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Domination with respect to nondegenerate properties: vertex and edge removal


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Abstract. In this paper we present results on changing and unchanging of the domination number with respect to the nondegenerate property $P$, denoted by $\gamma_P(G)$, when a graph $G$ is modified by deleting a vertex or deleting edges. A graph $G$ is $(\gamma_P(G), k)_P$-critical if $\gamma_P(G - S) < \gamma_P(G)$ for any set $S \subseteq V(G)$ with $|S| = k$. Properties of $(\gamma_P, k)_P$-critical graphs are studied. The plus bondage number with respect to the property $P$, denoted $b_P^+(G)$, is the cardinality of the smallest set of edges $U \subseteq E(G)$ such that $\gamma_P(G - U) > \gamma_P(G)$. Some known results for ordinary domination and bondage numbers are extended to $\gamma_P(G)$ and $b_P^+(G)$. Conjectures concerning $b_P^+(G)$ are posed.

Keywords: dominating set, domination number, bondage number, additive graph property, hereditary graph property, induced-hereditary graph property

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1. Introduction

All graphs considered in this article are finite, undirected, without loops or multiple edges. For the graph theory terminology not presented here, we follow Haynes et al. [10]. We denote the vertex set and the edge set of a graph $G$ by $V(G)$ and $E(G)$, respectively. The subgraph induced by $S \subseteq V(G)$ is denoted by $\langle S, G \rangle$. For a vertex $x$ of $G$, $N(x, G)$ denotes the set of all neighbors of $x$ in $G$, $N[x, G] = N(x, G) \cup \{x\}$ and the degree of $x$ is $\deg(x, G) = |N(x, G)|$. The maximum and minimum degrees of vertices in the graph $G$ are denoted by $\Delta(G)$ and $\delta(G)$, respectively.

Let $\mathcal{G}$ denote the set of all mutually nonisomorphic graphs. A graph property is any nonempty subset of $\mathcal{G}$. We say that a graph $G$ has the property $P$ whenever there exists a graph $H \in P$ which is isomorphic to $G$. For example, we list some graph properties:

$\triangleright \mathcal{I} = \{H \in \mathcal{G}: H$ is totally disconnected$\}$;
\[ F = \{ H \in G : \text{H is a forest} \}; \]
\[ UK = \{ H \in G : \text{each component of H is complete} \}. \]

A graph property \( P \) is called: (a) *hereditary* (induced-hereditary), if from the fact that a graph \( G \) has property \( P \), it follows that all subgraphs (induced subgraphs) of \( G \) also belong to \( P \); (b) *nondegenerate* if \( I \subseteq P \), and (c) *additive* if it is closed under taking disjoint unions of graphs. Note that: (i) \( I \) and \( F \) are nondegenerate, additive and hereditary properties, and (ii) \( UK \) is nondegenerate, additive, induced-hereditary and is not hereditary.

A dominating set for a graph \( G \) is a set of vertices \( D \subseteq V(G) \) such that every vertex of \( G \) is either in \( D \) or is adjacent to an element of \( D \). The domination number \( \gamma(G) \) of a graph \( G \) is the minimum cardinality of a dominating set of \( G \). A dominating set \( D \) is called an efficient dominating set if the distance between any two vertices in \( D \) is at least three. Not all graphs have efficient dominating sets; however, if a graph \( G \) has an efficient dominating set, then the cardinality of any efficient dominating set equals the domination number of \( G \) [2].

Any set \( S \subseteq V(G) \) such that the subgraph \( \langle S, G \rangle \) possesses the property \( P \) is called a \( P \)-set. The domination number with respect to the property \( P \), denoted by \( \gamma_P(G) \), is the smallest cardinality of a dominating \( P \)-set of \( G \). Observe that if \( I \subseteq P_2 \subseteq P_1 \subseteq G \) then [8] \( \gamma(G) = \gamma_I(G) \leq \gamma_{P_1}(G) \leq \gamma_{P_2}(G) \leq \gamma_{I}(G) = i(G) \), where \( i(G) \) is the independent domination number of \( G \). The concept of domination with respect to any property \( P \) was introduced by Goddard et al. [8]. Michalak [11] has considered this parameter when the property is additive and induced-hereditary.

It is often of interest to know how the value of a graph parameter is affected when a small change is made in a graph. In this connection, in [14], the present author began an investigation on effects on \( \gamma_P \) when a graph is modified by deleting a vertex or by adding an edge. We continue this work here and present results on changing \( \gamma_P(G) \) when an edge or a vertex is removed from \( G \).

### 2. Definitions and known results

Let \( G \) be a graph and let \( P \subseteq G \) be nondegenerate. Any minimum dominating \( P \)-set of \( G \) is called a \( \gamma_P(G) \)-set. Let \( G \) be a graph and \( v \in V(G) \). A vertex \( v \) of the graph \( G \) is said to be

(a) [6] \( \gamma_P \)-good, if \( v \) belongs to some \( \gamma_P(G) \)-set;
(b) [6] \( \gamma_P \)-bad, if \( v \) belongs to no \( \gamma_P(G) \)-set;
(c) [18] \( \gamma_P \)-fixed if \( v \) belongs to every \( \gamma_P(G) \)-set;
(d) [18] \( \gamma_P \)-free if \( v \) belongs to some \( \gamma_P(G) \)-set but not to all \( \gamma_P(G) \)-sets.

We also need the following sets:
\(G_P(G) = \{x \in V(G) : x \text{ is } \gamma_P\text{-good}\};
\)
\(B_P(G) = \{x \in V(G) : x \text{ is } \gamma_P\text{-bad}\};
\)
\(F_P(G) = \{x \in V(G) : x \text{ is } \gamma_P\text{-fixed}\};
\)
\(F_{\bar{P}}(G) = \{x \in F_P(G) : \gamma_P(G - x) = \gamma_P(G) - 1\};
\)
\(F_{\overline{\bar{P}}}(G) = \{x \in F_P(G) : \gamma_P(G - x) = \gamma_P(G)\};
\)
\(F_{\overline{\overline{\bar{P}}}}(G) = \{x \in F_P(G) : \gamma_P(G - x) = \gamma_P(G) + p, p \text{ is integer}\};
\)
\(V_P^0(G) = \{x \in V(G) : \gamma_P(G - x) = \gamma_P(G)\};
\)
\(V_P^-(G) = \{x \in V(G) : \gamma_P(G - x) < \gamma_P(G)\};
\)
\(V_P^+(G) = \{x \in V(G) : \gamma_P(G - x) > \gamma_P(G)\}.
\)

Clearly \(\{G_P(G), B_P(G)\}\) and \(\{V_P^-(G), V_P^+(G)\}\) are partitions of \(V(G)\), and \(\{F_P(G), F_{\overline{\overline{\bar{P}}}}(G)\}\) is a partition of \(G_P(G)\). Moreover:

**Observation 2.1** ([14]). Let \(G\) be a graph of order \(n \geq 2\) and let \(\mathcal{H} \subseteq \mathcal{G}\) be nondegenerate and closed under the union with \(K_1\). Then
1. \(\{F_{\mathcal{H}}^1(G), F_{\mathcal{H}}^0(G)\}\) is a partition of \(F_{\mathcal{H}}(G)\);  
2. \(\{F_{\mathcal{H}}^1(G), F_{\mathcal{H}}^0(G), \ldots, F_{\mathcal{H}}^{n-2}(G)\}\) is a partition of \(F_{\mathcal{H}}(G)\);  
3. \(\{F_{\mathcal{H}}^1(G), F_{\mathcal{H}}^0(G)\}\) is a partition of \(V_{\mathcal{H}}(G)\);  
4. \(\{F_{\mathcal{H}}^1(G), F_{\mathcal{H}}^0(G), B_{\mathcal{H}}(G)\}\) is a partition of \(V_{\mathcal{H}}^0(G)\);  
5. \(\{F_{\mathcal{H}}^1(G), F_{\mathcal{H}}^2(G), \ldots, F_{\mathcal{H}}^{n-2}(G)\}\) is a partition of \(V_{\mathcal{H}}^0(G)\).

For each nondegenerate property \(P \subseteq \mathcal{G}\) we define the following classes of graphs \(G\):
- \((C^kR_P) \gamma_P(G - S) < \gamma_P(G)\) for any set \(S \subseteq V(G)\) with \(|S| = k\),
- \((C^+E_P) \gamma_P(G - e) > \gamma_P(G)\) for all \(e \in E(G)\)

For convenience we omit the subscript \(\mathcal{G}\). For a survey on results concerning the classes \(CV^1R\) and \(C^+E\) see for instance [10, Chapter 5], [19] and the bibliography in [10]. We define a graph \(G\) to be \((\gamma_P(G), k_P)\text{-critical}\) if \(G\) is in \(C^kR_P\). The \((\gamma(G), k)\text{-critical graphs provided } k \geq 2\) are introduced by Brigham et al [5]. Further results on these graphs can be found in [12], [13].

**Lemma 2.2** ([14]). Let \(G\) be a graph of order at least two, \(v \in V_{\mathcal{H}}^-(G)\) and let \(\mathcal{H} \subseteq \mathcal{G}\) be nondegenerate and closed under the union with \(K_1\). Then \(N(v, G) \subseteq B_{\mathcal{H}}(G - v) - F_{\mathcal{H}}(G)\). If \(M\) is a \(\gamma_{\mathcal{H}}(G - v)\text{-set then } M \cup \{v\}\) is a \(\gamma_{\mathcal{H}}(G)\text{-set.}\)

**Lemma 2.3** ([14]). Let \(x\) and \(y\) be two different and nonadjacent vertices in a graph \(G\). Let \(\mathcal{H} \subseteq \mathcal{G}\) be hereditary and closed under the union with \(K_1\). If \(\gamma_{\mathcal{H}}(G + xy) < \gamma_{\mathcal{H}}(G)\) then \(\gamma_{\mathcal{H}}(G + xy) = \gamma_{\mathcal{H}}(G) - 1\). Moreover, \(\gamma_{\mathcal{H}}(G + xy) = \gamma_{\mathcal{H}}(G) - 1\) if and only if at least one of the following conditions holds:
1. \(x \in V_{\mathcal{H}}^-(G)\) and \(y\) is a \(\gamma_{\mathcal{H}}\text{-good vertex of } G - x\);  
2. \(x\) is a \(\gamma_{\mathcal{H}}\text{-good vertex of } G - y\) and \(y \in V_{\mathcal{H}}^-(G)\).
Lemma 2.4 ([14]). Let $\mathcal{H} \subseteq \mathcal{G}$ be nondegenerate and closed under the union with $K_1$ and let $x$ be a $\gamma^+_P$-fixed vertex of a graph $G$. Then $N(x, G) \subseteq B_{\mathcal{H}}(G - x) \cap (V^0_{\mathcal{H}}(G) \cup \mathbf{F}^1_{\mathcal{H}}(G))$ and for each $y \in N(x, G)$, $\gamma_{\mathcal{H}}(G - \{x, y\}) = \gamma_{\mathcal{H}}(G)$.

One measure of stability of the domination number with respect to the property $\mathcal{P}$ under edge removal is the bondage number $b$. For every graph $G$ with at least one edge and every nondegenerate property $\mathcal{P}$, the plus bondage number with respect to the property $\mathcal{P}$, denoted by $b^+_P(G)$, is the cardinality of the smallest set of edges $U \subseteq E(G)$ such that $\gamma_{\mathcal{P}}(G - U) > \gamma_{\mathcal{P}}(G)$. Since $\gamma_{\mathcal{P}}(G - E(G)) = |V(G)| > \gamma_{\mathcal{P}}(G)$ for every graph $G$ with at least one edge and every nondegenerate property $\mathcal{P}$, it follows that $b^+_P(G)$ always exists. Note that $b_0^+(G) = b^+_C(G) = b(G)$—the ordinary bondage number. The bondage number of graphs belonging to $CV^1 R$ is examined for instance in [9], [20], [21], [16]. The next result shows that the class $CV^1 R^P$ plays an important role in the study of the plus bondage number with respect to $\mathcal{P}$.

Lemma 2.5 ([17]). Let $G$ be a graph and let $\mathcal{H} \subseteq \mathcal{G}$ be nondegenerate and induced-hereditary. If $b^+_P(G) > \Delta(G)$ then $G$ is in $CV^1 R^*_H$.

3. Edge removal

An edge $e$ of a graph $G$ is $\gamma^+_P$-ER-critical if $\gamma_{\mathcal{P}}(G - e) > \gamma_{\mathcal{P}}(G)$. We begin with necessary and sufficient conditions for an edge of a graph to be $\gamma^+_P$-ER-critical.

Theorem 3.1 ([15] when $\mathcal{H} = \mathcal{G}$). Let $x_1$ and $x_2$ be adjacent vertices in a graph $G$ and let $G_{12} = G - x_1 x_2$. Let $\mathcal{H} \subseteq \mathcal{G}$ be hereditary and closed under the union with $K_1$. Then $x_1 x_2$ is $\gamma^+_P$-ER-critical if and only if one of the following conditions holds:

(R1) $x_i \in B_{\mathcal{H}}(G), x_j \in \mathbf{F}^1_{\mathcal{H}}(G), x_i \in V^-(G_{12})$ and $x_j \in \mathbf{F}^{q-1}_{\mathcal{H}}(G_{12})$ where $\{i, j\} = \{1, 2\}$ and $q \geq 1$;

(R2) $x_i \in B_{\mathcal{H}}(G), x_j \in \mathbf{F}^1_{\mathcal{H}}(G), x_i \in V^-(G_{12})$ and $x_j \in \mathbf{F}^0_{\mathcal{H}}(G_{12}) \cap G_{\mathcal{H}}(G - x_i)$ where $\{i, j\} = \{1, 2\}$;

(R3) $x_i \in B_{\mathcal{H}}(G), x_j \in \mathbf{F}^0_{\mathcal{H}}(G), x_i \in V^-(G_{12}) \cap B_{\mathcal{H}}(G - x_j)$ and $x_j \in V^-(G_{12}) \cap G_{\mathcal{H}}(G - x_i)$ where $\{i, j\} = \{1, 2\}$;

(R4) $x_1, x_2 \in \mathbf{F}^0_{\mathcal{H}}(G), x_1 \in V^-(G_{12}) \cap G_{\mathcal{H}}(G - x_2)$ and $x_2 \in V^-(G_{12}) \cap G_{\mathcal{H}}(G - x_1)$.

Proof. Sufficiency: Let (R1) hold and let $M$ be a $\gamma_{\mathcal{H}}(G_{12} - x_i)$-set. By Lemma 2.2 (applied to $G_{12}$), $M \cup \{x_1\}$ is a $\gamma_{\mathcal{H}}(G_{12})$-set. Since $x_j \in \mathbf{F}^1_{\mathcal{H}}(G_{12})$, $x_j \in G_{\mathcal{H}}(G - x_i)$. Now, if one of (R1)–(R4) is satisfied then the result immediately follows by Lemma 2.3 (applied to $G_{12}$).

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Necessity: Let $\gamma_H(G) < \gamma_H(G_{12})$. By Lemma 2.3 it follows that $\gamma_H(G) = \gamma_H(G_{12}) - 1$ and without loss of generality we may assume that $x_1 \in V_H^- (G_{12})$. Note that no $\gamma_H(G)$-set contains both $x_1$ and $x_2$. Indeed, if $M$ is a $\gamma_H(G)$-set with $x_1, x_2 \in M$ then since $H$ is hereditary, $M$ is a dominating $H$-set of $G_{12}$—a contradiction.

(a) Let $x_2 \in \text{Fi}^2_H(G_{12})$, $q \geq 1$. We have $\gamma_H(G-x_2) = \gamma_H(G_{12}-x_2) = \gamma_H(G_{12}) + q - 1 = \gamma_H(G) + q$. Then $x_2 \in \text{Fi}^2_H(G)$, which implies $x_1 \in \text{B}_H(G)$.

(b) Let $x_2 \in \text{Fr}^0_H(G_{12}) \cap G_H(G-x_1)$. In this case $\gamma_H(G-x_2) = \gamma_H(G_{12}-x_2) = \gamma_H(G_{12}) - 1 = \gamma_H(G)$, it follows that $x_2 \in V_H^0(G)$. Assume there is a $\gamma_H(G)$-set $M$ with $x_2 \notin M$. Then $M$ is a dominating $H$-set of $G - x_2$ with $\lvert M \rvert = \gamma_H(G) = \gamma_H(G - x_2)$. Hence $M$ is a $\gamma_H(G-x_2)$-set. Since $x_1 \in \text{B}_H(G-x_2)$ we have $x_1, x_2 \notin M$. But then $M$ is a dominating $H$-set of $G_{12}$ with $\lvert M \rvert < \gamma_H(G_{12})$—a contradiction. Since $x_2 \in V_H^0(G)$, $x_2 \in \text{Fi}^1_H(G)$. Thus $x_1 \in \text{B}_H(G)$.

(c) Let without loss of generality $x_1 \in \text{B}_H(G-x_2) \text{ and } x_2 \in V_H^0(G_{12}) \cap G_H(G-x_1)$. Since $\gamma_H(G-x_2) = \gamma_H(G_{12}-x_2) = \gamma_H(G_{12}) - 1 = \gamma_H(G)$ it follows that $x_2 \in V_H^0(G)$.

(d) Let $M_1$ be a $\gamma_H(G-x_2)$-set with $x_1 \in M_1$ and $M_2$ a $\gamma_H(G-x_1)$-set with $x_2 \in M_2$. Then $M_1$ and $M_2$ are dominating $H$-sets of $G$ and $\lvert M_i \rvert = \gamma_H(G-x_i) = \gamma_H(G_{12} - x_i) = \gamma_H(G_{12}) - 1 = \gamma_H(G)$ for $i = 1, 2$. Hence $M_1$ and $M_2$ are $\gamma_H(G)$-sets and $x_1, x_2 \in \text{Fi}^0_H(G) \cup \text{Fr}^0_H(G)$. Since $x_1 \notin M_2$ and $x_2 \notin M_1$, it follows that $x_1, x_2 \in \text{Fr}^0_H(G)$.

There are no other possibilities because of Lemma 2.3.

Recall that a vertex cover of a graph $G$ is a set of vertices such that each edge of $G$ is incident to at least one vertex of the set.

**Corollary 3.2.** Let $H \subseteq G$ be hereditary and closed under the union with $K_1$. Let a graph $G$ have at least one edge.

(i) If $v \in V_H^-(G)$ then for every edge $e \in E(G)$ incident to $v$, $\gamma_H(G - e) \leq \gamma_H(G)$.

(ii) If $V_H^-(G)$ is a vertex cover then for every edge $e \in E(G)$, $\gamma_H(G - e) \leq \gamma_H(G)$.

Now, we give a characterization of the class $C^+ ER_P$.

**Theorem 3.3** ([22] and [3] when $H = G$; [1] when $H = I$). Let $H \subseteq G$ be nondegenerate and hereditary. The graph $G$ is in $C^+ ER_H$ if and only if $G$ has at least one edge and is a disjoint union of stars.

**Proof.** Sufficiency: Let $G$ be a disjoint union of stars $T_1, T_2, \ldots, T_k$ and let $t_i$ be a central vertex of $T_i$, $i = 1, \ldots, k$. Clearly $\{t_1, t_2, \ldots, t_k\}$ is a $\gamma_H(G)$-set. For every edge $e$ of $G$, the graph $G - e$ has exactly $k + 1$ components and hence $\gamma_H(G - e) \geq k + 1 > \gamma_H(G)$. 79
Necessity: Let for every two adjacent vertices $x$ and $y$, $\gamma_{\mathcal{H}}(G - xy) > \gamma_{\mathcal{H}}(G)$. Let $S$ be a $\gamma_{\mathcal{H}}(G)$-set. If $|S \cap \{x, y\}| \neq 1$ then since $\mathcal{H}$ is hereditary, $S$ is a dominating $\mathcal{H}$-set of $G - xy$. This implies $\gamma_{\mathcal{H}}(G - xy) \leq \gamma_{\mathcal{H}}(G)$—a contradiction. Thus both $S$ and $V(G) - S$ are independent. Assume there are $u, v \in S$ with a common neighbor, say $w$. Then $S$ is a dominating $\mathcal{H}$-set of $G - uw$, which leads to $\gamma_{\mathcal{H}}(G - uw) \leq \gamma_{\mathcal{H}}(G)$—again a contradiction. Thus $G$ is a union of stars. □

4. VERTEX REMOVAL

In this section we investigate some basic properties of $(\gamma_{\mathcal{P}}(G), k)_p$-critical graphs.

Observation 4.1. Let $\mathcal{H} \subseteq \mathcal{G}$ be nondegenerate and let $G$ be a graph with $\gamma_{\mathcal{H}}(G) \geq 2$.

(i) $G$ is in $CV^k R_{\mathcal{H}}$ for all $k$ for which $|V(G)| - \gamma_{\mathcal{H}}(G) + 1 \leq k \leq |V(G)| - 1$.

(ii) If $G$ is in $CV^k R_{\mathcal{H}}$ then $k \not\in \{s : s = \deg(x, G) \text{ for some } x \in V(G)\}$.

Proof. (i) Obvious.

(ii) For any $x \in V(G)$ with $\deg(x, G) > 0$, any $\gamma_{\mathcal{H}}(G - N(x, G))$-set is also a dominating $\mathcal{H}$-set of $G$. □

Observation 4.2. Let $G$ be a graph and let $\mathcal{H} \subseteq \mathcal{G}$ be nondegenerate and closed under the union with $K_1$. If $S = \{x_1, \ldots, x_k\} \subseteq V(G)$ then $\gamma_{\mathcal{H}}(G) - k \leq \gamma_{\mathcal{H}}(G - S)$. If equality holds then $\gamma_{\mathcal{H}}(G) - 1 \geq k$, $S$ is independent, $S \subseteq V^-_{\mathcal{H}}(G)$ and for any $x \in S$ and any $S_x \subseteq S - \{x\}$, $x \in V^-_{\mathcal{H}}(G - S_x)$. In particular, if $G$ is in $CV^k R_{\mathcal{H}}$ then $\gamma_{\mathcal{H}}(G) - k \leq \gamma_{\mathcal{H}}(G - S) \leq \gamma_{\mathcal{H}}(G) - 1$.

Proof. Because of Observation 2.1(3) it remains to prove that $S$ is independent when equality holds. Suppose to the contrary, $x_1 x_2 \in E(G)$. Then $x_1 \in V^-_{\mathcal{H}}(G)$ and by Lemma 2.2 it follows that $x_2 \in B_{\mathcal{H}}(G - x_1)$ contradicting $x_2 \in V^-_{\mathcal{H}}(G - x_1)$. □

Proposition 4.3. Let $\mathcal{H} \subseteq \mathcal{G}$ be nondegenerate and closed under the union with $K_1$. Let a graph $G$ be in $CV^2 R_{\mathcal{H}}$.

(i) Then $V(G) = V^-_{\mathcal{H}}(G) \cup \Fr^0_{\mathcal{H}}(G) \cup B_{\mathcal{H}}(G)$.

(ii) If $\mathcal{H} = \mathcal{G}$ then $V(G) = V^-_{\mathcal{G}}(G) \cup \Fr^0(G)$.

Proof. (i) Since the removal of a vertex can decrease $\gamma_{\mathcal{H}}(G)$ by at most one (Observation 2.1(3)), $V^+_{\mathcal{H}}(G)$ is empty. If $v \in \Fr^0_{\mathcal{H}}(G)$ then $\gamma_{\mathcal{H}}(G - \{u, v\}) = \gamma_{\mathcal{H}}(G)$ for any $u \in N(v, G)$ because of Lemma 2.4.

(ii) Suppose $v \in B(G)$ and $u \in N(v, G)$. Since $\gamma(G - \{u, v\}) < \gamma(G)$, adding $v$ to any $\gamma(G - \{u, v\})$-set produces a $\gamma(G)$-set containing $v$—a contradiction. □
**Proposition 4.4.** Let $G$ be a graph of order $n \geq 2$ and let $\mathcal{H} \subseteq \mathcal{G}$ be induced-hereditary and closed under the union with $K_1$.

(i) $G$ is in $CV^1 R_{\mathcal{H}}$ if and only if $\gamma_{\mathcal{H}}(G - v) \neq \gamma_{\mathcal{H}}(G)$ for all $v \in V(G)$.

(ii) $G$ is in $CV^1 R_{\mathcal{H}}$ if and only if $\gamma_{\mathcal{H}}(G - v) = \gamma_{\mathcal{H}}(G) - 1$ for all $v \in V(G)$.

(iii) If $G$ is in $CV^1 R_{\mathcal{H}}$ then $\text{Fi}^{-1}_{\mathcal{H}}(G) = \{x \in V(G) : \text{deg}(x, G) = 0\}$.

**Proof.** Clearly $\mathcal{H}$ is nondegenerate. (i) **Necessity:** Obvious.

**Sufficiency:** Assume $V^+_{\mathcal{H}}(G)$ is not empty. By Lemma 2.2 and Observation 2.1(5), no vertex in $V^+_{\mathcal{H}}(G)$ is adjacent to a vertex in $V^-_{\mathcal{H}}(G)$. Hence for every vertex $x \in V^+_{\mathcal{H}}(G)$, $N[x, G] \subseteq V^+_{\mathcal{H}}(G)$. This implies $\text{deg}(x, G) = 0$ for every $x \in V^+_{\mathcal{H}}(G)$ ($\mathcal{H}$ is induced-hereditary). But then $V^+_{\mathcal{H}}(G) \subseteq V^-_{\mathcal{H}}(G)$—a contradiction. Thus $V(G) = V^-_{\mathcal{H}}(G)$.

(ii) **Sufficiency:** Obvious.

**Necessity:** The result immediately follows by Observation 2.1(3).

(iii) If $x \in \text{Fi}^{-1}_{\mathcal{H}}(G)$ then clearly $N(x, G) \subseteq B_{\mathcal{H}}(G)$. \hfill $\square$

**Observation 4.5** ([4] when $\mathcal{H} = \mathcal{G}$). Let $\mathcal{H} \subseteq \mathcal{G}$ be nondegenerate. A graph $G$ with $\gamma_{\mathcal{H}}(G) = 2$ is in $CV^1 R_{\mathcal{H}}$ if and only if it is isomorphic to $K_{2n}$ with a 1-factor removed for some $n \geq 1$.

**Example 4.6.** Let $\mathcal{H} \subseteq \mathcal{G}$ be nondegenerate.

(1) $\overline{K_n}$, $n \geq 2$, is the unique graph of order $n$ which is in $CV^k R_{\mathcal{H}}$ for all $k = 1, 2, \ldots, n - 1$ (by Observation 4.1).

(2) $K_{2n}$ minus a 1-factor is in $CV^k R_{\mathcal{H}}$ if and only if $k$ is odd and $1 \leq k \leq 2n - 1$.

(3) $K_{m,m}$, $m \geq 2$ is in $CV^k R_{\mathcal{H}}$ if and only if $k \in \{1, 2, \ldots, 2m - 1\} - \{m\}$.

(4) If $K_2 \in \mathcal{H}$ then $K_{m,m}$, $m \geq 2$ is in $CV^k R_{\mathcal{H}}$ if and only if either $m = 2$ and $k \in \{1, 3\}$ or $m \geq 3$ and $k = 2m - 1$.

**Proposition 4.7.** Let $\mathcal{H} \subseteq \mathcal{G}$ be hereditary and closed under the union with $K_1$. If a graph $G$ is in $CV^1 R_{\mathcal{H}}$ and $G$ has at least one edge then $b^+_{\mathcal{H}}(G) \geq 2$.

**Proof.** The result immediately follows by Corollary 3.2(ii) and Proposition 4.4. \hfill $\square$

Our next result is an upper bound on the order of $(\gamma_{\mathcal{P}}, k)_{\mathcal{P}}$-critical graphs in terms of $\Delta$ and $\gamma_{\mathcal{P}}$. Some properties of the extremal graphs are obtained.
Theorem 4.8. Let $\mathcal{H} \subseteq \mathcal{G}$ be induced-hereditary and closed under the union with $K_1$ and let $G$ be in $CV^1R_\mathcal{H}$. Then $|V(G)| \leq (\Delta(G) + 1)(\gamma_{\mathcal{H}}(G) - 1) + 1$. If equality holds then:

(i) if $x \in V(G)$ and $v \in G_{\mathcal{H}}(G - x)$ then $x \in \text{Fi}_{\mathcal{H}}(G - v)$ and $v \in \text{Fi}_{\mathcal{H}}(G - x)$;
(ii) for every $x \in V(G)$, $G_{\mathcal{H}}(G - x) = \text{Fi}_{\mathcal{H}}(G - x) - \text{Fi}_{\mathcal{H}}^1(G - x)$ and $G_{\mathcal{H}}(G - x)$ is an efficient dominating set of $G - x$;
(iii) ([7] when $\mathcal{H} = \mathcal{G}$) $G$ is regular;
(iv) $\gamma(G) = i(G)$;
(v) let $\mathcal{U} \subseteq \mathcal{G}$ be induced-hereditary and closed under union with $K_1$. Then $G$ is in $CV^1R_\mathcal{U}$;
(vi) ([16] when $\mathcal{H} = \mathcal{G}$) $b^+_{\mathcal{H}}(G) \leq \Delta(G) + 1 = \delta(G) + 1$ provided $\Delta(G) \geq 1$.

We need the following observation to prove Theorem 4.8.

Observation 4.9. Let $\mathcal{H} \subseteq \mathcal{G}$ be nondegenerate and let $G$ be a graph. Then $|V(G)| \leq (1 + \Delta(G))\gamma_{\mathcal{H}}(G)$. The equality holds if and only if each $\gamma_{\mathcal{H}}(G)$-set is efficient dominating and each $\gamma_{\mathcal{H}}$-good vertex of $G$ has the maximum degree.

Proof. Let $M = \{x_1, \ldots, x_k\}$ be a $\gamma_{\mathcal{H}}(G)$-set. Then $|V(G)| = \sum_{i=1}^{k} N[x_i, G] \leq \sum_{i=1}^{k} (\deg(x_i, G) + 1) \leq k(\Delta(G) + 1) = \gamma_{\mathcal{H}}(G)(\Delta(G) + 1)$. The equality holds if and only if $\deg(x_i, G) = \Delta(G)$, $i = 1, 2, \ldots, k$ and $\{N[x_1, G], N[x_2, G], \ldots, N[x_k, G]\}$ is a partition of $V(G)$. □

Proof of Theorem 4.8. If $G$ has no edges then the results are obvious. So, let $G$ have edges. Clearly $\Delta(G) \geq 2$ and $\gamma_{\mathcal{H}}(G) \geq 2$. Let $v \in V(G)$. Using Observation 4.9 we have $|V(G)| = |V(G - v)| + 1 \leq (1 + \Delta(G - v))\gamma_{\mathcal{H}}(G - v) + 1 \leq (1 + \Delta(G))(\gamma_{\mathcal{H}}(G) - 1) + 1$. Let equality hold and let $M = \{x_1, \ldots, x_k\}$ be a $\gamma_{\mathcal{H}}(G - v)$-set. It follows by Observation 4.9 that $M$ is an efficient dominating set of $G - v$ and $\deg(x_i, G - v) = \Delta(G)$, $i = 1, \ldots, k$. Hence to prove (iii) it suffices to prove (i).

(i) and (ii): Let $M$ be an efficient dominating set of $G - v$ and let $Q$ be an efficient dominating set of $G - x$ with $v \in Q$. Since $|Q| = \gamma_{\mathcal{H}}(G) - 1 = |M|$ it follows: (a) each vertex in $Q - \{v\}$ dominates a unique vertex of $M$, and (b) there exists exactly one vertex in $M$, say $w$, which is not dominated by $Q - v$. Since $M \cup \{v\}$ is independent (by Lemma 2.2), it follows that $w = x$. Therefore $x \in \text{Fi}_{\mathcal{H}}(G - v)$ and by symmetry, $v \in \text{Fi}_{\mathcal{H}}(G - x)$. Thus (i) holds and to prove (ii) it remains to show that $\text{Fi}_{\mathcal{H}}^{-1}(G - x)$ is empty. Suppose to the contrary $u \in \text{Fi}_{\mathcal{H}}^{-1}(G - x)$. By Observation 4.9 it follows that $|V(G)| - 2 = |V((G - x) - u)| \leq (\Delta(G) + 1)(\gamma_{\mathcal{H}}(G) - 2) = |V(G)| - \Delta(G) - 1$—a contradiction with $\Delta(G) \geq 2$.
(iv) Let \( v \in V(G) \) and let \( M \) be a \( \gamma_H(G - v) \)-set. Since \( M \) is independent (by (ii)), it follows by Lemma 2.2 that \( M \cup \{v\} \) is an independent \( \gamma_H(G) \)-set. Hence \( \gamma_H(G) = i(G) \).

(v) By (iv) it follows that \( \gamma(G) = \gamma_U(G) = i(G) \). By (ii) applied to the property \( U \) we have \( \gamma_U(G - v) = i(G - v) = i(G) - 1 = \gamma_U(G) - 1 \) for each \( v \in V(G) \).

(vi) Let \( v \in V(G) \) and let \( M \) be the unique \( \gamma_H(G - v) \)-set. Let \( x \in M \) and let \( y \in V(G) \) be adjacent. Consider the graph \( G_1 = (G - v) - xy \). Assume \( \gamma_H(G_1) \leq \gamma_H(G - v) \). Since \( |V(G_1)| = |V(G - v)| \), it follows by Observation 4.9 that \( \Delta(G_1) = \Delta(G - v) = \Delta(G) \), \( \gamma_H(G_1) = \gamma_H(G - v) \) and if \( M_1 \) is a \( \gamma_H(G_1) \)-set then (a) \( M_1 \) is efficient dominating, and (b) each vertex in \( M_1 \) has degree \( \Delta(G) \). Hence \( x \notin M_1 \). But then \( M_1 \neq M \) is a \( \gamma_H(G - v) \)-set—a contradiction with (ii). Thus \( \gamma_H(G_1) > \gamma_H(G - v) \).

Let \( G_v \) be the graph obtained from \( G \) after deleting all edges incident with \( v \) in \( G \). Since \( \mathcal{H} \) is induced-hereditary and closed under the union with \( K_1 \), \( \gamma_H(G_v - xy) = \gamma_H(G_1) + 1 > \gamma_H(G - v) + 1 = \gamma_H(G) \). Therefore \( b_H^+(G) \leq \Delta(G) + 1 \). \( \square \)

Examples of \( CV^1R \)-graphs \( G \) of order \((\Delta(G) + 1)(\gamma(G) - 1) + 1 \) may be found in [4], [10, p. 140] and [19].

**Proposition 4.10** ([13] when \( \mathcal{H} = \mathcal{G} \)). Let \( \mathcal{H} \subseteq \mathcal{G} \) be induced-hereditary and closed under the union with \( K_1 \) and let \( G \) be in \( CV^kR_H \). Then \( |V(G)| \leq (\Delta(G) + 1) \times (\gamma_H(G) - 1) + k \).

**Proof.** We proceed by induction on \( k \). If \( k = 1 \) then the result follows by Theorem 4.8. So, let \( G \) be in \( CV^kR_H \), \( k \geq 2 \), and not in \( CV^1R_H \). If \( x \in V_H^+(G) \) then there is \( y \in N(x, G) - V_H^+(G) \) (\( \mathcal{H} \) is induced-hereditary) and by Lemma 2.2, \( y \in V_H^0(G) \). Hence \( \gamma_H(G - y) = \gamma_H(G) \) and \( G - y \) is in \( CV^{k-1}R_H \). The result now follows by the inductive hypothesis. \( \square \)

The next conjecture concerning the case \( \mathcal{P} = \mathcal{G} \) is the main outstanding conjecture on the ordinary bondage number.

**Conjecture 4.11** (Teschner [20] when \( \mathcal{P} = \mathcal{G} \)). Let \( \mathcal{P} \subseteq \mathcal{G} \) be additive and hereditary. For any graph \( G \) which is in \( CV^1R_P \), \( b_P^+(G) \leq 1.5\Delta(G) \).

Particular support for this conjecture is the fact that \( b_P(C_{3k+1}) = 3 = 1.5\Delta(C_{3k+1}) \) [17]. Now let \( \mathcal{P} = \mathcal{G} \). Teschner [20] has shown that Conjecture 4.11 is true when \( \gamma(G) \leq 3 \). Observe that if \( G = K_t \times K_t \) for a positive integer \( t \geq 2 \), then \( b(G) = 1.5\Delta(G) \) as was found independently by Hartnell and Rall [9] and by Teschner [21].

Motivated by Theorem 4.8 and Lemma 2.5 we state the following:
Conjecture 4.12. Let $G$ be in $CV^1R_P$ where $P \subseteq G$ is induced-hereditary and closed under the union with $K_1$. If $\Delta(G) \geq 1$ and $|V(G)| = (\Delta(G) + 1)(\gamma_P(G) - 1) + 1$ then (a) $b_P^+(G) = \Delta(G) + 1$, and (b) $G$ is not in $CV^2R_P$.

References

[21] U. Teschner: The bondage number of a graphs $G$ can be much greater than $\Delta(G)$. Ars Comb. 43 (1996), 81–87.


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