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Kybernetika, Vol. 49 (2013), No. 1, 128--140

Persistent URL: `http://dml.cz/dmlcz/143244`

KYBERNETIKA — VOLUME *49* (2013), NUMBER 1, PAGES 128–140

ON THE WEAK ROBUSTNESS OF FUZZY MATRICES

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A matrix A in $(\max, \min)$-algebra (fuzzy matrix) is called weakly robust if $A^k \otimes x$ is an eigenvector of A only if x is an eigenvector of A. The weak robustness of fuzzy matrices are studied and its properties are proved. A characterization of the weak robustness of fuzzy matrices is presented and an $O(n^2)$ algorithm for checking the weak robustness is described.

Keywords: weak robustness, fuzzy matrices

Classification: 08A72, 90B35, 90C47

1. INTRODUCTION

Fuzzy matrices (the addition and the multiplication are formally replaced by operations of maximum and minimum) are appropriate for expressing applications of fuzzy discrete dynamic systems, graph theory, scheduling, knowledge engineering, cluster analysis, fuzzy systems and for describing diagnosis of technical devices [21, 22], medical diagnosis [17, 18] or fuzzy logic programs [9]. The problem studied in [17] and recently in [14, 15, 16] leads to the problem of finding the greatest invariants of a fuzzy system.

Consider a system $\mathcal{S}$ which supports web users buying objects $o_1, \dots, o_n$. Let $R = (r_{ij})$ denote the binary relation, where entry r_{ij} describes the level of preference of the object o_i to the object o_j (note that preference itself is a binary relation, while r_{ij} are real numbers expressing levels).

The question is: What are the maximum levels of interest for the objects which are not influenced by R? The question is studied for one web user (fuzzy matrix) and great number of web users (interval fuzzy matrix) and leads to the problem of finding the maximum level of interest of objects (greatest eigenvector of the (interval) fuzzy matrix) with entries corresponding to the fuzzy relation R.

The eigenproblem of fuzzy matrices and its connection to paths in digraphs were investigated in [2, 8]. Fuzzy matrices with interval coefficients are also of practical importance, see [11, 12, 13, 15].

The aim of this paper is to describe matrices for which $A^k \otimes x$ is an eigenvector of A for any x and any k only when x is eigenvector of A and to find polynomial algorithms for verifying the equivalent conditions and the corresponding properties of fuzzy matrices. The questions considered in this paper are analogous to those in [1], where *weakly stable* (robust) matrices in *max-plus* algebra are studied.

2. BACKGROUND OF THE PROBLEM

The fuzzy algebra $\mathcal{B}$ is a triple $(B, \oplus, \otimes)$, where $(B, \leq)$ is a bounded linearly ordered set with binary operations *maximum* and *minimum*, denoted by $\oplus$, $\otimes$.

The least element in B will be denoted by O, the greatest one by I.

By $\mathbb{N}$ we denote the set of all natural numbers and by $\mathbb{N}_0$ the set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The greatest common divisor of a set $S \subseteq \mathbb{N}$ is denoted by $\gcd S$. For a given natural $n \in \mathbb{N}$, we use the notations N and M for the set of all smaller or equal positive natural numbers, i. e., $N = \{1, 2, \ldots, n\}$ and $M = \{1, 2, \ldots, m\}$, respectively.

For any $n \in \mathbb{N}$, $B(n, n)$ denotes the set of all square matrices of order n and $B(n)$ the set of all n-dimensional column vectors over $\mathcal{B}$. If each entry of a matrix $A \in B(n, n)$ is equal to O we shall denote it as $A = O$. The matrix operations over $\mathcal{B}$ are defined formally in the same manner (with respect to $\oplus$, $\otimes$) as matrix operations over any field.

Let $x = (x_1, \ldots, x_n)^\top \in B(n)$ and $y = (y_1, \ldots, y_n)^\top \in B(n)$ be vectors. We write $x \leq y$ $(x < y)$ if $x_i \leq y_i$ $(x_i < y_i)$ holds for each $i \in N$. For a given square matrix $A \in B(n, n)$ let us denote $s_k(A) = A \oplus A^2 \oplus \cdots \oplus A^k$ where A^i stands for the i-fold iterated product $A \otimes A \otimes \ldots \otimes A$.

A square matrix is called diagonal if all its diagonal entries are elements of B and off-diagonal entries are O. A diagonal matrix with all diagonal entries equal to I is called a unit matrix and denoted by U. A matrix obtained from a diagonal matrix (unit matrix) by permuting the rows and/or columns is called a permutation matrix (unit permutation matrix) and denoted by P (P_U). If $C = P_U^T \otimes A \otimes P_U$ for some unit permutation matrix P_U then we say that A and C are equivalent (denoted by $A \approx C$), whereby the matrix P_U^T is the transpose of the matrix P_U (if $P_U = (p_{ij})$ then $P_U^T = (p_{ji})$) and $P_U^T \otimes P_U = P_U \otimes P_U^T = U$.

For a matrix $A \in B(n, n)$ the symbol $G(A) = (N, E)$ stands for a complete, arc-weighted digraph associated with A, i. e., the node set of $G(A)$ is N, and the capacity of any arc (i, j) is a_{ij}. In addition, for given $h \in B$, the *threshold digraph* $G(A, h)$ is the digraph with the node set N and with the arc set $E = \{(i, j);\ i, j \in N,\ a_{ij} \geq h\}$. A path in the digraph $G(A) = (N, E)$ is a sequence of nodes $p = (i_1, \ldots, i_{k+1})$ such that $(i_j, i_{j+1}) \in E$ for $j = 1, \ldots, k$. The number k is the length of the path p and is denoted by $\ell(p)$. If $i_1 = i_{k+1}$, then p is called a cycle and it is called an elementary cycle if moreover $i_j \neq i_m$ for $j, m = 1, \ldots, k$. A digraph $G = (N, E)$ without cycles is called *acyclic*. If $G = (N, E)$ contains at least one cycle G is called *cyclic*.

A matrix A is called *generalized Hamiltonian permutation* if all nonzero entries of A lie on a Hamiltonian cycle (the threshold digraph $G(A, h)$, $h = \min\{a_{ij};\ a_{ij} > O \wedge i, j \in N\}$ is elementary cycle containing all nodes).

By a *strongly connected component* $\mathcal{K}$ of $G(A, h) = (N, E)$ we mean a subdigraph $\mathcal{K}$ generated by a non-empty subset $K \subseteq N$ such that any two distinct nodes $i, j \in K$ are contained in a common cycle and K is a maximal subset with this property. A strongly connected component $\mathcal{K}$ of a digraph is called non-trivial, if there is a cycle of positive length in $\mathcal{K}$. For any non-trivial strongly connected component $\mathcal{K}$ the *period* of $\mathcal{K}$ is defined as

$$\operatorname{per} \mathcal{K} = \gcd \{ \ell(c);\ c \text{ is a cycle in } \mathcal{K},\ \ell(c) > 0 \}.$$

If $\mathcal{K}$ is trivial, then $\operatorname{per} \mathcal{K} = 1$. By $\mathrm{SCC}^\star(G)$ we denote the set of all non-trivial strongly connected components of G. The set of all strongly connected components of G is denoted by SCC(G).

Let $A \in B(n,n)$ and $x \in B(n)$. The orbit $O(A,x)$ of $x = x^{(0)}$ generated by A is the sequence

$$x^{(0)}, x^{(1)}, x^{(2)}, \dots, x^{(n)}, \dots,$$

where $x^{(r)} = A^r \otimes x^{(0)}$ for each $r \in \mathbb{N}$.

For a given matrix $A \in B(n,n)$, the number $\lambda \in B$ and the n-tuple $x \in B(n)$ are the so-called *eigenvalue* of A and λ-*eigenvector* of A, respectively, if

$$A \otimes x = \lambda \otimes x.$$

The *eigenspace* $V(A,\lambda)$ is defined as the set of all λ-eigenvectors of A with associated eigenvalue λ, i. e.,

$$V(A,\lambda) = \{x \in B(n);\ A \otimes x = \lambda \otimes x\}.$$

There is a well-known connection between the entries in powers of matrices and paths in associated digraphs: (i,j)th entry a^k_{ij} in A^k is equal to the maximum capacity of a path from $\mathcal{P}^k_{ij}$, where $\mathcal{P}^k_{ij}$ is the set of all paths of length k beginning at node i and ending at node j. If $\mathcal{P}_{ij}$ denotes the set of all paths from i to j, then $a^*_{ij} = \bigoplus_{k=1} a^k_{ij}$ is the maximum capacity of a path from $\mathcal{P}_{ij}$ and a^*_{jj} is the maximum capacity of a cycle containing node j.

Theorem 2.1. (Zimmermann [23]) Let $(B,\oplus,\otimes)$ be a fuzzy algebra and $\lambda \in B$. Then each vector u_j with components $\lambda \otimes a^*_{ij} \otimes a^*_{jj}$ for $i \in N$ is a λ-eigenvector of A.

Let a matrix $A = (a_{ij}) \in B(n,n)$ and $\lambda \in B$. Let us define the greatest λ-eigenvector $x^*(A,\lambda)$ corresponding to a matrix A and λ as

$$x^*(A,\lambda) = \bigoplus_{x \in V(A,\lambda)} x.$$

In [19, 20], it has been proved that for a given eigenvalue λ of A the greatest λ-eigenvector exists and in [2] it was stated that the greatest I-eigenvector $x^*(A,I)$ exists for every matrix A and its entries are given by the formula $x^*_i(A,I) = \bigoplus_j a^*_{ij} \otimes a^*_{jj}$. The greatest I-eigenvector $x^*(A,I)$ can be computed by the following iterative procedure (see [2]). Let us denote $x^1_i(A) = \bigoplus_{j \in N} a_{ij}$ for each $i \in N$ and $x^{k+1}(A) = A \otimes x^k(A)$ for all $k \in \{1,2,\dots\}$. Then

$$x^{k+1}(A) \le x^k(A) \ \text{ and } x^*(A,I) = x^n(A). \tag{1}$$

For every matrix $A \in B(n,n)$ denote

$$c(A) = \bigotimes_{i \in N} \bigoplus_{j \in N} a_{ij}, \quad \text{and} \quad c^*(A) = (c(A),\dots,c(A))^T \in B(n).$$

A matrix $A \in B(n,n)$ is *ultimately periodic* if there is a natural number p such that the following holds for some $\lambda \in B$ and natural number R :

$$A^{k+p} = \lambda \otimes A^k \ \text{ for all } \ k \ge R.$$

The smallest natural number p with above property is called the period of A, denoted by $\operatorname{per}(A,\lambda)$. The smallest R with above property is called the *defect* of A, denoted by $\operatorname{def}(A,\lambda)$. For $\lambda = I$ let us denote $\operatorname{per}(A,I)$ by abbreviation $\operatorname{per} A$. A matrix A with $\operatorname{per} A = 1$ is called a *stationary matrix*.

By linearity of B, any element of any power of the matrix A is equal to some element of A. Therefore, the sequence of powers of A contains only finitely many different matrices with entries of A.

Theorem 2.2. (Gavalec [5]) Let $A \in B(n,n)$. Then

$$\operatorname{per} A \;=\; \operatorname{lcm}\{\, \operatorname{per}\mathcal{K};\ \mathcal{K} \in \mathrm{SCC}^{\star}(G)\,\}.$$

Let us denote

$$T(A,\lambda) = \{x \in B(n);\ O(A,x) \cap V(A,\lambda) \neq \emptyset\},$$

$$T^{*}(A,\lambda) = \{x \in B(n);\ x^{*}(A,\lambda) \in O(A,x)\}.$$

The set $T(A,\lambda)$ ($T^{*}(A,\lambda)$) allows us to describe matrices for which a λ-eigenvector (the greatest λ-eigenvector) is reached with any start vector. It is easily seen that $x^{*}(A,\lambda) \geq c^{*}(A)$ holds true and $x^{*}(A,\lambda)$ can not be reached with a vector $x \in B(n)$, $x < c^{*}(A)$.

Let us denote the following set by $M(A) = \{x \in B(n);\ x < c^{*}(A)\}$.

Definition 2.1. Let $A \in B(n,n)$ be a matrix and $\lambda \in B$. Then A is called λ-robust if $T(A,\lambda) = B(n)$ and A is called strongly λ-robust if $T^{*}(A,\lambda) = B(n) \setminus M(A)$, respectively.

Theorem 2.3. (Plavka and Szabó [14]) Let $A = (a_{ij}) \in B(n,n)$, $\lambda > \max_{i,j\in N} a_{ij}$. Then A is λ-robust if and only if $\operatorname{per}(A,\lambda) = 1$.

Theorem 2.4. (Plavka and Szabó [14]) Let $A \in B(n,n)$ and $\lambda > c(A)$. Then A is strongly λ-robust if and only if $x^{*}(A,\lambda) = c^{*}(A)$ and $G(A,c(A))$ is a strongly connected digraph with period equal to 1.

3. SOLVABILITY OF A SYSTEM OF FUZZY LINEAR EQUATIONS

In this section we shall suppose that A is a square matrix and recall the crucial results concerning a system of fuzzy linear equations $A \otimes x = b$ (see [3, 4, 6, 22]). We use the notation introduced in [3, 6] adapted for square matrices.

For any $j \in N$ denote

$\overline{x}_j(A,b) = \min\{b_i;\ a_{ij} > b_i\}$, whereby $\min\emptyset = I$ by definition,

$L_j(A,b) = \{i \in N; a_{ij} \otimes \overline{x}_j = b_i\}$,

$S(A,b) = \{x \in B(n);\ A \otimes x = b\}$.

Unique solvability can be characterized using the notion of minimal covering. If D is a set and $\mathcal{E} \subseteq \mathcal{P}(D)$ is a set of subsets of D, then $\mathcal{E}$ is said to be a covering of D, if $\bigcup\mathcal{E} = D$ and a covering $\mathcal{E}$ of D is called minimal, if $\bigcup(\mathcal{E} - F) \neq D$ holds for every $F \in \mathcal{E}$.

Theorem 3.1. (Cechlárová [3]) Let $A \in B(n,n)$ be a fuzzy matrix and $b \in B(n)$ be a vector. Then the following conditions are equivalent:

(i) $S(A,b) \neq \emptyset$,

(ii) $\overline{x}(A,b) \in S(A,b)$,

(iii) $\bigcup_{j \in N} L_j(A,b) = N$.

Theorem 3.2. (Cechlárová [3]) Let $A \in B(n,n)$ be a fuzzy matrix and $b \in B(n)$ be a vector. Then $S(A,b) = \{\overline{x}(A,b)\}$ if and only if $\{L_1,\ldots,L_n\}$ is a minimal covering of the form $L_{\pi(i)} = \{i\}$ for a permutation $\pi \in P_n$ and for all i with $a_{i\pi(i)} = b_i$ it holds $b_i = I$.

4. WEAK λ-ROBUSTNESS OF FUZZY MATRICES

It follows from the definitions of $V(A,\lambda)$ and $T(A,\lambda)$ that $x \in V(A,\lambda)$ implies $A \otimes x \in V(A,\lambda)$ and $V(A,\lambda) \subseteq T(A,\lambda) \subseteq B(n)$ is fulfilled for every matrix $A \in B(n,n)$ and $\lambda \in B$.

The next lemma describes a universal criterion for weak robustness (that is, for any operator in any "extremal" algebra, see [1, 14]).

Lemma 4.1. Let $A = (a_{ij}) \in B(n,n)$, $\lambda \in B$. Then $T(A,\lambda) = V(A,\lambda)$ if and only if for every $x \in B(n):\ A \otimes x \in V(A,\lambda) \Leftrightarrow x \in V(A,\lambda)$.

Proof. Let us notice first that $x \in V(A,\lambda) \Rightarrow A \otimes x \in V(A,\lambda)$ and $V(A,\lambda) \subseteq T(A,\lambda)$ hold true for every matrix A and every λ. Suppose now that $V(A,\lambda) = T(A,\lambda)$ and $A \otimes x \in V(A,\lambda)$. Then $x \in T(A,\lambda)$ and hence $x \in V(A,\lambda)$.

For the converse implication, let us assume that $A \otimes x \in V(A,\lambda) \Rightarrow x \in V(A,\lambda)$ holds for every $x \in B(n)$ and $x \in T(A,\lambda)$. Then $A^k \otimes x \in V(A,\lambda)$ for some k implies $A^k \otimes x \in V(A,\lambda)$, $A^{k-1} \otimes x \in V(A,\lambda)$,..., $x \in V(A,\lambda)$. $\square$

In general, $T(A,\lambda) \neq V(A,\lambda)$. Let us consider $B = [0,10]$, $\lambda = 10$ and the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix}.$$

Vector $x = (5,5,5)^T$ does not belong to $V(A,10)$ but $A \otimes x = (1,2,2)^T \in V(A,10)$ which means that $T(A,10) \neq V(A,10)$.

Definition 4.1. Let $A = (a_{ij}) \in B(n,n)$, $\lambda \in B$. A matrix A is called *weakly λ-robust* if $T(A,\lambda) = V(A,\lambda)$.

Lemma 4.2. Let $A = (a_{ij}) \in B(n,n)$, $A = O$ and $\lambda \in B$. Then A is weakly λ-robust if and only if $\lambda = O$.

Proof. Let us suppose that $A = O$, A is weakly λ-robust and $\lambda > O$. Then it is easy to see that $(A \otimes (A \otimes (I, \dots, I)^\top)) = (A \otimes (O, \dots, O)^\top) = (O, \dots, O)^\top$, i.e. $A \otimes (I, \dots, I)^\top \in V(A, \lambda)$ and $(I, \dots, I)^\top \notin V(A, \lambda)$, a contradiction with Lemma 4.1. The converse implication trivially follows. $\square$

Let us denote

$$C_A = A \begin{pmatrix} i_1 & i_2 & \dots & i_k \\ j_1 & j_2 & \dots & j_k \end{pmatrix} = \begin{pmatrix} a_{i_1 j_1} & \dots & a_{i_1 j_k} \\ \vdots & & \vdots \\ a_{i_k j_1} & \dots & a_{i_k j_k} \end{pmatrix}$$

for $i_1, \dots, i_k, j_1, \dots, j_k \in N$ the $k \times k$ matrix, $1 \le k \le n$, which arose from the matrix A by deleting O columns and corresponding rows, hence $(i_1, \dots, i_k) = (j_1, \dots, j_k)$.

The eigenspace $V(A, O)$ consists of vectors $x = (x_1, \dots, x_n)^\top \in B(n)$, where

$$x_i = \begin{cases} \alpha \in B, & \text{for } \ i \in \{j \in N;\ \max_{k \in N} a_{kj} = O\} \\ O, & \text{for } i \notin \{j \in N;\ \max_{k \in N} a_{kj} = O\}. \end{cases} \tag{2}$$

Lemma 4.3. If $A = (a_{ij}) \in B(n, n), A \neq O$ and $\lambda = O$ then A is weakly λ-robust if and only if C_A contains no O columns.

Proof. Let us suppose that $A = (a_{ij}) \in B(n, n)$, $A \neq O$, $\lambda = O$, $C_A \in B(k, k)$, C_A contains no O columns and $A \otimes x \in V(A, O)$, i.e. $A \otimes x = (\alpha_1, \dots, \alpha_n)^\top$, where $(\alpha_1, \dots, \alpha_n)^\top$ has a form of (2). The vector $(O, \dots, O)^\top \in B(k)$ is the only solution of the system $C_A \otimes y = (O, \dots, O)^\top$ and then each solution x of the system $A \otimes x = (\alpha_1, \dots, \alpha_n)^\top$, where $(\alpha_1, \dots, \alpha_n)^\top$ of the form (2) is an element of $V(A, O)$.

To prove the converse implication suppose that A is weakly λ-robust,

$$C_A = A \begin{pmatrix} 1 & 2 & \dots & k \\ 1 & 2 & \dots & k \end{pmatrix},$$

$1 \le k < n$ and C_A contains O column, say first. Denote the set $\{i \in N; a_{i1} > O\}$ by $J^{(1)}$. The set $J^{(1)}$ is subset of $\{k+1, \dots, n\}$, $|J^{(1)}| \ge 1$ and for the vector $\tilde{x} = (\tilde{x}_1, \dots, \tilde{x}_n)^\top$ such that

$$\tilde{x}_j = \begin{cases} I, & \text{if } \ j = 1 \\ O, & \text{if } \ j > 1 \end{cases} \quad \text{we get} \quad (A \otimes \tilde{x})_i = \begin{cases} a_{i1}, & \text{if } \ i \in J^{(1)} \\ O, & \text{otherwise.} \end{cases}$$

But then $A \otimes \tilde{x} \in V(A, O)$ and $\tilde{x} \notin V(A, O)$. This is a contradiction with the weak λ-robustness of A. $\square$

Lemma 4.4. Let $A \neq O$ and $\lambda > O$. If A is weakly λ-robust then A contains no O column and no O row.

Proof. Let us suppose that $A \neq O$, $\lambda > O$, A is weakly λ-robust and A contains rows with O entries. Denote

$$J^{(2)} = \{i \in N; \max_{j \in N} a_{ij} = O\} \quad \text{and} \quad \tilde{a} = \lambda \otimes \bigotimes_{a_{ij} > O} a_{ij} > O.$$

Then the equalities $(A \otimes (A \otimes x))_i = (A \otimes x)_i = O$ hold for each $i \in J^{(2)}$ and for each $x = (x_1, \ldots, x_n)^\top \in B(n)$. Specially, the greatest λ-eigenvector $x^*(A,\lambda)$ has a form $x^*(A,\lambda) = (x_1^*(A,\lambda), \ldots, x_n^*(A,\lambda))^\top$ with $x_i^*(A,\lambda) = O$ for $i \in J^{(2)}$ and if $x_i^*(A,\lambda) > O$ then $x_i^*(A,\lambda) \geq \tilde{a}$. For an arbitrary but fixed index $i \in J^{(2)}$ let us construct the vector

$$\tilde{x} = (x_1^*(A,\lambda), \ldots, x_{i-1}^*(A,\lambda), \tilde{a}, x_{i+1}^*(A,\lambda), \ldots, x_n^*(A,\lambda))^\top.$$

If $x^*(A,\lambda) = (O, \ldots, O)^\top$ then the vector $\tilde{x}$ contradicts Lemma 4.1.

Suppose now that $x^*(A,\lambda) \neq (O, \ldots, O)^\top$ and denote $J^{(3)} = \{k \in N; x_k^*(A,\lambda) > O\}$. Then for each $k \in J^{(3)}$ we get

$$\begin{aligned}(A \otimes \tilde{x})_k &= \bigoplus_{j \neq i} a_{kj} \otimes x_j^*(A,\lambda) \oplus a_{ki} \otimes \tilde{a} = \bigoplus_{j \neq i} a_{kj} \otimes x_j^*(A,\lambda \\ &= \bigoplus_{j \neq i} a_{kj} \otimes x_j^*(A,\lambda) \oplus a_{ki} \otimes O = x_k^*(A,\lambda).\end{aligned}$$

Hence, the equalities $A \otimes (A \otimes \tilde{x}) = A \otimes x^*(A,\lambda) = \lambda \otimes x^*(A,\lambda) = \lambda \otimes (A \otimes \tilde{x})$ and the inequality $A \otimes \tilde{x} \neq \lambda \otimes \tilde{x}$ hold true. However, this is a contradiction with Lemma 4.1.

Assume now that nth column of A is O column. Then the greatest λ-eigenvector $x^*(A,\lambda)$ has a form $x^*(A,\lambda) = (x_1^*(A,\lambda), \ldots, x_{n-1}^*(A,\lambda), I)^\top$ and the vector

$$\tilde{x} = (x_1^*(A,\lambda), \ldots, x_{n-1}^*(A,\lambda), \tilde{x}_n)^\top \text{ with } \tilde{x}_n < \lambda$$

is not a λ-eigenvector of A and the equality $A \otimes (A \otimes \tilde{x}) = \lambda \otimes (A \otimes \tilde{x})$ contradicts Lemma 4.1. □

Theorem 4.1. Let $A \neq O$ and $\lambda > O$. If A is weakly λ-robust then A is a permutation matrix.

Proof. Let us suppose that $A \neq O$, $\lambda > O$, A is weakly λ-robust and matrix A is not permutation, i.e. A contains no O column and O row, by Lemma 4.4 and there exists at least one column, say nth, such that at least two entries are not equal to O. Let us denote the vector $\tilde{b} = (\tilde{a}, \ldots, \tilde{a})^\top$ with $O < \tilde{a} = \lambda \otimes \bigotimes_{a_{ij} > O} a_{ij}$.

Let us consider the system of fuzzy equations $A \otimes x = \tilde{b}$. By Theorem 3.2 the system $A \otimes x = \tilde{b}$ is solvable, $\overline{x}(A,\tilde{b}) = \tilde{b}$, $\bigcup_{j \in N} L_j(A,\tilde{b}) = N$ and the system of sets $\{L_1(A,\tilde{b}), \ldots, L_n(A,\tilde{b})\}$ is not a minimal covering of N ($|L_n(A,\tilde{b})| \geq 2$ and A contains no O column and no O row). Moreover there exists $k \in N, k \neq n$ such that $\bigcup_{j \in N - \{k\}} L_j(A,\tilde{b}) = N$. Let us define the vector $x = (x_1, \ldots, x_n)^\top$ with $x_j = \tilde{a}$ for $j \neq k$ and $x_k = O$. Then we get $x \in S(A,\tilde{b})$, $A \otimes (A \otimes x) = \lambda \otimes (A \otimes x)$ and $A \otimes x \neq \lambda \otimes x$, i.e. $A \otimes x \in V(A,\lambda)$ and $x \notin V(A,\lambda)$, a contradiction with Lemma 4.1. □

Lemma 4.5. If $A \otimes x \in V(A,\lambda)$ then $\lambda \otimes (A \otimes x) = A^2 \otimes x = A^3 \otimes x = \ldots$.

Proof. Let us suppose that $A \otimes x \in V(A,\lambda)$ and compute now the kth power of A multiplying by x for $k \geq 3$, i.e. $A^k \otimes x = A^{k-2} \otimes (A^2 \otimes x) = \lambda \otimes (A^{k-1} \otimes x) = \cdots = \lambda \otimes (A \otimes x)$. □

Denote

$$c^+(A) = \min_{i,j \in N} \{a_{ij}; a_{ij} > c(A)\}.$$

We assume that the minimum of the empty set is equal to I.

Theorem 4.2. (Plavka and Szabó [14]) Let $A \in B(n,n)$, $\lambda \in B$, $\lambda > c(A)$. Then $x^*(A,\lambda) = c^*(A)$ if and only if $G(A, c^+(A))$ is an acyclic digraph.

Theorem 4.3. [2] Let $A \in B(n,n)$ and $\lambda = I$. Every constant vector $x = (\alpha, \ldots, \alpha)^\top$ with $\alpha \le c(A)$ is an eigenvector of A, and no constant vector with entries $\alpha > c(A)$ is an eigenvector of A.

Let $A = (a_{ij}) \in B(n,n)$ be generalized Hamiltonian permutation, $\beta \in B$. Let us denote

$$m_A = \max_{i,j \in N} a_{ij}, \quad J_A = \{j \in N;\ \max_{i \in N} a_{ij} = c(A)\}.$$

The set of trivial β-eigenvectors and non-trivial $c(A)$-eigenvectors of A is denoted by

$$V^{(1)}(A,\beta) = \{(\alpha, \ldots, \alpha)^\top;\ \alpha \in B \wedge \alpha \le \beta\}$$

and

$$V^{(2)}(A, c(A)) = \{(\alpha_1, \ldots, \alpha_n)^\top;\ c(A) \le \alpha_i \le I \text{ for } i \in J_A \ \wedge\ \alpha_i = c(A) \text{ for } i \notin J_A\},$$

respectively.

Theorem 4.4. Let $A = (a_{ij}) \in B(n,n)$ be generalized Hamiltonian permutation and $\lambda \in B$. Then

$$V(A,\lambda) = \begin{cases} V^{(1)}(A,\lambda), & \text{if } \lambda < c(A) \\ V^{(1)}(A,c(A)) \cup V^{(2)}(A,c(A)), & \text{if } \lambda = c(A) \\ V^{(1)}(A,c(A)), & \text{if } \lambda > c(A). \end{cases}$$

P r o o f. Let us suppose that $A = (a_{ij}) \in B(n,n)$ is generalized Hamiltonian permutation, $\lambda \in B$ and $x \in V(A,\lambda)$. Then $A \otimes x = \lambda \otimes x$ is the system of fuzzy linear equalities $a_{i\pi(i)} \otimes x_{\pi(i)} = \lambda \otimes x_i$, $i \in N$. Since A is generalized Hamiltonian permutation then there exists $i \in N$, say $i = 1$, such that $a_{1\pi(1)} = c(A)$, i.e. $\pi(1) \in J_A$ and

$$a_{1\pi(1)} \otimes x_{\pi(1)} = \lambda \otimes x_1 \Leftrightarrow c(A) \otimes x_{\pi(1)} = \lambda \otimes x_1.$$

If $\lambda > c(A)$ then the equality $c(A) \otimes x_{\pi(1)} = x_1$ implies $x_1 \le c(A)$. Moreover the matrix A is generalized Hamiltonian permutation then there exists $j_1 \in N$, $j_1 \ne 1$ such that $a_{j_1 1} \otimes x_1 = \lambda \otimes x_{j_1} \Leftrightarrow x_{j_1} = x_1$ because $a_{j_1 1} \ge c(A)$. Similarly, we can show that $x_{j_1} = x_{j_2} = \cdots = x_{j_{n-1}} = x_1 \le c(A)$ and $x \in V^{(1)}(A, c(A))$.

If $\lambda < c(A)$ from the above we get $x_{\pi(1)} = \lambda \otimes x_1$ which implies $x_{\pi(1)} \le \lambda$. The matrix A is generalized Hamiltonian permutation then there exists $j_1 \in N$, $j_1 \ne \pi(1)$ such that $a_{\pi(1) j_1} \otimes x_{j_1} = \lambda \otimes x_{\pi(1)} \Leftrightarrow x_{j_1} = x_{\pi(1)}$ because $a_{j_1 1} \ge c(A)$ and $x_{\pi(1)} \le \lambda$. Similarly, we can show that $x_{j_1} = x_{j_2} = \cdots = x_{j_{n-1}} = x_{\pi(1)} \le \lambda$ and $x \in V^{(1)}(A,\lambda)$.

Let us consider $\lambda = c(A)$ and $\pi(1) \in J_A$. Then the equality $c(A) \otimes x_{\pi(1)} = c(A) \otimes x_1$ holds true and there exists $j_1 \in N$, $j_1 \neq \pi(1)$ such that $a_{\pi(1)j_1} \otimes x_{j_1} = c(A) \otimes x_{\pi(1)}$. Now we shall consider two cases.

Case 1: $j_1 \in J_A$, i.e. $a_{\pi(1)j_1} = c(A) = \lambda$ then

$$c(A) \otimes x_{j_1} = c(A) \otimes x_{\pi(1)} \Leftrightarrow x_{j_1} = x_{\pi(1)} < c(A) \vee (x_{j_1} \geq c(A) \wedge x_{\pi(1)} \geq c(A)).$$

Case 2: $j_1 \notin J_A$, i.e. $a_{\pi(1)j_1} > c(A) = \lambda$ then

$$x_{j_1} = c(A) \otimes x_{\pi(1)} \Leftrightarrow x_{j_1} = x_{\pi(1)} < c(A) \vee (x_{j_1} = c(A) \wedge x_{\pi(1)} \geq c(A)).$$

Similarly, applying the above process for coordinates $x_{j_2}, \dots, x_{j_{n-1}}$ of x we can conclude that $x \in V^{(1)}(A, c(A))$ or $x \in V^{(2)}(A, c(A))$. □

Theorem 4.5. Let $A = (a_{ij}) \in B(n,n)$, $A \neq O$ be an generalized Hamiltonian permutation matrix and $\lambda > O$. Then A is weakly λ-robust if and only if $\lambda < c(A)$ or all entries on the Hamiltonian cycle are equal to λ (i.e. $m_A = c(A) = \lambda$).

Proof. Let us suppose that $A = (a_{ij}) \in B(n,n)$, $A \neq O$ is generalized Hamiltonian permutation, $\lambda > O$, A is weakly λ-robust, $\lambda \geq c(A)$ and all entries on the Hamiltonian cycle are not equal to λ. Then the last two conditions can be equivalently rewritten as follows

$$\lambda \geq c(A) \wedge (m_A \neq c(A) \ \vee \ c(A) \neq \lambda) \Leftrightarrow (\lambda \geq c(A) \ \wedge \ c(A) < m_A) \ \vee \ \lambda > c(A).$$

(i) Let us assume that A is weakly λ-robust and $\lambda \geq c(A) \wedge \ c(A) < m_A$ (hence $n \geq 2$). The matrix A contains in each row and each column just one no O entry by the Theorem 4.1. We shall consider two cases.

CASE 1. A is weakly λ-robust and $\lambda > c(A) \wedge \ c(A) < m_A$. The eigenspace $V(A, \lambda)$ is equal to the set $\{(\alpha, \dots, \alpha)^\top; \alpha \leq c(A)\}$ by Theorem 4.4. Let us consider the vector $x = (x_1, \dots, x_n)^\top$, where $x_i = c(A)$ for $i \notin J_A$ and $x_i = m_A$ for $i \in J_A$. Then the vector x is a solution of the system $A \otimes x = \lambda \otimes c^*(A) = c^*(A)$. Moreover, the equalities $\lambda \otimes x_i = \lambda \otimes m_A > c(A) = (A \otimes x)_i$ hold for $i \in J_A$ and hence we obtain $A \otimes x \neq \lambda \otimes x$. The equalities

$$A \otimes (A \otimes x) = A \otimes c^*(A) = \lambda \otimes c^*(A) = \lambda \otimes (A \otimes x)$$

imply $A \otimes x \in V(A, \lambda)$ and $x \notin V(A, \lambda)$ what is a contradiction with Lemma 4.1.

CASE 2. A is weakly λ-robust and $\lambda = c(A) \wedge \ c(A) < m_A$. Each solution x of the system $A \otimes x = \lambda \otimes c^*(A) = c^*(A)$ has the form $x = (x_1, \dots, x_n)^\top$, where $x_i = c(A)$ for $i \notin J_A$ and $x_i = \alpha \geq c(A), \alpha \in B$ for $i \in J_A$. Since $c(A) < m_A$ and $G(A, c(A))$ is a Hamiltonian cycle then there exists $r, s \in N$ such that $a_{rs} > c(A)$ and $r \in J_A, \ s \notin J_A$. Denote the vectors $x', \tilde{x}$ as follows:

$$x' = (x'_1, \dots, x'_n)^\top, \text{ where } x'_k = \begin{cases} a_{rs}, & \text{for } \ k = s \\ c(A), & \text{otherwise,} \end{cases}$$

$$\tilde{x} = (\tilde{x}_1, \dots, \tilde{x}_n)^\top, \text{ where } \tilde{x}_k = \begin{cases} a_{rs}, & \text{for } k = r \\ c(A), & \text{otherwise.} \end{cases}$$

Then $A \otimes x' \neq \lambda \otimes x'$ and the equalities

$$A \otimes (A \otimes x') = A \otimes \tilde{x} = \lambda \otimes \tilde{x} = \lambda \otimes (A \otimes x')$$

imply $A \otimes x' \in V(A, \lambda)$ and $x' \notin V(A, \lambda)$, what is a contradiction with Lemma 4.1.

(ii) Now, let us assume that A is weakly λ-robust and $\lambda > c(A) = m_A$ (if $\lambda > c(A) \wedge\ c(A) < m_A$ see CASE 1). Then $x = (I, \dots, I)^\top$ is a solution of the system $A \otimes (A \otimes x) = \lambda \otimes (A \otimes x)$ and $A \otimes x \neq \lambda \otimes x$, a contradiction with Lemma 4.1.

To prove the converse implication let us consider $\lambda < c(A)$ or $\lambda = c(A) = m_A$. If $\lambda < c(A)(\leq m_A)$ then $V(A, \lambda) = \{(\alpha, \dots, \alpha)^\top; \alpha \leq \lambda\}$ by Theorem 4.4. Let us suppose that $A \otimes x \in V(A, \lambda)$ then $A \otimes x = (\alpha, \dots, \alpha)^\top$, $\alpha \leq \lambda$. The system $A \otimes x = (\alpha, \dots, \alpha)^\top$ has just one solution $\overline{x} = \overline{x}(A, (\alpha, \dots, \alpha)^\top) = (\alpha, \dots, \alpha)^\top$ by Theorem 3.2. Moreover the equality $A \otimes \overline{x} = \overline{x} = \lambda \otimes \overline{x}$ implies $\overline{x} \in V(A, \lambda)$ and the assertion follows.

Consider now that $\lambda = c(A) = m_A$. The eigenspace $V(A, \lambda)$ is equal to the union of sets $V^{(1)}(A, c(A))$ and $V^{(2)}(A, c(A))$ by Theorem 4.4. If $A \otimes x \in V^{(1)}(A, c(A))$ then the system $A \otimes x = \lambda \otimes x = (\alpha, \dots, \alpha)^\top$ for $\alpha < \lambda = c(A)$ has just one solution $x = (\alpha, \dots, \alpha)^\top \in V^{(1)}(A, \lambda)$ by Theorem 3.2. If $A \otimes x \in V^{(2)}(A, c(A))$ then either $[A \otimes x]_i = c(A)$ and $\overline{x}_i = c(A)$ for $i \notin J_A$ or $[A \otimes x]_i \in [c(A), I]$ and $\overline{x}_i = I$ for $i \in J_A$. Hence we get $x \in V^{(2)}(A, \lambda)$ and the matrix A is λ-robust. $\square$

Now, let us suppose that $A = (a_{ij}) \in B(n, n)$ is a permutation matrix and $\lambda \in B$. Then the digraph $G(A, c(A)$ is the set of Hamiltonian cycles, say $c_i = (k_1^i, \dots, k_{l_i}^i)$ for $i \in S = \{1, \dots, s\}$. Without loss of generality the matrix A can be considered in block-diagonal form (denoted by $A = (A_1, \dots, A_s)$)

$$A = \begin{pmatrix} A_1 & O & \dots & O \\ O & A_2 & \dots & O \\ \vdots & & & \\ O & O & \dots & A_s \end{pmatrix}, \tag{3}$$

where each submatrix A_i is generalized Hamiltonian permutation and corresponds to the cycle $c_i = (k_1^i, \dots, k_{l_i}^i)$. By Lemma 4.4 the eigenspace of the matrix A_i can be described as follows

$$V(A_i, \lambda) = \begin{cases} V^{(1)}(A_i, \lambda), & \text{if } \lambda < c(A_i) \\ V^{(1)}(A_i, c(A_i)) \cup V^{(2)}(A_i, c(A_i)), & \text{if } \lambda = c(A_i) \\ V^{(1)}(A_i, c(A_i)), & \text{if } \lambda > c(A_i). \end{cases}$$

Moreover, a matrix A is permutation if and only if $A_1, \dots, A_s$ are generalized Hamiltonian permutation, $c(A) \leq c(A_i)$ and there exists at least one $i \in \{1, \dots, s\}$ such that $c(A_i) = c(A)$.

As a consequence of the above consideration the following corollaries result.

Corollary 4.1. Let $A \in B(n,n)$, $A \neq O$, $A = (A_1, \ldots, A_s)$, $s \geq 2$ be a block-diagonal permutation matrix and $\lambda \in B$. Then

$$V(A,\lambda) = \{(\alpha_1^1, \ldots, \alpha_{l_1}^1, \ldots, \alpha_1^s, \ldots, \alpha_{l_s}^s)^\top; (\alpha_1^i, \ldots, \alpha_{l_i}^i)^\top \in V(A_i, \lambda),\ i \in S\}.$$

Corollary 4.2. Let $A \in B(n,n)$, $A \neq O$, $A = (A_1, \ldots, A_s)$, $s \geq 2$ be a block-diagonal permutation matrix and $\lambda \in B$. Then A is weakly λ-robust if and only if $(\forall i \in S)[\lambda < c(A_i)\ \vee\ \lambda = c(A_i) = m(A_i)]$.

Theorem 4.6. Let $A \in B(n,n)$, $A \neq O$ be a permutation matrix, $\lambda \in B$ and $C = P_U^T \otimes A \otimes P_U (= (A_1, \ldots, A_s))$. Then A is weakly λ-robust if and only if C is weakly λ-robust.

Proof. It is clear that $x \in V(A,\lambda) \Leftrightarrow P_U^T \otimes x \in V(C,\lambda)$. Let us suppose that $x \in B(n)$ is an arbitrary but fixed vector such that $x \in V(A,\lambda) \Leftrightarrow A \otimes x \in V(A,\lambda)$ and $C \otimes y \in V(C,\lambda)$. Then for $x = P_U \otimes y$ we get the following result

$$C \otimes y = (P_U^T \otimes A \otimes P_U) \otimes y = (P_U^T \otimes A \otimes P_U) \otimes P_U^T \otimes x = P_U^T \otimes A \otimes x \in V(C,\lambda).$$

Moreover, $A \otimes x \in V(A,\lambda)$ implies $x \in V(A,\lambda)$, hence $P_U^T \otimes x \in V(C,\lambda)$ and we conclude that C is weak λ-robust. $\square$

Theorem 4.7. Let $A = (a_{ij}) \in B(n,n)$, $\lambda \in B$ and k be natural number. If A is weakly λ-robust then A^k is weakly λ-robust for each k.

Proof. Let us suppose that $A = (a_{ij}) \in B(n,n), \lambda \in B$ and k is a natural number. It is known that a power of a permutation matrix is again a permutation. The assertion follows from the fact that $c(A) = c(A^k)$. $\square$

We can use the obtained results to derive an algorithm for checking the weakly λ-robustness of a given matrix.

Algorithm **Weak Robustness**

Input. $A = (a_{ij})$, $\lambda \in B$.
Output. 'yes' in variable wr if A is weakly λ-robust; 'no' in wr otherwise.
begin

if $A = O$ and $\lambda = O$ **then** wr:='yes' **else** wr:='no'

if $A \neq O$ and $\lambda = O$ and C_A contains no O column **then** wr:='yes' **else** wr:='no'

if $A \neq O$ and $\lambda > O$ and $A \approx (A_1, \ldots, A_s)$ and

$(\forall i \in \{1, \ldots, s\})[\lambda < c(A_i)\ \vee\ \lambda = c(A_i) = m(A_i)]$ **then** wr:='yes' **else** wr:='no'

end

Theorem 4.8. Let A be a fuzzy matrix and $\lambda \in B$. The algorithm **Weak Robustness** correctly decides whether a matrix A is weakly λ-robust in $O(n^2)$ arithmetic operations.

Proof. The number of operations for checking the equivalence A to a block-diagonal permutation matrix is $O(n^2)$. Thus, the complexity of all steps of the algorithm is $3.O(n^2) = O(n^2)$. □

We conclude the section by a table which describes the efficient solvable cases of the reachability and present the main results on the strong λ-robustness, λ-robustness [14] and the weak λ-robustness of matrices.

Type$\|\lambda$	$\lambda < c(A)$	$\lambda = c(A) \leq m_A$	$c(A) < \lambda$	$m_A < \lambda$
weak λ-robustness	$O(n^2)$	$O(n^2)$	open	open
λ-robustness	open	open	open	$O(n^3)$ [14]
strong λ-robustness	open	open	$O(n^3)$ [14]	$O(n^3)$ [14]

Remark 4.1. In max-plus algebra, a matrix is weakly robust (i. e. weakly stable) if and only if each spectral class is initial and its critical graph is a Hamiltonian cycle, see [1] for the background and explanation. The main result of our paper (Theorem 4.5) presents a similar description of weakly robust matrices in max-min algebra.

ACKNOWLEDGEMENTS

The author would like to thank the referees for valuable remarks and suggestions that helped to increase the clarity of arguments in the paper.

This work was supported by the Slovak Cultural and Education Agency No. 032TUKE-4/2013 and is also the result of the project implementation Development of the Center of Information and Communication Technologies for Knowledge Systems (project number: 26220120030) supported by the Research & Development Operational Program funded by the ERDF.

(Received January 10, 2012)

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