad}\}$;
$\mathbf{Fi}_{\mathcal{P}}(G) = \{x \in V(G)\colon\ x \text{ is } \gamma_{\mathcal{P}}\text{-fixed}\}$;
$\mathbf{Fr}_{\mathcal{P}}(G) = \{x \in V(G)\colon\ x \text{ is } \gamma_{\mathcal{P}}\text{-free}\}$;
$\mathbf{Fr}^{-}_{\mathcal{P}}(G) = \{x \in \mathbf{Fr}_{\mathcal{P}}(G)\colon\ \gamma_{\mathcal{P}}(G-x) = \gamma_{\mathcal{P}}(G) - 1\}$;
$\mathbf{Fr}^{0}_{\mathcal{P}}(G) = \{x \in \mathbf{Fr}_{\mathcal{P}}(G)\colon\ \gamma_{\mathcal{P}}(G-x) = \gamma_{\mathcal{P}}(G)\}$;
$\mathbf{Fi}^{p}_{\mathcal{P}}(G) = \{x \in \mathbf{Fi}_{\mathcal{P}}(G)\colon\ \gamma_{\mathcal{P}}(G-x) = \gamma_{\mathcal{P}}(G) + p\}$, p is integer;
$\mathbf{V}^{0}_{\mathcal{P}}(G) = \{x \in V(G)\colon\ \gamma_{\mathcal{P}}(G-x) = \gamma_{\mathcal{P}}(G)\}$;
$\mathbf{V}^{-}_{\mathcal{P}}(G) = \{x \in V(G)\colon\ \gamma_{\mathcal{P}}(G-x) < \gamma_{\mathcal{P}}(G)\}$;
$\mathbf{V}^{+}_{\mathcal{P}}(G) = \{x \in V(G)\colon\ \gamma_{\mathcal{P}}(G-x) > \gamma_{\mathcal{P}}(G)\}$.

Clearly $\{\mathbf{G}_{\mathcal{P}}(G), \mathbf{B}_{\mathcal{P}}(G)\}$ and $\{\mathbf{V}^{-}_{\mathcal{P}}(G), \mathbf{V}^{0}_{\mathcal{P}}(G), \mathbf{V}^{+}_{\mathcal{P}}(G)\}$ are partitions of $V(G)$, and $\{\mathbf{Fi}_{\mathcal{P}}(G), \mathbf{Fr}_{\mathcal{P}}(G)\}$ is a partition of $\mathbf{G}_{\mathcal{P}}(G)$. Moreover:

Observation 2.1 ([14]). *Let G be a graph of order $n \geqslant 2$ and let $\mathcal{H} \subseteq \mathcal{G}$ be nondegenerate and closed under the union with K_1. Then*

(1) $\{\mathbf{Fr}^{-}_{\mathcal{H}}(G), \mathbf{Fr}^{0}_{\mathcal{H}}(G)\}$ *is a partition of* $\mathbf{Fr}_{\mathcal{H}}(G)$*;*
(2) $\{\mathbf{Fi}^{-1}_{\mathcal{H}}(G), \mathbf{Fi}^{0}_{\mathcal{H}}(G), \ldots, \mathbf{Fi}^{n-2}_{\mathcal{H}}(G)\}$ *is a partition of* $\mathbf{Fi}_{\mathcal{H}}(G)$*;*
(3) $\{\mathbf{Fi}^{-1}_{\mathcal{H}}(G), \mathbf{Fr}^{-}_{\mathcal{H}}(G)\}$ *is a partition of* $\mathbf{V}^{-}_{\mathcal{H}}(G)$*;*
(4) $\{\mathbf{Fi}^{0}_{\mathcal{H}}(G), \mathbf{Fr}^{0}_{\mathcal{H}}(G), \mathbf{B}_{\mathcal{H}}(G)\}$ *is a partition of* $\mathbf{V}^{0}_{\mathcal{H}}(G)$*;*
(5) $\{\mathbf{Fi}^{1}_{\mathcal{H}}(G), \mathbf{Fi}^{2}_{\mathcal{H}}(G), \ldots, \mathbf{Fi}^{n-2}_{\mathcal{H}}(G)\}$ *is a partition of* $\mathbf{V}^{+}_{\mathcal{H}}(G)$*.*

For each nondegenerate property $\mathcal{P} \subseteq \mathcal{G}$ we define the following classes of graphs G:
$(CV^kR_{\mathcal{P}})$ $\gamma_{\mathcal{P}}(G-S) < \gamma_{\mathcal{P}}(G)$ for any set $S \subsetneq V(G)$ with $|S| = k$,
$(C^+ER_{\mathcal{P}})$ $\gamma_{\mathcal{P}}(G-e) > \gamma_{\mathcal{P}}(G)$ for all $e \in E(G)$

For convenience we omit the subscript $\mathcal{G}$. For a survey on results concerning the classes CV^1R and C^+ER see for instance [10, Chapter 5], [19] and the bibliography in [10]. We define a graph G to be $(\gamma_{\mathcal{P}}(G), k)_{\mathcal{P}}$*-critical* if G is in $CV^kR_{\mathcal{P}}$. The $(\gamma(G), k)$-critical graphs provided $k \geqslant 2$ are introduced by Brigham et al [5]. Further results on these graphs can be found in [12], [13].

Lemma 2.2 ([14]). *Let G be a graph of order at least two, $v \in V^{-}_{\mathcal{H}}(G)$ and let $\mathcal{H} \subseteq \mathcal{G}$ be nondegenerate and closed under the union with K_1. Then $N(v, G) \subseteq \mathbf{B}_{\mathcal{H}}(G-v) - \mathbf{Fi}_{\mathcal{H}}(G)$. If M is a $\gamma_{\mathcal{H}}(G-v)$-set then $M \cup \{v\}$ is a $\gamma_{\mathcal{H}}(G)$-set.*

Lemma 2.3 ([14]). *Let x and y be two different and nonadjacent vertices in a graph G. Let $\mathcal{H} \subseteq \mathcal{G}$ be hereditary and closed under the union with K_1. If $\gamma_{\mathcal{H}}(G+xy) < \gamma_{\mathcal{H}}(G)$ then $\gamma_{\mathcal{H}}(G+xy) = \gamma_{\mathcal{H}}(G) - 1$. Moreover, $\gamma_{\mathcal{H}}(G+xy) = \gamma_{\mathcal{H}}(G) - 1$ if and only if at least one of the following conditions holds:*

(i) *$x \in \mathbf{V}^{-}_{\mathcal{H}}(G)$ and y is a $\gamma_{\mathcal{H}}$-good vertex of $G-x$;*
(ii) *x is a $\gamma_{\mathcal{H}}$-good vertex of $G-y$ and $y \in \mathbf{V}^{-}_{\mathcal{H}}(G)$.*

Lemma 2.4 ([14]). *Let $\mathcal{H} \subseteq \mathcal{G}$ be nondegenerate and closed under the union with K_1 and let x be a $\gamma^0_{\mathcal{H}}$-fixed vertex of a graph G. Then $N(x,G) \subseteq \mathbf{B}_{\mathcal{H}}(G-x) \cap (\mathbf{V}^0_{\mathcal{H}}(G) \cup \mathbf{Fi}^1_{\mathcal{H}}(G))$ and for each $y \in N(x,G)$, $\gamma_{\mathcal{H}}(G - \{x,y\}) = \gamma_{\mathcal{H}}(G)$.*

One measure of stability of the domination number with respect to the property $\mathcal{P}$ under edge removal is the bondage number [17]. For every graph G with at least one edge and every nondegenerate property $\mathcal{P}$, the *plus bondage number with respect to the property* $\mathcal{P}$, denoted $b^+_{\mathcal{P}}(G)$, is the cardinality of the smallest set of edges $U \subseteq E(G)$ such that $\gamma_{\mathcal{P}}(G-U) > \gamma_{\mathcal{P}}(G)$. Since $\gamma_{\mathcal{P}}(G - E(G)) = |V(G)| > \gamma_{\mathcal{P}}(G)$ for every graph G with at least one edge and every nondegenerate property $\mathcal{P}$, it follows that $b^+_{\mathcal{P}}(G)$ always exists. Note that $b_{\mathcal{G}}(G) = b^+_{\mathcal{G}}(G) = b(G)$—the ordinary bondage number. The bondage number of graphs belonging to CV^1R is examined for instance in [9], [20], [21], [16]. The next result shows that the class $CV^1R_{\mathcal{P}}$ plays an important role in the study of the plus bondage number with respect to $\mathcal{P}$.

Lemma 2.5 ([17]). *Let G be a graph and let $\mathcal{H} \subseteq \mathcal{G}$ be nondegenerate and induced-hereditary. If $b^+_{\mathcal{H}}(G) > \Delta(G)$ then G is in $CV^1R_{\mathcal{H}}$.*

3. Edge removal

An edge e of a graph G is $\gamma^+_{\mathcal{P}}$-ER-*critical* if $\gamma_{\mathcal{P}}(G-e) > \gamma_{\mathcal{P}}(G)$. We begin with necessary and sufficient conditions for an edge of a graph to be $\gamma^+_{\mathcal{P}}$-ER-critical.

Theorem 3.1 ([15] when $\mathcal{H} = \mathcal{G}$). *Let x_1 and x_2 be adjacent vertices in a graph G and let $G_{12} = G - x_1x_2$. Let $\mathcal{H} \subseteq \mathcal{G}$ be hereditary and closed under the union with K_1. Then x_1x_2 is $\gamma^+_{\mathcal{P}}$-ER-critical if and only if one of the following conditions holds:*

- (R1) *$x_i \in \mathbf{B}_{\mathcal{H}}(G)$, $x_j \in \mathbf{Fi}^q_{\mathcal{H}}(G)$, $x_i \in V^-_{\mathcal{H}}(G_{12})$ and $x_j \in \mathbf{Fi}^{q-1}_{\mathcal{H}}(G_{12})$ where $\{i,j\} = \{1,2\}$ and $q \geqslant 1$;*
- (R2) *$x_i \in \mathbf{B}_{\mathcal{H}}(G)$, $x_j \in \mathbf{Fi}^1_{\mathcal{H}}(G)$, $x_i \in V^-_{\mathcal{H}}(G_{12})$ and $x_j \in \mathbf{Fr}^0_{\mathcal{H}}(G_{12}) \cap \mathbf{G}_{\mathcal{H}}(G - x_i)$ where $\{i,j\} = \{1,2\}$;*
- (R3) *$x_i \in \mathbf{B}_{\mathcal{H}}(G)$, $x_j \in \mathbf{Fi}^0_{\mathcal{H}}(G)$, $x_i \in V^-_{\mathcal{H}}(G_{12}) \cap \mathbf{B}_{\mathcal{H}}(G - x_j)$ and $x_j \in V^-_{\mathcal{H}}(G_{12}) \cap \mathbf{G}_{\mathcal{H}}(G - x_i)$ where $\{i,j\} = \{1,2\}$;*
- (R4) *$x_1, x_2 \in \mathbf{Fr}^0_{\mathcal{H}}(G)$, $x_1 \in V^-_{\mathcal{H}}(G_{12}) \cap \mathbf{G}_{\mathcal{H}}(G - x_2)$ and $x_2 \in V^-_{\mathcal{H}}(G_{12}) \cap \mathbf{G}_{\mathcal{H}}(G - x_1)$.*

Proof. *Sufficiency*: Let (R1) hold and let M be a $\gamma_{\mathcal{H}}(G_{12} - x_i)$-set. By Lemma 2.2 (applied to G_{12}), $M \cup \{x_i\}$ is a $\gamma_{\mathcal{H}}(G_{12})$-set. Since $x_j \in \mathbf{Fi}_{\mathcal{H}}(G_{12})$, $x_j \in \mathbf{G}_{\mathcal{H}}(G - x_i)$. Now, if one of (R1)–(R4) is satisfied then the result immediately follows by Lemma 2.3 (applied to G_{12}).

Necessity: Let $\gamma_{\mathcal{H}}(G) < \gamma_{\mathcal{H}}(G_{12})$. By Lemma 2.3 it follows that $\gamma_{\mathcal{H}}(G) = \gamma_{\mathcal{H}}(G_{12}) - 1$ and without loss of generality we may assume that $x_1 \in \mathbf{V}^-_{\mathcal{H}}(G_{12})$. Note that no $\gamma_{\mathcal{H}}(G)$-set contains both x_1 and x_2. Indeed, if M is a $\gamma_{\mathcal{H}}(G)$-set with $x_1, x_2 \in M$ then since $\mathcal{H}$ is hereditary, M is a dominating $\mathcal{H}$-set of G_{12}—a contradiction.

(a) Let $x_2 \in \mathbf{Fi}^{q-1}_{\mathcal{H}}(G_{12})$, $q \geqslant 1$. We have $\gamma_{\mathcal{H}}(G - x_2) = \gamma_{\mathcal{H}}(G_{12} - x_2) = \gamma_{\mathcal{H}}(G_{12}) + q - 1 = \gamma_{\mathcal{H}}(G) + q$. Then $x_2 \in \mathbf{Fi}^q_{\mathcal{H}}(G)$, which implies $x_1 \in \mathbf{B}_{\mathcal{H}}(G)$.

(b) Let $x_2 \in \mathbf{Fr}^0_{\mathcal{H}}(G_{12}) \cap \mathbf{G}_{\mathcal{H}}(G - x_1)$. In this case $\gamma_{\mathcal{H}}(G - x_2) = \gamma_{\mathcal{H}}(G_{12} - x_2) = \gamma_{\mathcal{H}}(G_{12}) = \gamma_{\mathcal{H}}(G) + 1$. Hence $x_2 \in \mathbf{Fi}^1_{\mathcal{H}}(G)$, which implies $x_1 \in \mathbf{B}_{\mathcal{H}}(G)$.

(c) Let without loss of generality $x_1 \in \mathbf{B}_{\mathcal{H}}(G - x_2)$ and $x_2 \in \mathbf{V}^-_{\mathcal{H}}(G_{12}) \cap \mathbf{G}_{\mathcal{H}}(G - x_1)$. Since $\gamma_{\mathcal{H}}(G - x_2) = \gamma_{\mathcal{H}}(G_{12} - x_2) = \gamma_{\mathcal{H}}(G_{12}) - 1 = \gamma_{\mathcal{H}}(G)$ it follows that $x_2 \in \mathbf{V}^0_{\mathcal{H}}(G)$. Assume there is a $\gamma_{\mathcal{H}}(G)$-set M with $x_2 \notin M$. Then M is a dominating $\mathcal{H}$-set of $G - x_2$ with $|M| = \gamma_{\mathcal{H}}(G) = \gamma_{\mathcal{H}}(G - x_2)$. Hence M is a $\gamma_{\mathcal{H}}(G - x_2)$-set. Since $x_1 \in \mathbf{B}_{\mathcal{H}}(G - x_2)$ we have $x_1, x_2 \notin M$. But then M is a dominating $\mathcal{H}$-set of G_{12} with $|M| < \gamma_{\mathcal{H}}(G_{12})$—a contradiction. Since $x_2 \in \mathbf{V}^0_{\mathcal{H}}(G)$, $x_2 \in \mathbf{Fi}^0_{\mathcal{H}}(G)$. Thus $x_1 \in \mathbf{B}_{\mathcal{H}}(G)$.

(d) Let M_1 be a $\gamma_{\mathcal{H}}(G - x_2)$-set with $x_1 \in M_1$ and M_2 a $\gamma_{\mathcal{H}}(G - x_1)$-set with $x_2 \in M_2$. Then M_1 and M_2 are dominating $\mathcal{H}$-sets of G and $|M_i| = \gamma_{\mathcal{H}}(G - x_i) = \gamma_{\mathcal{H}}(G_{12} - x_i) = \gamma_{\mathcal{H}}(G_{12}) - 1 = \gamma_{\mathcal{H}}(G)$ for $i = 1, 2$. Hence M_1 and M_2 are $\gamma_{\mathcal{H}}(G)$-sets and $x_1, x_2 \in \mathbf{Fi}^0_{\mathcal{H}}(G) \cup \mathbf{Fr}^0_{\mathcal{H}}(G)$. Since $x_1 \notin M_2$ and $x_2 \notin M_1$, it follows that $x_1, x_2 \in \mathbf{Fr}^0_{\mathcal{H}}(G)$.

There are no other possibilities because of Lemma 2.3. □

Recall that a *vertex cover* of a graph G is a set of vertices such that each edge of G is incident to at least one vertex of the set.

Corollary 3.2. *Let $\mathcal{H} \subseteq \mathcal{G}$ be hereditary and closed under the union with K_1. Let a graph G have at least one edge.*

(i) *If $v \in \mathbf{V}^-_{\mathcal{H}}(G)$ then for every edge $e \in E(G)$ incident to v, $\gamma_{\mathcal{H}}(G - e) \leqslant \gamma_{\mathcal{H}}(G)$.*
(ii) *If $\mathbf{V}^-_{\mathcal{H}}(G)$ is a vertex cover then for every edge $e \in E(G)$, $\gamma_{\mathcal{H}}(G - e) \leqslant \gamma_{\mathcal{H}}(G)$.*

Now, we give a characterization of the class $C^+ER_{\mathcal{P}}$.

Theorem 3.3 ([22] and [3] when $\mathcal{H} = \mathcal{G}$; [1] when $\mathcal{H} = \mathcal{I}$)**.** *Let $\mathcal{H} \subseteq \mathcal{G}$ be nondegenerate and hereditary.The graph G is in $C^+ER_{\mathcal{H}}$ if and only if G has at least one edge and is a disjoint union of stars.*

Proof. *Sufficiency*: Let G be a disjoint union of stars $T_1, T_2, \ldots, T_k$ and let t_i be a central vertex of T_i, $i = 1, \ldots, k$. Clearly $\{t_1, t_2, \ldots, t_k\}$ is a $\gamma_{\mathcal{H}}(G)$-set. For every edge e of G, the graph $G - e$ has exactly $k + 1$ components and hence $\gamma_{\mathcal{H}}(G - e) \geqslant k + 1 > \gamma_{\mathcal{H}}(G)$.

Necessity: Let for every two adjacent vertices x and y, $\gamma_{\mathcal{H}}(G-xy) > \gamma_{\mathcal{H}}(G)$. Let S be a $\gamma_{\mathcal{H}}(G)$-set. If $|S \cap \{x,y\}| \neq 1$ then since $\mathcal{H}$ is hereditary, S is a dominating $\mathcal{H}$-set of $G-xy$. This implies $\gamma_{\mathcal{H}}(G-xy) \leqslant \gamma_{\mathcal{H}}(G)$—a contradiction. Thus both S and $V(G)-S$ are independent. Assume there are $u, v \in S$ with a common neighbor, say w. Then S is a dominating $\mathcal{H}$-set of $G-uw$, which leads to $\gamma_{\mathcal{H}}(G-uw) \leqslant \gamma_{\mathcal{H}}(G)$—again a contradiction. Thus G is a union of stars. □

4. Vertex removal

In this section we investigate some basic properties of $(\gamma_{\mathcal{P}}(G), k)_{\mathcal{P}}$-critical graphs.

Observation 4.1. *Let $\mathcal{H} \subseteq \mathcal{G}$ be nondegenerate and let G be a graph with $\gamma_{\mathcal{H}}(G) \geqslant 2$.*

(i) *G is in $CV^kR_{\mathcal{H}}$ for all k for which $|V(G)| - \gamma_{\mathcal{H}}(G) + 1 \leqslant k \leqslant |V(G)| - 1$.*

(ii) *If G is in $CV^kR_{\mathcal{H}}$ then $k \notin \{s\colon\ s = \deg(x,G)$ for some $x \in V(G)\}$.*

Proof. (i) Obvious.

(ii) For any $x \in V(G)$ with $\deg(x,G) > 0$, any $\gamma_{\mathcal{H}}(G - N(x,G))$-set is also a dominating $\mathcal{H}$-set of G. □

Observation 4.2. *Let G be a graph and let $\mathcal{H} \subseteq \mathcal{G}$ be nondegenerate and closed under the union with K_1. If $S = \{x_1, \ldots, x_k\} \subsetneq V(G)$ then $\gamma_{\mathcal{H}}(G) - k \leqslant \gamma_{\mathcal{H}}(G-S)$. If equality holds then $\gamma_{\mathcal{H}}(G) - 1 \geqslant k$, S is independent, $S \subseteq \mathbf{V}^-_{\mathcal{H}}(G)$ and for any $x \in S$ and any $S_x \subseteq S - \{x\}$, $x \in \mathbf{V}^-_{\mathcal{H}}(G - S_x)$. In particular, if G is in $CV^kR_{\mathcal{H}}$ then $\gamma_{\mathcal{H}}(G) - k \leqslant \gamma_{\mathcal{H}}(G-S) \leqslant \gamma_{\mathcal{H}}(G) - 1$.*

Proof. Because of Observation 2.1(3) it remains to prove that S is independent when equality holds. Suppose to the contrary, $x_1x_2 \in E(G)$. Then $x_1 \in \mathbf{V}^-_{\mathcal{H}}(G)$ and by Lemma 2.2 it follows that $x_2 \in \mathbf{B}_{\mathcal{H}}(G - x_1)$ contradicting $x_2 \in \mathbf{V}^-_{\mathcal{H}}(G - x_1)$. □

Proposition 4.3. *Let $\mathcal{H} \subseteq \mathcal{G}$ be nondegenerate and closed under the union with K_1. Let a graph G be in $CV^2R_{\mathcal{H}}$.*

(i) *Then $V(G) = \mathbf{V}^-_{\mathcal{H}}(G) \cup \mathbf{Fr}^0_{\mathcal{H}}(G) \cup \mathbf{B}_{\mathcal{H}}(G)$.*

(ii) *If $\mathcal{H} = \mathcal{G}$ then $V(G) = \mathbf{V}^-(G) \cup \mathbf{Fr}^0(G)$.*

Proof. (i) Since the removal of a vertex can decrease $\gamma_{\mathcal{H}}(G)$ by at most one (Observation 2.1(3)), $\mathbf{V}^+_{\mathcal{H}}(G)$ is empty. If $v \in \mathbf{Fi}^0_{\mathcal{H}}(G)$ then $\gamma_{\mathcal{H}}(G - \{u,v\}) = \gamma_{\mathcal{H}}(G)$ for any $u \in N(v,G)$ because of Lemma 2.4.

(ii) Suppose $v \in \mathbf{B}(G)$ and $u \in N(v,G)$. Since $\gamma(G - \{u,v\}) < \gamma(G)$, adding v to any $\gamma(G - \{u,v\})$-set produces a $\gamma(G)$-set containing v—a contradiction. □

Proposition 4.4. *Let G be a graph of order $n \geqslant 2$ and let $\mathcal{H} \subseteq \mathcal{G}$ be induced-hereditary and closed under the union with K_1.*

(i) *G is in $CV^1R_{\mathcal{H}}$ if and only if $\gamma_{\mathcal{H}}(G-v) \neq \gamma_{\mathcal{H}}(G)$ for all $v \in V(G)$.*
(ii) *G is in $CV^1R_{\mathcal{H}}$ if and only if $\gamma_{\mathcal{H}}(G-v) = \gamma_{\mathcal{H}}(G) - 1$ for all $v \in V(G)$.*
(iii) *If G is in $CV^1R_{\mathcal{H}}$ then $\mathbf{Fi}_{\mathcal{H}}^{-1}(G) = \{x \in V(G) \colon \deg(x, G) = 0\}$.*

Proof. Clearly $\mathcal{H}$ is nondegenerate. (i) *Necessity*: Obvious.

Sufficiency: Assume $\mathbf{V}_{\mathcal{H}}^{+}(G)$ is not empty. By Lemma 2.2 and Observation 2.1(5), no vertex in $\mathbf{V}_{\mathcal{H}}^{+}(G)$ is adjacent to a vertex in $\mathbf{V}_{\mathcal{H}}^{-}(G)$. Hence for every vertex $x \in \mathbf{V}_{\mathcal{H}}^{+}(G)$, $N[x, G] \subseteq \mathbf{V}_{\mathcal{H}}^{+}(G)$. This implies $\deg(x, G) = 0$ for every $x \in \mathbf{V}_{\mathcal{H}}^{+}(G)$ ($\mathcal{H}$ is induced-hereditary). But then $\mathbf{V}_{\mathcal{H}}^{+}(G) \subseteq \mathbf{V}_{\mathcal{H}}^{-}(G)$—a contradiction. Thus $V(G) = \mathbf{V}_{\mathcal{H}}^{-}(G)$.

(ii) *Sufficiency*: Obvious.

Necessity: The result immediately follows by Observation 2.1(3).

(iii) If $x \in \mathbf{Fi}_{\mathcal{H}}^{-1}(G)$ then clearly $N(x, G) \subseteq \mathbf{B}_{\mathcal{H}}(G)$. □

Observation 4.5 ([4] when $\mathcal{H} = \mathcal{G}$)**.** *Let $\mathcal{H} \subseteq \mathcal{G}$ be nondegenerate. A graph G with $\gamma_{\mathcal{H}}(G) = 2$ is in $CV^1R_{\mathcal{H}}$ if and only if it is isomorphic to K_{2n} with a 1-factor removed for some $n \geqslant 1$.*

Example 4.6. Let $\mathcal{H} \subseteq \mathcal{G}$ be nondegenerate.

(1) $\overline{K_n}$, $n \geqslant 2$, is the unique graph of order n which is in $CV^kR_{\mathcal{H}}$ for all $k = 1, 2, \ldots, n-1$ (by Observation 4.1).
(2) K_{2n} minus a 1-factor is in $CV^kR_{\mathcal{H}}$ if and only if k is odd and $1 \leqslant k \leqslant 2n-1$.
(3) $K_{m,m}$, $m \geqslant 2$ is in $CV^kR_{\mathcal{I}}$ if and only if $k \in \{1, 2, \ldots, 2m-1\} - \{m\}$.
(4) If $K_2 \in \mathcal{H}$ then $K_{m,m}$, $m \geqslant 2$ is in $CV^kR_{\mathcal{H}}$ if and only if either $m = 2$ and $k \in \{1, 3\}$ or $m \geqslant 3$ and $k = 2m-1$.

Proposition 4.7. *Let $\mathcal{H} \subseteq \mathcal{G}$ be hereditary and closed under the union with K_1. If a graph G is in $CV^1R_{\mathcal{H}}$ and G has at least one edge then $b_{\mathcal{H}}^{+}(G) \geqslant 2$.*

Proof. The result immediately follows by Corollary 3.2(ii) and Proposition 4.4. □

Our next result is an upper bound on the order of $(\gamma_{\mathcal{P}}, k)_{\mathcal{P}}$-critical graphs in terms of Δ and $\gamma_{\mathcal{P}}$. Some properties of the extremal graphs are obtained.

Theorem 4.8. *Let $\mathcal{H} \subseteq \mathcal{G}$ be induced-hereditary and closed under the union with K_1 and let G be in $CV^1R_{\mathcal{H}}$. Then $|V(G)| \leqslant (\Delta(G)+1)(\gamma_{\mathcal{H}}(G)-1)+1$. If equality holds then:*

(i) *if $x \in V(G)$ and $v \in \mathbf{G}_{\mathcal{H}}(G-x)$ then $x \in \mathbf{Fi}_{\mathcal{H}}(G-v)$ and $v \in \mathbf{Fi}_{\mathcal{H}}(G-x)$;*
(ii) *for every $x \in V(G)$, $\mathbf{G}_{\mathcal{H}}(G-x) = \mathbf{Fi}_{\mathcal{H}}(G-x) - \mathbf{Fi}^{-1}_{\mathcal{H}}(G-x)$ and $\mathbf{G}_{\mathcal{H}}(G-x)$ is an efficient dominating set of $G-x$;*
(iii) *([7] when $\mathcal{H} = \mathcal{G}$) G is regular;*
(iv) *$\gamma(G) = i(G)$;*
(v) *let $\mathcal{U} \subseteq \mathcal{G}$ be induced-hereditary and closed under union with K_1. Then G is in $CV^1R_{\mathcal{U}}$;*
(vi) *([16] when $\mathcal{H} = \mathcal{G}$) $b^+_{\mathcal{H}}(G) \leqslant \Delta(G)+1 = \delta(G)+1$ provided $\Delta(G) \geqslant 1$.*

We need the following observation to prove Theorem 4.8.

Observation 4.9. *Let $\mathcal{H} \subseteq \mathcal{G}$ be nondegenerate and let G be a graph. Then $|V(G)| \leqslant (1+\Delta(G))\gamma_{\mathcal{H}}(G)$. The equality holds if and only if each $\gamma_{\mathcal{H}}(G)$-set is efficient dominating and each $\gamma_{\mathcal{H}}$-good vertex of G has the maximum degree.*

Proof. Let $M = \{x_1, \ldots, x_k\}$ be a $\gamma_{\mathcal{H}}(G)$-set. Then $|V(G)| = \left|\bigcup_{i=1}^{k} N[x_i, G]\right| \leqslant \Sigma_{i=1}^{k}(\deg(x_i, G)+1) \leqslant k(\Delta(G)+1) = \gamma_{\mathcal{H}}(G)(\Delta(G)+1)$. The equality holds if and only if $\deg(x_i, G) = \Delta(G)$, $i = 1, 2, \ldots, k$ and $\{N[x_1, G], N[x_2, G], \ldots, N[x_k, G]\}$ is a partition of $V(G)$. □

Proof of Theorem 4.8. If G has no edges then the results are obvious. So, let G have edges. Clearly $\Delta(G) \geqslant 2$ and $\gamma_{\mathcal{H}}(G) \geqslant 2$. Let $v \in V(G)$. Using Observation 4.9 we have $|V(G)| = |V(G-v)|+1 \leqslant (1+\Delta(G-v))\gamma_{\mathcal{H}}(G-v)+1 \leqslant (1+\Delta(G))(\gamma_{\mathcal{H}}(G)-1)+1$. Let equality hold and let $M = \{x_1, \ldots, x_k\}$ be a $\gamma_{\mathcal{H}}(G-v)$-set. It follows by Observation 4.9 that M is an efficient dominating set of $G-v$ and $\deg(x_i, G-v) = \Delta(G)$, $i = 1, \ldots, k$. Hence to prove (iii) it suffices to prove (i).

(i) and (ii): Let M be an efficient dominating set of $G-v$ and let Q be an efficient dominating set of $G-x$ with $v \in Q$. Since $|Q| = \gamma_{\mathcal{H}}(G)-1 = |M|$ it follows: (a) each vertex in $Q-\{v\}$ dominates a unique vertex of M, and (b) there exists exactly one vertex in M, say w, which is not dominated by $Q-v$. Since $M \cup \{v\}$ is independent (by Lemma 2.2), it follows that $w = x$. Therefore $x \in \mathbf{Fi}_{\mathcal{H}}(G-v)$ and by symmetry, $v \in \mathbf{Fi}_{\mathcal{H}}(G-x)$. Thus (i) holds and to prove (ii) it remains to show that $\mathbf{Fi}^{-1}_{\mathcal{H}}(G-x)$ is empty. Suppose to the contrary $u \in \mathbf{Fi}^{-1}_{\mathcal{H}}(G-x)$. By Observation 4.9 it follows that $|V(G)|-2 = |V((G-x)-u)| \leqslant (\Delta(G)+1)(\gamma_{\mathcal{H}}(G)-2) = |V(G)|-\Delta(G)-1$—a contradiction with $\Delta(G) \geqslant 2$.

(iv) Let $v \in V(G)$ and let M be a $\gamma_{\mathcal{H}}(G-v)$-set. Since M is independent (by (ii)), it follows by Lemma 2.2 that $M \cup \{v\}$ is an independent $\gamma_{\mathcal{H}}(G)$-set. Hence $\gamma_{\mathcal{H}}(G) = i(G)$.

(v) By (iv) it follows that $\gamma(G) = \gamma_{\mathcal{U}}(G) = i(G)$. By (ii) applied to the property $\mathcal{U}$ we have $\gamma_{\mathcal{U}}(G-v) = i(G-v) = i(G) - 1 = \gamma_{\mathcal{U}}(G) - 1$ for each $v \in V(G)$.

(vi) Let $v \in V(G)$ and let M be the unique $\gamma_{\mathcal{H}}(G-v)$-set. Let $x \in M$ and let $y \in V(G)$ be adjacent. Consider the graph $G_1 = (G-v) - xy$. Assume $\gamma_{\mathcal{H}}(G_1) \leqslant \gamma_{\mathcal{H}}(G-v)$. Since $|V(G_1)| = |V(G-v)|$, it follows by Observation 4.9 that $\Delta(G_1) = \Delta(G-v) = \Delta(G)$, $\gamma_{\mathcal{H}}(G_1) = \gamma_{\mathcal{H}}(G-v)$ and if M_1 is a $\gamma_{\mathcal{H}}(G_1)$-set then (a) M_1 is efficient dominating, and (b) each vertex in M_1 has degree $\Delta(G)$. Hence $x \notin M_1$. But then $M_1 \neq M$ is a $\gamma_{\mathcal{H}}(G-v)$-set—a contradiction with (ii). Thus $\gamma_{\mathcal{H}}(G_1) > \gamma_{\mathcal{H}}(G-v)$.

Let G_v be the graph obtained from G after deleting all edges incident with v in G. Since $\mathcal{H}$ is induced-hereditary and closed under the union with K_1, $\gamma_{\mathcal{H}}(G_v - xy) = \gamma_{\mathcal{H}}(G_1) + 1 > \gamma_{\mathcal{H}}(G-v) + 1 = \gamma_{\mathcal{H}}(G)$. Therefore $b^+_{\mathcal{H}}(G) \leqslant \Delta(G) + 1$. □

Examples of CV^1R-graphs G of order $(\Delta(G)+1)(\gamma(G)-1)+1$ may be found in [4], [10, p. 140] and [19].

Proposition 4.10 ([13] when $\mathcal{H} = \mathcal{G}$). *Let $\mathcal{H} \subseteq \mathcal{G}$ be induced-hereditary and closed under the union with K_1 and let G be in $CV^kR_{\mathcal{H}}$. Then $|V(G)| \leqslant (\Delta(G)+1) \times (\gamma_{\mathcal{H}}(G) - 1) + k$.*

P r o o f. We proceed by induction on k. If $k = 1$ then the result follows by Theorem 4.8. So, let G be in $CV^kR_{\mathcal{H}}$, $k \geqslant 2$, and not in $CV^1R_{\mathcal{H}}$. If $x \in \mathbf{V}^+_{\mathcal{H}}(G)$ then there is $y \in N(x,G) - \mathbf{V}^+_{\mathcal{H}}(G)$ ($\mathcal{H}$ is induced-hereditary) and by Lemma 2.2, $y \in \mathbf{V}^0_{\mathcal{H}}(G)$. Hence $\gamma_{\mathcal{H}}(G-y) = \gamma_{\mathcal{H}}(G)$ and $G-y$ is in $CV^{k-1}R_{\mathcal{H}}$. The result now follows by the inductive hypothesis. □

The next conjecture concerning the case $\mathcal{P} = \mathcal{G}$ is the main outstanding conjecture on the ordinary bondage number.

Conjecture 4.11 (Teschner [20] when $\mathcal{P} = \mathcal{G}$). *Let $\mathcal{P} \subseteq \mathcal{G}$ be additive and hereditary. For any graph G which is in $CV^1R_{\mathcal{P}}$, $b^+_{\mathcal{P}}(G) \leqslant 1.5\Delta(G)$.*

Particular support for this conjecture is the fact that $b_{\mathcal{P}}(C_{3k+1}) = 3 = 1.5\Delta(C_{3k+1})$ [17]. Now let $\mathcal{P} = \mathcal{G}$. Teschner [20] has shown that Conjecture 4.11 is true when $\gamma(G) \leqslant 3$. Observe that if $G = K_t \times K_t$ for a positive integer $t \geqslant 2$, then $b(G) = 1.5\Delta(G)$ as was found independently by Hartnel and Rall [9] and by Teschner [21].

Motivated by Theorem 4.8 and Lemma 2.5 we state the following:

Conjecture 4.12. *Let G be in $CV^1R_{\mathcal{P}}$ where $\mathcal{P} \subseteq \mathcal{G}$ is induced-hereditary and closed under the union with K_1. If $\Delta(G) \geqslant 1$ and $|V(G)| = (\Delta(G)+1)(\gamma_{\mathcal{P}}(G)-1)+1$ then* (a) $b^+_{\mathcal{P}}(G) = \Delta(G)+1$, *and* (b) *$G$ is not in $CV^2R_{\mathcal{P}}$.*

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